Let $A_a^{(k)}$ for $a = 1, 2, \cdots b_k$ be linearly independent k-cycles forming a basis of $\mathcal{H}_k(\mathbb{R})$ or $\mathcal{H}_k(\mathbb{C})$.

$\Rightarrow$ Generic element of $\mathcal{H}_k(\mathbb{R})$ or $\mathcal{H}_k(\mathbb{C})$

$$\sum_{a=1}^{b_k} \alpha_a A_a^{(k)} \qquad \alpha_a \in \mathbb{R} \text{ or } \mathbb{C}.$$

$$\sum_{a=1}^{b_k} \alpha_a A_a^{(k)} = 0 \quad \text{iff} \quad \alpha_a = 0 \ \forall\, a$$

We shall mostly deal with $\mathcal{H}_k(\mathbb{R})$ or $\mathcal{H}_k(\mathbb{C})$ & drop $\mathbb{R}$ or $\mathbb{C}$ from the argument of $\mathcal{H}_k$.

b_k: betti numbers

Given a k-cycle A and $(2n-k)$-cycle B, $A \cap B$ is a set of points.

\# of such points counted with orientation $\equiv (A, B)$.

Given $A = \sum_{a=1}^{b_k} \alpha_a A_a^{(k)} \in \mathcal{H}_k$, $B \equiv \sum_{b=1}^{b_{2n-k}} \beta_b A_b^{(2n-k)} \in \mathcal{H}_{2n-k}$

$$(A, B) \equiv \sum_{a,b} \alpha_a \beta_b \left(A_a^{(k)}, A_b^{(2n-k)}\right).$$

(A, B) depends only on the homology class of A, B.

(Follows from the fact that if we have a k-cycle $S_k = \partial R_{k+1}$ then

$$S_k \cap T_{2n-k} = 0 \quad \text{for any } (2n-k) \text{ cycle } T_{2n-k})$$

One can show that $(A_a^{(k)}, A_b^{(2n-k)})$ is a square, non-degenerate matrix.

($\mathcal{H}_k$ & $\mathcal{H}_{2n-k}$ have same dimension and are dual vector spaces).

$$b_k = b_{2n-k}$$

Given a vector space V, its dual is the set of linear maps from V to $\mathbb{R}$ or $\mathbb{C}$.

Consider a differential form of degree k:

$$A = \frac{1}{k!} \underbrace{A_{m_1 \cdots m_k}}_{\text{Totally anti-symmetric}} dx^{m_1} \wedge \cdots dx^{m_k}$$

$$dA \equiv \frac{1}{k!} \partial_{\cancel{k}m_0} A_{m_1 \cdots m_k} dx^{m_0} \wedge \cdots dx^{m_k}$$

$$d(dA) = 0.$$

k form $\omega^{(k)}_{\cancel{k}}$ is closed iff

$$d\omega^{(k)}_{\cancel{k}} = 0.$$

$\omega^{(k)}_{\cancel{k}}$ is exact iff

$$\omega^{(k)}_{\cancel{k}} = d\phi^{(k-1)}_{\cancel{k-1}}$$ for some $(k-1)$-form ϕ_{k-1}.

$$\omega^{(k)}_{\cancel{k}} \equiv \tilde{\omega}^{(k)}_{\cancel{k}} \quad \text{if} \quad \omega^{(k)}_{\cancel{k}} - \tilde{\omega}^{(k)}_{\cancel{k}} \text{ is exact.}$$

Equivalence classes of closed k-forms with real/complex coefficient functions form real/complex cohomology group H^k.

$\rightarrow$ a group under addition.

If $\omega^{(k)}_1, \cdots \omega^{(k)}_s$ denote a basis of linearly independent k-forms then general element of H^k is

$$\sum_{a=1}^{s} \beta_a \omega^{(k)}_a \qquad \beta_a \in \mathbb{R} \text{ or } \mathbb{C}$$

A k-form can be integrated over a k-cycle.

Given $A = \sum_a \alpha_a A_a^{(k)} \in \mathcal{H}_k$

$\omega = \sum_b \beta_b \omega_b^{(k)} \in \mathcal{H}^k$

define

$$(A, \omega) \equiv \sum_{a,b} \alpha_a \beta_b \int_{A_a^{(k)}} \omega_b^{(k)}$$

$\rightarrow$ depends only on the homology class of A and cohomology class of ω.

Follows from:

$$\int_A d\omega = 0 \quad , \quad \int_{\partial A} \omega = 0.$$

$\int_{A_a^{(k)}} \omega_b^{(k)}$ is a non-degenerate square matrix and depends only on the equivalence classes of $A_a^{(k)}$, $\omega_b^{(k)}$.

$\Rightarrow \quad s = b_k.$

$\mathcal{H}_k$ and $\mathcal{H}^k$ are dual vector spaces.

But $\mathcal{H}_k$ is also dual to $\mathcal{H}_{2n-k}$.

$\Rightarrow \mathcal{H}^k$ and $\mathcal{H}_{2n-k}$ are isomorphic.

~~dual~~ $\quad\quad\quad \Downarrow$

dual to $\mathcal{H}^{2n-k}$

$\Rightarrow \mathcal{H}^k$ and $\mathcal{H}^{2n-k}$ are dual.

Given $\omega_1 \in \mathcal{H}^k$, $\omega_2 \in \mathcal{H}^{2n-k}$

we define

$$(\omega_1, \omega_2) = \int_M \omega_1 \wedge \omega_2$$

$\rightarrow$ depends only of the cohomology classes of ω_1 and ω_2 and gives a non-degenerate pairing.

So far we have not assumed the existence of a metric.

For a manifold with metric we can choose representative elements of the cohomology to be harmonic forms.

$$\Delta \omega^{(k)}_b = 0 \qquad b = 1, \ldots b_k.$$

$$\Downarrow$$

$$(d\delta + \delta d) \qquad \Rightarrow \quad b_k = \# \text{ of harmonic } k\text{-forms.}$$

$$\delta \equiv * d *.$$

Note: Cohomology is a topological property of M and is independent of the choice of metric.

On a ~~Calabi-Yau~~ Complex manifold we can choose finer structure.

(p, q) form:

$$\omega^{(p,q)} = \frac{1}{p!\,q!} A_{\alpha_1 \ldots \alpha_p \bar\beta_1 \ldots \bar\beta_q} dz^{\alpha_1} \wedge \ldots dz^{\alpha_p} d\bar z^{\beta_1} \wedge \ldots d\bar z^{\beta_q}$$

$\mathcal{H}^{(p,q)}$ is isomorphic to the space of harmonic (p,q) forms.

$\mathcal{H}^{(p,q)}$: dual to $\mathcal{H}^{(n-p,n-q)}$

$$\left(\omega^{(p,q)}, \omega^{(n-p,n-q)}\right)$$

$$= \int_M \omega^{(p,q)} \wedge \omega^{(n-p,n-q)}$$

$h_{p,q}$: dimension of $\mathcal{H}_{p,q}$

$\rightarrow$ Hodge numbers of M.

$$b_k = \sum_{p=0}^{n} h_{p,k-p}.$$

$$h_{p,q} = h_{n-p,n-q}$$