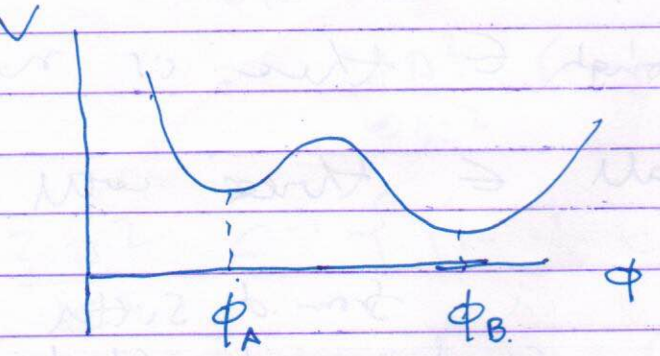
Vacuum decay in QFT

$$S = \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]$$



Goal: Compute the probability per unit volume per unit time for decay from false to true vacuum.

Euclidean action:

$$S_E = \int d^4x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right]$$

Decay probability $\propto \exp(-C\,(S_E(\text{bounce}) - S_E(\phi_A)))$

- Bounce: a soln. satisfying:

$\phi(\tau, \vec{x}) \to \phi_A$ for $\tau \to \pm\infty$

$\phi(\tau, x) \to \phi_A$ for $|\vec{x}| \to \infty$.

ϕ_A : The soln. $\phi(\tau, \vec{x}) = \phi_A$.

The condition $\phi(\tau, \vec{x}) \to \phi_A$ for $\tau \to \pm\infty$

$$\Rightarrow \int \left(\frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) + V(\phi_A) \right)$$

$$\to 0 \quad \text{as } \tau \to \pm\infty$$

$\Rightarrow$ Guarantees that it will remain 0 at all times.

At $\tau = 0$. $\partial_\tau \phi = 0$ ($\tau \to -\tau$ symmetry)

$\phi(\tau = 0, \vec{x})$: a zero energy configuration (relative to false vacuum)

$\Rightarrow$ the bubble which then expands classically.

We shall assume that the bounce with minimal S_E is spherically symmetric.

$$\phi = \phi(\rho) \qquad \rho = \sqrt{\tau^2 + \vec{x}^2}.$$

ϕ eq. simplifies:

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = V'(\phi)$$

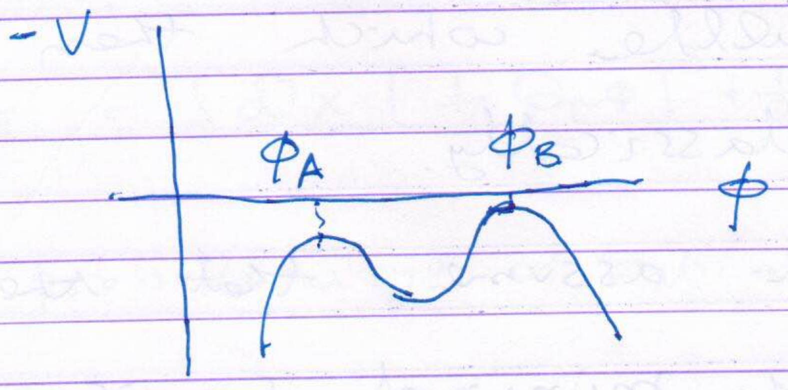
$$S_E = 2\pi^2 \int_0^\infty \rho^3 \, d\rho \left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V(\phi) \right]$$

Decay rate: $\exp(-B)$

$B = S_E(\text{bounce}) - S_E(\phi_A)$

Interpretation of ϕ eq:

→ Particle moving in a potential $-V(\phi)$ with friction coefficient $\frac{3}{\rho}$. | ρ = "time"



Strategy for finding the soln.:

Need: $\rho \to \infty$, $\phi \to \phi_A$

At $\rho = 0$ we need to start at a point between ϕ_A, ϕ_B.

$\Rightarrow$ Need the particle to lose just enough energy due to friction so that it comes to rest at ϕ_A.

Is this possible?

① If $\phi(\rho=0)$ is too close to ϕ_A the particle will undershoot.

② Suppose we begin close to ϕ_B.

By beginning sufficiently close to ϕ_B we can ensure that the particle stays close to ϕ_B for sufficiently long "time".

$\rightarrow$ The particle will begin rolling effectively at large ρ

$\Rightarrow$ - Friction term negligible

$\Rightarrow$ The particle will overshoot.

Thus there must be some point between ϕ_A and ϕ_B so that if we start there at $\rho = 0$, the particle will reach ϕ_A

$\Rightarrow$ bounce solution.

Thin wall approximation:

$$V(\phi_A) - V(\phi_B) = \text{small} = \epsilon$$

$\Rightarrow$ Add some potential of order ϵ to $V(\phi)$ we get a new potential $U(\phi)$ such that

$$U(\phi_A) = U(\phi_B)$$

$$V(\phi) = U(\phi) + \epsilon W(\phi).$$

$U(\phi)$ has a soln. $\phi_0(\rho)$ if

$$\frac{d^2\phi_0}{d\rho^2} = U'(\phi)$$

that begins at ϕ_A and ends at ϕ_B.

$\phi = F(\rho)$ is the soln.

$$F(\rho = \infty) = \phi_A$$
$$F(\rho = -\infty) = \phi_B.$$

Note $F(\rho + c)$ is also soln. for any constant c. ~~except that it does not obey b.c.~~

Define $F(\rho)$ such that

$$F(0) = \frac{\phi_A + \phi_B}{2}$$

$$F(\rho) = \phi_B \text{ for } \rho << 0$$
$$= \phi_B \text{ for } \rho >> 0$$

Now consider the bounce:

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = U'(\phi) + \epsilon W'(\phi)$$

Since $-V(\phi_B)$ is only slightly large that $-V(\phi_A)$ we cannot afford to lose much energy due to friction.

$\Rightarrow$ Have to reduce $\frac{3}{\rho}$.

$\Rightarrow$ Need to adjust things so that ϕ stays sufficiently close to ϕ_B for long "time" and then begins rolling. so that the effect of the friction term is small.

When it rolls significantly it satisfies.

$$\frac{d^2\phi}{d\rho^2} \simeq U'(\phi)$$

Soln. $\phi = F(\rho - \bar{\rho})$ for some large $\bar{\rho}$.

For $\rho \to \infty$. $\phi \to F(\infty) = \phi_A$

$\rho \to 0$ $\quad \phi \to F(-\bar{\rho}) \simeq \phi_B$ for large $\bar{\rho}$.

Remaining task: Find $\bar{\rho}$.

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V(\phi) \right]$$

S_E (bounce) $- S_E(\phi_A)$

$$= 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(F'(\rho-\bar\rho)\right)^2 + V(F(\rho-\bar\rho)) - V(\phi_A) \right]$$

3 separate contributions:

① $\rho - \bar\rho \gg 0$. $F \Rightarrow \phi_A$. $\Rightarrow$ integrand $=0$.

② $\bar\rho - \rho \gg 0$: $F \to \phi_B$.

$$2\pi^2 \int_0^{\bar\rho} \rho^3 d\rho \left[V(\phi_B) - V(\phi_A) \right]$$

$$= -2\pi^2 \epsilon \frac{\bar\rho^4}{4}$$

③ $|\rho-\bar\rho| \sim 1$ (within finite range of $\bar\rho$):

$$2\pi^2 \bar\rho^3 \int_0^\infty \left[\frac{1}{2} F'(x)^2 + U(F(x)) - U(\phi_A) \right]$$

$= \sigma$ (surface tension of bubble.)

Net contribution: $2\pi^2 \left(-\epsilon \frac{\bar\rho^4}{4} + \bar\rho^3 \sigma\right)$.

$\bar{\rho}$ should extremize this:

$$-\epsilon \bar{\rho}^3 + 3\bar{\rho}^2 \sigma = 0$$

$$\bar{\rho} = 3\sigma/\epsilon$$

$$B = S_E(\text{bounce}) - S_E(\phi_A)$$

$$= 2\pi^2 \left(-\epsilon \frac{\bar{\rho}^4}{4} + \bar{\rho}^3 \sigma\right)$$

$$= \frac{27\pi^2 \sigma^4}{2\epsilon^2}$$

Now recall that the solution at $\tau = 0$ is supposed to describe the bubble that will evolve classically.

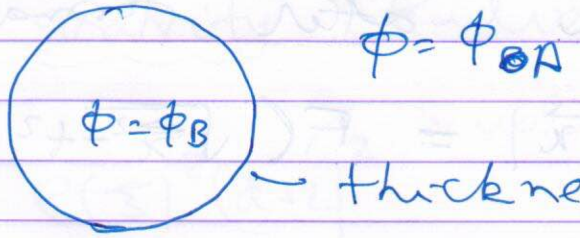
soln. at $\tau = 0$. $\rho = \sqrt{\tau^2 + |\vec{x}|^2} = |\vec{x}| = r$

$$\Rightarrow \phi(t=0, \vec{x}) = F(|\vec{x}| - \bar{\rho})$$

For $|\vec{x}| - \bar{\rho} \gg 0$, $\phi \to F(\infty) = \phi_A$

For $-|\vec{x}| + \bar{\rho} \gg 0$, $\phi \to F(-\infty) = \phi_B$

For $|\vec{x}| - \rho \sim 1$, F is the interpolating soln.



Calculate the energy (~~Minkowski~~ Lorentzian theory).

$\partial_t \phi = 0$ at $t = 0$. (since $\partial_\tau \phi = 0$)

$$E = \int d^3 r \left(\frac{1}{2} |\vec{\nabla} \phi|^2 + V(\phi) - V(\phi_A) \right)$$

energy relative to False vacuum.

$$= 4\pi \int r^2 dr \left(\frac{1}{2} F'(r - \bar{\rho})^2 + V(F(r - \bar{\rho})) - V(\phi_A) \right)$$

$$= 4\pi \bar{\rho}^2 \sigma - 4\pi \int_0^{\bar{\rho}} r^2 dr \, \epsilon$$

$$= 4\pi \left(-\epsilon \frac{\bar{\rho}^3}{3} + \sigma \bar{\rho}^2 \right)$$

→ vanishes for $\bar{\rho} = \frac{3\sigma}{\epsilon}$

→ same condition we had found by classical analysis.

Growth after formation:

$$\phi(t,\vec{x}) = F\left(\sqrt{\vec{x}^2-t^2} - \bar{\rho}\right).$$

~~For $\vec{x}^2-t^2$~~ ϕ changes from ϕ_A to ϕ_B across $\vec{x}^2-t^2=R^2$

$$\vec{x}^2 = t^2+R^2$$

An expanding bubble of radius

$$\sqrt{R^2+t^2}$$

For large t, $|\vec{x}| \to t$

$\Rightarrow$ expansion at the speed of light.

Inclusion of gravity (Euclidean)

$$ds^2 = -d\xi^2 + \rho(\xi)^2 (d\Omega)^2$$ vol. of unit 3-sphere

eqs: $$\phi'' + \frac{3\rho'}{\rho}\phi' = \frac{dV}{d\phi}$$

$$\rho'^2 = 1 + \frac{1}{3}\kappa\rho^2\left(\frac{1}{2}\phi'^2 - V\right)$$

$$\kappa = 8\pi G$$

ξ: radial coordinate.

$$S_E = 2\pi^2 \int d\xi \left(\rho^3\left(\frac{1}{2}\phi'^2 + V\right) + \frac{3}{\kappa}\left(\rho^2\rho'' + \rho\rho'^2 - \rho\right)\right)$$

Decay probability: $\propto e^{-B}$

$$B = S_E(\text{bounce}) - S_E(\phi_A)$$

$\hookrightarrow \phi = \phi_A$

$\rho(\xi) = dS/AdS/$ Minkowski

ϕ eq: only change

① variable ξ

② Friction: $3\rho'/\rho$ instead of $3/\rho$

neglected in the thin wall approximation anyway.