Topological definition:
\[ \omega^{(k, \ell)} = \frac{1}{b!} \sum_{\sigma} \epsilon(\sigma) A_{\alpha_1} \cdots A_{\alpha_k} \frac{dx_{\beta_1}}{b_1} \wedge \cdots \wedge \frac{dx_{\beta_\ell}}{b_\ell} \]

\[ \bar{\omega}^{(k, \ell)} = \frac{1}{b!} (-1)^{b_1+1} \sum_{\sigma} \epsilon(\sigma) A_{\alpha_1} \cdots A_{\alpha_k} \frac{d\beta_1}{b_1} \wedge \cdots \wedge \frac{d\beta_\ell}{b_\ell} \]

\[ \bar{\omega}^{(k, \ell)} \] is totally anti-symmetric.

\[ \bar{\omega}^{(k, \ell)} \] is closed \((k, \ell)\) form: \( \bar{\omega}^{(k, \ell)} = 0 \)

\[ \bar{\omega}^{(k, \ell)} \] is exact \((k, \ell)\) form: \( \omega^{(k, \ell)} = \bar{\omega}^{(k, 2-1)} \)

\[ \bar{\omega}^2 = 0 \]

Equivalence relation:
\[ \omega^{(k, \ell)} = \omega^{(k, \ell)} \iff \]
\[ \omega^{(k, \ell)} - \bar{\omega}^{(k, 2-1)} = \bar{\omega}^{(k, 2-1)} \text{ for some } \omega^{(k, 2-1)} \]

Dimension of brane cohomology.
Representative elements can be taken to satisfy:

\[ (\bar{e} \star \bar{e} \star + \star \bar{e} \star \bar{e} ) \omega^{(b, p)} = 0 \]

Similar definitions can be made for cohomology using \( a \).

On a Kähler manifold:

\[ d \star d \star + \star d \star d = 2 (\bar{e} \star \bar{e} \star + \star \bar{e} \star \bar{e} ) \]

\[ = 2 (a \star a \star + \star a \star a ) \]

Same representative elements \( \omega^{(b, p)} \) represent cohomology of \( a \), \( \bar{a} \) and \( a \).

We can simply solve:

\[ (d \star d \star + \star d \star d) \omega^{(b, p)} = 0 \]

Number of such solutions = \( h_{b, p} \).

Hodge numbers.
Focus on Kahler manifolds from now on.

Consider a Kahler metric:

\[ g_{\phi} = \partial \bar{\partial} \phi \]

\[ J = g_{\phi} \partial \bar{\partial} \phi \]

\[ \partial \bar{\partial} J = 0. \quad (\text{also } \partial J = 0) \]

\[ \phi \] is a \((1,1)\) form in \( \Omega^1 \cap \Omega^1 \).

Kahler class.

Conversely, given an element of the Kahler class we can define a Kahler metric.

\( \exists \) Kahler moduli space is \( h_{1,1} \)-dimensional.

(For given complex structure)

Is there a similar understanding for the complex structure moduli space?
Caldesi's conjecture & Yau's theorem:

On a Kahler manifold we can introduce another \((1,1)\) form
\[
\omega = R e^\phi \, d\overline{z}^\alpha \wedge d\overline{z}^\beta.
\]
\(\omega\) is closed (also \(\omega\) closed).

\(\omega\) is an element of \(H^{1,1}\).

First Chern class.

If \(\omega\) is a trivial element of \(H^{1,1}\), i.e. \(\omega = E^\wedge (1,0)\) for some \((1,0)\) form \(E^\wedge (1,0)\), then we say that it has vanishing first Chern class.

Caldesi-Yau: A Kahler manifold with vanishing first Chern class admits a Ricci flat metric (a metric of \(SU(n)\) holonomy).
Complex structure moduli space:
Consider a Ricci flat Kähler metric $g_{\bar{\alpha} \bar{\beta}} \equiv dz^\alpha \, d\bar{z}^\beta$ on a CY manifold.

Now regard this as a real manifold keep the metric fixed, and deform the complex structure.

To first order the deformed metric is:

$$g_{\bar{\alpha} \bar{\beta}} \left( g_{\bar{\alpha} \bar{\beta}} + 8 g_{\bar{\alpha} \bar{\beta}} \, dz^\alpha \, d\bar{z}^\beta \right)$$

represent deformation of complex structure.
On a 2n dimensional Calabi-Yau manifold $h_{0,0} = 1$, $\Omega h \sigma_{n} = 1$.

\[ \sum_{\alpha} \omega_{\alpha} \wedge dx_{1} \wedge \cdots \wedge dx_{n} \wedge \partial \sigma = 0 \]

\[ \bar{\partial} \omega = 0 \Leftrightarrow \bar{\partial}_{\beta} \omega_{\alpha_{1} \cdots \alpha_{n}} = 0 \] harmonic

$\omega_{\alpha_{1} \cdots \alpha_{n}}$: Holomorphic $(n, 0)$ form.

Similarly $\bar{\partial}_{\beta_{1} \cdots \beta_{n}}$: Anti-holomorphic $(0, n)$ harmonic form.

Consider

\[ \omega_{\alpha_{1} \cdots \alpha_{n-1} \alpha} \]

$\omega_{\alpha_{1} \cdots \alpha_{n-1} \alpha}$ is an $(n-1, 1)$ form.

It can be shown to be harmonic $\in H^{(n-1, 1)}$.

Conversely, given an element of $H^{(n-1, 1)}$ we can construct $\omega_{\alpha_{1} \cdots \alpha_{n-1} \alpha}$.
Deformation of complex structure

Elements of $H^{n-1,1}$ and $H^{1,n-1}$.

Dimension of complex structure moduli space = $h_{n-1,1}$ complex.

Calabi-Yau 3-fold: (6-dimensional)

$\text{real}$

$\begin{align*}
h_{3,0} &= h_{0,3} = 1 \\
h_{1,2} &= h_{2,1}, & h_{11} = h_{22} & \text{not fixed.} \\
h_{1,0} &= h_{2,0} = h_{0,1} = h_{0,2} = h_{2,3} = h_{3,2} \\
&= h_{1,3} = h_{3,1} = 0
\end{align*}$

Calabi-Yau manifolds admit covariantly constant spinor.

E.g. take 6-d CY manifold.

$\text{SO}(6)$: tangent space group $\cong \text{SU}(4)$

Vectors: 6 of $\text{SO}(6)$

Spinors: 4 of $\text{SO}(6)$
Spinor field \( \tilde{\Sigma} \) is continued.

\[
(D_m \tilde{\Sigma})^a = (\tilde{\Sigma}^a + \Delta m + \frac{i}{4} \Lambda_{ab}(\tilde{T}^{ab})_{st} + \tilde{\Sigma}^s)^+ \tilde{\Sigma}^t
\]

\( \text{So}(6) \) generators in spinor representation.

Holonomy group \( \text{SU}(3) \).

Spins in first 3x3 block if we regard \( \text{SO}(6) \) as \( \text{SU}(4) \).

\[
4 = 3 + 1
\]

\[
\tilde{\Sigma} \downarrow \text{Singlet}
\]

One component does not transform under \( \text{SU}(3) \).

\[
\omega_{ab}(\tilde{T}^{ab})_{st} \tilde{\Sigma}^t = 0 \quad \text{if}
\]

\( \tilde{\Sigma} \) is taken to be a singlet.

\[
\vartheta_m \tilde{\Sigma}^t = 0 \quad \Rightarrow \quad \tilde{\Sigma}^t = \text{constant}
\]

Covariantly constant spinor.
Explicit construction:

1. Pick a point $P \in CY_3$.

2. Pick the basis in spinor representation $SU(3)$ such that the holonomy group at $P$ is:

$$
\begin{pmatrix}
SU(3) & 0 \\
0 & 1
\end{pmatrix}
$$

3. Take $\mathfrak{g}(P) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and define $\mathfrak{g}(Q)$ at other point $Q$ by parallel transport along a curve joining $P$ and $Q$.

The result is independent of the choice of basis since the holonomy group leaves $\mathfrak{g}(P)$ invariant.
Consequence: Take a 10-d Majorana-Weyl spinor.

\[ 16 \rightarrow (4 \otimes \overline{4})_{\text{left-handed Weyl}}_{\text{SO}(6)} + (4 \otimes \overline{4})_{\text{right-handed Weyl}}_{\text{SO}(6)} \]

Existence of a covariantly constant spinor on CY3.

\[ \Rightarrow \text{Existence of a } 16-d \text{ spinor satisfying:} \]

\[ D_m \dot{\phi} = 0 \quad D_m \dot{\phi} = 0 \]

Choose the SU(3) singlet component of 4.

\[ \Rightarrow 4 \text{ parameter } \text{SO}(6) \]

\{ Left handed Weyl + right handed Weyl \}

\[ 2 \text{ complex } = 4 \text{ real.} \]
Heterotic type I on CY_3

\[ N = 1 \text{ SUSY in } D = 4. \]

Type IIA / IIB on CY_3

\[ N = 2 \text{ SUSY in } D = 4. \]