11-d Sugra
(M)

\[ \text{D=10} \quad \text{IIA} \quad \text{IIB} \quad \text{E}_8 \times \text{E}_8 \quad \text{SO(32)} \leftrightarrow \text{I} \]

D=9

More such equivalence relations exist.
Introduction to Calabi-Yau manifolds:

Ref: Complex manifolds without potential theory (S.S. Chern)

Real k-dimensional manifold: Covered by coordinate charts \( \{ U_i \} \). - open sets.

On each \( U_i \), choose coordinates:
\[
(y^{(i)}_1, ..., y^{(i)}_k) \quad y^{(i)}_j < 1
\]

On \( U_i \cap U_j \)
\[
y^m_{(i)} = f^m_{(i)}(y^{(i)}_{(j)}) \quad 1 \leq m \leq k
\]

to some function.

A 2n-dimensional real manifold is a complex manifold if we can choose \( n \) complex coordinates \( z^{(i)}_1, ..., z^{(i)}_n \) on each coordinate chart \( U_i \) such that on \( U_i \cap U_j \)
\[
z^m_{(i)} = f^m_{(i)}(z^{(i)}_{(j)}) \quad 1 \leq m \leq n
\]

complex analytic function.
Complex structure $\mathbf{E}$: Multiplication by $i$

on the tangent space:

$$\frac{\partial}{\partial z^m} (x^i) \rightarrow -i \frac{\partial}{\partial z^m} (x^i) \quad m = 1, \ldots, n.$$ 

On $U_i \cap U_j$:

$$\frac{\partial}{\partial z^m} (x^i) \rightarrow i \frac{\partial}{\partial z^m} (x^i),$$

$$\frac{\partial}{\partial \bar{z}^m} (x^i) \rightarrow \frac{\partial}{\partial \bar{z}^m} (x^i).$$

(Would not be true if $\bar{\mathbb{R}}$ is a $\mathbb{C}$ of $\mathbb{C}(i)$ and $\mathbb{C}(i)$.)

Translated to the real coordinates $J$ is a $2n \times 2n$ matrix satisfying:

1. $J^2 = -1$

2. A differential eq. (reflection of Cauchy-Riemann condition)
Example: \( z = x + iy \)
\[ x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i} \]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{i}{\partial y} \right),
\quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial y} - \frac{i}{\partial x} \right)
\]

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial z} \\
\frac{\partial}{\partial \bar{z}}
\end{bmatrix}
\]

\[ J \]

Note: Same real manifold may admit different complex structures.

Example: Two dimensional torus \( T^2 \).
\[
(y', y^2) = (y' + 2\pi, y^2) = (y', y^2 + 2\pi).
\]

One choice: \( z = y' + iy^2 \).

\[ z \equiv z + i \equiv z + i 
\]

General choice: \( z = y' + r y^2 \)

\[ \text{Im} z > 0. \]

Example calculate \( J \).
Two manifolds which are identical as real manifolds may not be identical as complex manifolds.

Complex topology is more refined.

Metric on complex manifold \( M \):

\[
\mathrm{d}s^2 = g_{\alpha \overline{\beta}} \mathrm{d}z^\alpha \mathrm{d}\overline{z}^{\overline{\beta}} + g_{\alpha \beta} \mathrm{d}z^\alpha \mathrm{d}z^\beta
\]

Reality of \( \mathrm{d}s^2 \):

\[
\exists \ g_{\alpha \beta}^* = g_{\beta \alpha}^* , \ g_{\alpha \overline{\beta}}^* = g_{\overline{\beta} \alpha}^*
\]

Hermitean metric: \( g_{\alpha \beta} = 0 = g_{\overline{\beta} \alpha} \)

Kahler metric: Hermitean with

\[
g_{\alpha \overline{\beta}} = 2 \cdot \mathrm{d}z^\alpha \mathrm{d}\overline{z}^{\overline{\beta}} K
\]

on each coordinate chart.

On the overlap \( U_i \cap U_j \):

\[
K_i (\overline{z}_i^{(i)}, \overline{z}_i^{(j)}) = K_i (\overline{z}_i^{(j)}, \overline{z}_i^{(i)}) + f_{ij} (\overline{z}_i^{(j)}) + g_{ij} (\overline{z}_i^{(j)})
\]
Notion of holonomy:
Take a point \( P \in M \).

\( C \): a closed curve in \( M \) passing through \( P \).

1. Take an arbitrary \( 2n \) dimensional vector \( \vec{a} \) in the tangent space at \( P \).
2. Parallel transport it around \( C \).
   - new vector \( \vec{a}' = R(c) \vec{a} \).
   - \( 2n \times 2n \) matrix.

On Riemannian manifold \( \mathbb{R}^{2n} \):
\( R(c) \in SO(2n) \).

Collection of \( R(c) \) for all possible \( C \) passing through \( C \) form a group under multiplication.

\( C, C' \): closed curves passing through \( C \).
\( C \circ C' \): new closed curve that first goes along \( C' \) then along \( C \).
\( R(c) \ast R(c') = R(c \cdot c') \)

- A closed curve traversing \( c \) in opposite orientation.

\[ R(c) \ast R(c^{-1}) = R(c \cdot c^{-1}) = I \]

B Collection of \( R(c) \): Holonomy group \( G \).

\( G \) is a subgroup of \( SO(2n) \).

\( \ast \) independent of \( P \).

[Transport from \( P \) to \( P' \), then along \( C \), then back \( \ast \) to \( P' \) along the same curve.

\( \ast \) conjugation]

For a Kahler metric on a complex \( n \)-dimensional manifold:

\( G = U(n) \subset SO(2n) \).

\( \frac{\omega}{2\pi} \rightarrow U, \quad \frac{\partial}{\partial z} \rightarrow 0 \)}
Calabi–Yau manifold: Complex manifold which admit Kahler metric with $SU(n)$ holonomy $SU(n) \subset U(n)$.

Yau's theorem: Calabi–Yau manifolds admit One can show that $SU(n)$ holonomy $\Rightarrow$ vanishing Ricci tensor.

$\Rightarrow$ Satisfies Einstein's equation.

A Calabi–Yau manifold with a given complex structure typically admits a family of Kahler metrics with $SU(n)$ holonomy.

$\Rightarrow$ Kahler moduli space.

Both will be described in more detail later.
A Calabi-Yau manifold, regarded as a real manifold, admits a family of complex structures. A complex structure moduli space.

Example of a CY manifold:

- Torus.

\[ \mathbb{T} = \mathbb{T} + 1 = \mathbb{T} + \mathbb{T} \]

\[ ds^2 = a |d\bar{z}|^2 \text{ is invariant under constant } \mathbb{Z} \rightarrow \mathbb{Z} + 1, \mathbb{Z} \rightarrow \mathbb{Z} + \mathbb{Z}. \]

Holonomy: Trivial. \( = \mathbb{SU}(1) \).

- \( K = a \mathbb{Z} \)

Kähler moduli space: One real dimensional parameterized by \( a \).

Complex structure moduli space: One complex dimensional parameterized by \( \tau \).
Note: A manifold being Calabi-Yau is a topological property.

Two complex manifolds are topologically the same if they can be mapped to each other by analytic change of coordinates.

Calabi-Yau manifold: A complex manifold that admits a Kähler metric with $SU(n)$ holonomy.

It admits many other metrics which are not of $SU(n)$ holonomy. No important since in full string theory the metric on Calabi-Yau that is needed is Kähler but not of $SU(n)$ holonomy.

Ricci tensor $\neq 0$ effect of higher derivative terms.
Homology and Cohomology

A k-dimensional submanifold $C_k$ of $M$ is called a k-cycle if

$$2C_k = 0.$$ 

The boundary of $C_k$, $0 = \text{empty set}$.

$$\partial(2C_k) = 0 \text{ always}.$$ 

$C_k$ is called exact if

$$C_k = \partial B_{k+1}$$

for some $(k+1)$ dimensional subspace $B_{k+1}$ of $M$.

Given 2 k-cycles $C_k$, $\tilde{C}_k$ they are declared to be equivalent if

$$C_k = \tilde{C}_k + \partial B_{k+1}$$

for some $B_{k+1}$.

$H_k(\mathbb{Z})$: Integer cohomology $\pi_k$ of equivalent classes of $C_k$. 

$H_k(\mathbb{Z})$: group under addition.

If $C_k, \overline{C}_k \in H_k(\mathbb{Z})$ then

$C_k + \overline{C}_k = C_k \cup \overline{C}_k \in H_k(\mathbb{Z})$

$-C_k = C_k$ in opposite orientation.

Note: $H_k(\mathbb{Z})$ may contain element $A_k$ such that $nA_k = 0$ for some integer $n$.

Example: $M = 2$-dim. sphere with diametrically opposite points identified.

A curve $C$ connecting a point $P$ with antipodal point $P'$ has no boundary and is not contractible.

A non-trivial element of $H_k(\mathbb{Z})$.

But going around the point twice makes it contractible.

A trivial element of $H_k(\mathbb{Z})$. 
Real/complex homology: $H_k(R), H_k(C)$

- A vector space consisting of formal sums of $k$-cycles with real/complex coefficients.

$$A = \sum_k x_k A_k, \quad A_k: \text{$k$-cycle, } x_k \in \mathbb{R}, \mathbb{C}$$

Equivalence relation:

$$A \equiv B \iff A = B + \sum_j \beta_j \partial B_j$$

$E \subseteq \mathbb{R}$ or $E \subseteq \mathbb{C}$ subspace of $M$.

Note: If $nc$ is contractible then $nc = 2B$

$$c = \frac{1}{n} 2B$$

$C$ is a trivial element of $H_k(R)$ and $H_k(C)$.