11-d sugra
(M)

S^1

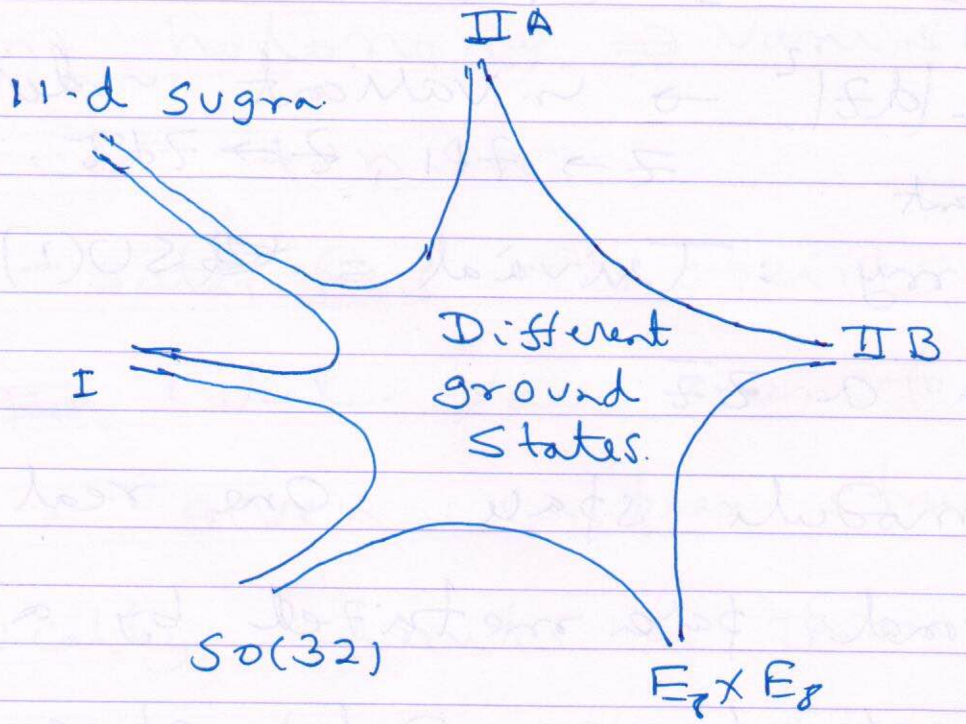
D=10   IIA   IIB $E_8 \times E_8$ $SO(32) \leftrightarrow$ I

S^1 S^1 S^1 S^1

$\longleftrightarrow$ $\longleftrightarrow$

D = 9

More such equivalence relations exist.



Introduction to Calabi-Yau manifolds:

Ref. Complex manifolds without potential theory (S. S. Chern)

Real k-dimensional manifold: Covered by coordinate charts $\{U_i\}$. — open sets.

On each U_i choose coordinates:

$$\vec{y}_{(i)} = (y^1_{(i)}, \dots y^k_{(i)}) \qquad \vec{y}^{\,2}_{(i)} < 1$$

On $U_i \cap U_j$

$$y^m_{(i)} = f^m_{(i)}(\vec{y}_{(j)}) \qquad 1 \le m \le k$$

↳ some function.

A 2n-dimensional real manifold is a complex manifold if we can choose n complex coordinates $\vec{z}_{(i)} = (z^1_{(i)}, \dots z^n_{(i)})$ on each coordinate chart U_i such that on $U_i \cap U_j$

$$z^m_{(i)} = f^m_{(i)}(\vec{z}_{(j)}) \qquad 1 \le m \le n$$

complex analytic function

Complex structure J: effect of multiplication by i on the tangent space.

$$\frac{\partial}{\partial z^m_{(i)}} \to -i \frac{\partial}{\partial z^m_{(i)}} \qquad m = 1, \cdots n.$$

$$\frac{\partial}{\partial \bar{z}^m_{(i)}} \to i \frac{\partial}{\partial \bar{z}^m_{(i)}}$$

On $U_i \cap U_j$

$$\frac{\partial}{\partial z^m_{(j)}} = \sum_l \frac{\partial f^l_{(i)}}{\partial z^m_{(j)}} \frac{\partial}{\partial z^l_{(i)}}$$

$$\frac{\partial}{\partial z^m_{(j)}} \to -i \frac{\partial}{\partial z^m_{(j)}}.$$

(Would not be true if $z^m_{(i)}$ is a fn. of $\vec{z}_{(j)}$ and $\bar{\vec{z}}_{(j)}$.)

Translated to the real coordinates J is a $2n \times 2n$ matrix satisfying

① $J^2 = -1$

② A differential eq. (reflection of Cauchy-Riemann condition)

Example: $z = x + iy \quad x = \frac{z+\bar{z}}{2} \quad y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z} \to -i\frac{\partial}{\partial z} \Rightarrow \frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \to -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x} \to -\frac{\partial}{\partial y} \quad \frac{\partial}{\partial y} \to \frac{\partial}{\partial x}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\Downarrow$$

$$J$$

Note: Same real manifold may admit different complex structures.

Example: Two dimensional torus T^2.

$$(y^1, y^2) \equiv (y^1 + 2\pi, y^2) \equiv (y^1, y^2 + 2\pi).$$

One choice: $z = y^1 + iy^2$.

$$z \equiv z + 1 \equiv z + i$$

General choice: $z = y^1 + \tau y^2$

τ complex no.

$\mathrm{Im}\,\tau > 0$.

$$z \equiv z + 1 \equiv z + \tau$$

Ex- Calculate J.

Two manifolds which are identical as real manifolds may not be identical as complex manifolds.

Complex topology → more refined.

Metric on complex manifold M

$$ds^2 = g_{\alpha\bar\beta}\, dz^\alpha d\bar z^\beta + g_{\alpha\beta}\, dz^\alpha dz^\beta + g_{\bar\alpha\bar\beta}\, d\bar z^\alpha d\bar z^\beta \quad \text{— one chart.}$$

Reality of ds^2

$$\Rightarrow g_{\alpha\bar\beta}^* = g_{\beta\bar\alpha}, \quad g_{\alpha\beta}^* = g_{\bar\alpha\bar\beta}$$

Hermitian metric: $g_{\alpha\beta} = 0 = g_{\bar\alpha\bar\beta}$.

Kahler metric: Hermitian with

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} K$$

on each coordinate chart.

On the overlap $U_i \cap U_j$

$$K_i(\vec z_{(i)}, \bar{\vec z}_{(i)}) = K_j(\vec z_{(j)}, \bar{\vec z}_{(j)}) + f_{ij}(\vec z_{(j)}) + g_{ij}(\bar{\vec z}_{(j)})$$

Notion of holonomy:

Take a point $P \in M$.

C: a closed curve in M passing through P.

(1) Take an arbitrary $2n$ dimensional vector $\vec{a}$ ~~at~~ in the tangent space at P.

(2) Parallel transport it around C.

$\Rightarrow$ new vector $\vec{a}' = R(C)\,\vec{a}$

$$\underset{2n \times 2n \text{ matrix.}}{R(C)}$$

On Riemannian manifold ~~R(C)~~

$$R(C) \in SO(2n).$$

Collection of $R(C)$ for all possible C passing through C form a group under multiplication.

C, C': closed curves passing through C

$C \circ C'$: new closed curve that first goes along C' then along C.

$R(C) \circ R(C') = R(C \circ C')$.

C^{-1} : A closed curve traversing C in opposite orientation.

$R(C) \circ R(C^{-1}) = R(C \circ C^{-1}) = I$

Collection of $R(C)$: Holonomy group G.

G : subgroup of $SO(2n)$

Independent of P.

[Transport from P^r to P^e, then along C, then back to P' along the same curve.

$\Rightarrow$ conjugation]

For a kahler metric on a complex n-dimensional manifold:

$G = U(n) \subset SO(2n)$.

$$\frac{\partial}{\partial z^i} \rightarrow U_i{}^j \frac{\partial}{\partial z^j}$$

Calabi-Yau manifold: Complex manifold which admit Kahler metric with $SU(n)$ holonomy

$SU(n) \subset U(n)$.

~~Yau's theorem: Calabi-Yau manifolds admit~~

One can show that

$SU(n)$ holonomy $\Rightarrow$ vanishing Ricci tensor.

$\Rightarrow$ satisfies Einstein's equation.

A ~~[illegible]~~ Calabi-Yau manifold with a given complex structure typically admits a family of Kahler metrics with $SU(n)$ holonomy.

$\Rightarrow$ Kahler moduli space

Both will be described in more detail later.

A Calabi-Yau manifold, regarded as a real manifold, admits a ~~family~~ family of complex structures.

$\Rightarrow$ Complex structure moduli space.

Example of a C-Y manifold:

$\Rightarrow$ Torus.

$z \equiv z+1 \equiv z+\tau$

$ds^2 = \underset{\substack{\uparrow \\ \text{constant}}}{a}\,|dz|^2 \rightarrow$ invariant under $z \rightarrow z+1$, $z \rightarrow z+\tau$.

Holonomy : Trivial. $=$ ~~SU~~ $SU(1)$.

$K = a\, z\bar{z}$

Kahler moduli space: One real dimensional parametrized by a.

Complex structure moduli space: One complex dimensional parametrized by τ.

Note: A manifold being Calabi-Yau is a topological property.

Two complex manifolds are topologically the same if they can be mapped to each other by analytic change of coordinates.

Calabi-Yau manifold: A complex manifold that <u>admits</u> a Kahler metric with $SU(n)$ holonomy.

It ~~can~~ admits many other metrics which are not of $SU(n)$ holonomy. So important since in full string theory the metric on Calabi-Yau that is needed is Kahler but not of $SU(n)$ holonomy.

Ricci $\to$ tensor $\neq 0$ $\to$ effect of higher derivative terms.

Homology and cohomology

A k-dimensional ~~x~~ submanifold C_k of M is called a k-cycle if

$$\partial C_k = 0.$$

Boundary of C_k. $0 \equiv$ empty set.

$$\partial(\partial C_k) = 0 \quad \text{always.}$$

C_k is called exact if

$$C_k = \partial B_{k+1}$$

for some $(k+1)$ dimensional subspace B_{k+1} of M.

Given 2 k-cycles $C_k, \tilde{C}_k$ they are declared to be equivalent iff

$$C_k = \tilde{C}_k + \partial B_{k+1}$$

for some B_{k+1}.

Integer cohomology $\mathcal{H}_k(\mathbb{Z})$: space of equivalence classes of C_k

$\mathcal{H}_k(\mathbb{Z})$:- group under addition.

If $C_k, \tilde{C}_k \in \mathcal{H}_k(\mathbb{Z})$ then

$C_k + \tilde{C}_k \equiv C_k \cup \tilde{C}_k \in \mathcal{H}_k(\mathbb{Z})$

$-C_k = C_k$ in opposite orientation.

Note: $\mathcal{H}_k(\mathbb{Z})$ may contain element

A_k such that $n A_k = 0$.

integer.

Example: M = 2-dim. sphere with

diametrically opposite points identified.

A curve C connecting a point P with

anti-podal point P' has no

boundary and is not contractible.

$\Rightarrow$ Non-trivial element of $\mathcal{H}_1(\mathbb{Z})$.

But going around the ~~point~~ curve twice

makes it contractible.

$\rightarrow$ A trivial element of $\mathcal{H}_1(\mathbb{Z})$.

Real / Complex homology: $\mathcal{H}_k(\mathbb{R})$, $\mathcal{H}_k(\mathbb{C})$

A vector space consisting of formal sums of k-cycles with real/complex coefficients.

$$A = \sum_\ell \alpha_\ell A_\ell \qquad A_\ell : \text{k-cycle.}, \ \alpha_\ell \in \mathbb{R}, \mathbb{C}$$

Equivalence relation:

$$A \equiv B \quad \text{iff} \quad A = B + \sum_s \beta_s \, \partial B_s$$

β_s: $\in \mathbb{R}$ or $\mathbb{C}$

B_s: subspaces of M.

Note: If nC is contractible then

$$nC = \partial B$$

$$C = \frac{1}{n} \partial B$$

$\Rightarrow$ C is a trivial element of $\mathcal{H}_k(\mathbb{R})$ and $\mathcal{H}_k(\mathbb{C})$