Topological definition:

$$\omega^{(p,q)} \Rightarrow \cancel{f(\vec{z}_{\alpha}, \vec{\bar{z}})}$$

$$\equiv \frac{1}{p!\,q!} A_{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \cdots \bar{\beta}_q}(\vec{z}, \vec{\bar{z}})\, dz^{\alpha_1} \wedge \cdots dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots d\bar{z}^{\beta_q}$$

$$\bar{\partial}\omega^{(p,q)} = \frac{1}{p!\,q!}(-1)^{p}\, \bar{\partial}_{\beta_0} A_{\alpha_1 \cdots \alpha_p \bar{\beta}_1 \cdots \bar{\beta}_q}$$

$$dz^{\alpha_1} \wedge \cdots dz^{\alpha_p} \wedge \underbrace{d\bar{z}^{\beta_0} \wedge \cdots d\bar{z}^{\beta_q}}_{\text{totally anti-symmetric}}$$

$\bar{\partial}$-closed (p,q) form: $\bar{\partial}\omega^{(p,q)} = 0$

$\bar{\partial}$-exact (p,q) form $\omega^{(p,q)} = \bar{\partial}\omega^{(p,q-1)}$.

$$\bar{\partial}^2 = 0$$

Equivalence relation:

$\omega^{(p,q)} \cong \tilde{\omega}^{(p,q)}$ iff

$\omega^{(p,q)} - \tilde{\omega}^{(p,q)} = \bar{\partial}\omega^{(p,q-1)}$ for some $\omega^{(p,q-1)}$.

$\rightarrow$ Dolbeault cohomology.

Representative elements can be taken to ~~be~~ satisfy:

$$(\bar{\partial} * \bar{\partial} * + * \bar{\partial} * \bar{\partial})\, \omega^{(p,q)} = 0$$

Similar defn. can be made for cohomology using ∂.

On a Kahler manifold:

$$d * d * + * d * d = 2(\bar{\partial} * \bar{\partial} * + * \bar{\partial} * \bar{\partial})$$

$$= 2(\partial * \partial * + * \partial * \partial)$$

$\Rightarrow$ Same representative elements $\omega^{(p,q)}$ represent cohomology of d, $\bar{\partial}$ and ∂.

We can simply solve

$$(d * d * + * d * d)\, \omega^{(p,q)} = 0.$$

# of such solutions = $h_{p,q}$.

Hodge numbers.

Focus on Kahler manifolds from now on.

Consider a Kahler metric:

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$$

$$J \equiv g_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$$

$\bar{\partial} J = 0$. (also $\partial J = 0$)

$\rightarrow$ a $(1,1)$ form $\in \mathcal{H}^{1,1}$.

$\downarrow$

Kahler class.

Conversely, given an element of the Kahler class we can define a Kahler metric

$\Rightarrow$ Kahler moduli space is $h_{1,1}$ dimensional.

(For given complex structure)

Is there a similar understanding for the complex structure moduli space?

Calabi's conjecture & Yau's theorem:

~~Con~~ On a Kahler manifold we can introduce another (1,1) form

$$\Omega^{(1,1)} = R_{\alpha\bar{\beta}}\, dz^{\alpha}\, d\bar{z}^{\beta}$$

↓

$\bar{\partial}$ closed (also ∂-closed)

$\Rightarrow$ an element of $\mathcal{H}^{1,1}$

↓

First Chern class.

If $\Omega^{(1,1)}$ is a trivial element of $\mathcal{H}^{(1,1)}$ i.e. $\Omega^{(1,1)} = \bar{\partial}\Lambda^{(1,0)}$ for some (1,0) form $\Lambda^{(1,0)}$, then we say that it has vanishing first Chern class.

Calabi-Yau: A Kahler manifold with vanishing first Chern class admits a Ricci flat metric (a metric of $SU(n)$ holonomy).

Complex structure moduli space:

Consider a Ricci flat Kahler metric

$g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta$ on a CY manifold.

Now regard this as a real manifold, keep the metric fixed, and deform the complex structure:

To first order the deformed metric is:

$$\underbrace{(g_{\alpha\bar\beta} + \delta g_{\alpha\bar\beta})\, dz^\alpha\, d\bar z^\beta}_{\text{deformation of Kahler structure}}$$

$$+ \;\delta g_{\alpha\beta}\, dz^\alpha\, dz^\beta + \delta g_{\bar\alpha\bar\beta}\, d\bar z^\alpha\, d\bar z^\beta$$

represent deformation of complex structure.

On a $2n$ dimensional Calabi-Yau manifold $h_{n,0} = 1$; $h_{0,n} = 1$

$$\Downarrow$$

$$\omega = \omega_{\alpha_1 \ldots \alpha_n} dz^{\alpha_1} \wedge dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_n} \quad \text{with}$$

$$\bar{\partial}\omega = 0 \Rightarrow \bar{\partial}_{\beta}\, \omega_{\alpha_1 \ldots \alpha_n} = 0$$

$\omega_{\alpha_1 \ldots \alpha_n}$: Holomorphic $(n,0)$ harmonic form.

Similarly $\overline{\omega}_{\bar{\beta}_1 \ldots \bar{\beta}_n}$: Anti-holomorphic $(0,n)$ harmonic form.

Consider

$$\omega_{\alpha_1 \ldots \alpha_{n-1} \alpha}\, g^{\alpha \bar{\beta}}\, \delta g_{\bar{\beta}\bar{\gamma}}$$

→ an $(n-1, 1)$ form.

→ Can be shown to be harmonic

$$\in \mathcal{H}^{(n-1,1)}.$$

Conversely, given an element of $\mathcal{H}^{(n-1,1)}$ we can construct $\delta g_{\bar{\beta}\bar{\gamma}}$.

$\Rightarrow$ Deformation of complex structure

$\Leftrightarrow$ elements of $\mathcal{H}^{n-1,1}$ & $\mathcal{H}^{1,n-1}$.

$\Rightarrow$ Dimension of complex structure

moduli space $= h_{n-1,1}$ complex.

Calabi-Yau 3-fold: (6-dimensional real)

$h_{3,0} = h_{0,3} = 1$

$h_{1,2} = h_{2,1}$, $h_{11} = h_{22}$ not fixed.

$h_{1,0} = h_{2,0} = h_{0,1} = h_{0,2} = h_{2,3} = h_{3,2}$

$= h_{1,3} = h_{3,1} = 0$

Calabi-Yau manifolds admit

covariantly constant spinor.

e.g. take 6-d CY manifold.

$SO(6)$: tangent space group. $\cong SU(4)$

Vectors: 6 of $SO(6)$

Spinors: 4 of $SO(6)$

Spinor field ξ:

$$(D_m \xi)^s = \left(\delta_{st}\, \partial_m + \frac{i}{4}\, \omega_m^{\ ab} (\Sigma^{ab})_{st}\right) \xi^t$$

SO(6) generators in spinor representation.

Holonomy group SU(3).

Sits in first 3x3 block if we regard SO(6) as SU(4)

$$4 = 3 + 1$$

singlet.

$\Rightarrow$ One component does not transform under SU(3).

$$\Rightarrow \quad \omega_m^{\ ab} (\Sigma^{ab})_{st}\, \xi^t = 0 \quad \text{if}$$

ξ is taken to be a singlet.

$$\Rightarrow \partial_m \xi^t = 0 \Rightarrow \xi^t = \text{constant}$$

Covariantly constant spinor.

Explicit construction:

1. Pick a point $P \in CY_3$.

2. Pick the basis in spinor representation s.t. holonomy group at P acts as:

$$\begin{pmatrix} SU(3) & 0 \\ 0 & 1 \end{pmatrix}$$

3. Take $\xi(P) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

& define $\xi(Q)$ at other point Q by $\parallel$ transport along a curve joining P and Q.

The result is independent of the choice of path since holonomy group leaves $\xi(P)$ invariant.

Consequence: Take a 10-d Majorana-Weyl spinor.

$$16 \to (4_{SO(6)}, \text{left-hand Weyl}_{SO(3,1)}) + (\bar{4}_{SO(6)}, \text{right-handed Weyl}_{SO(3,1)}).$$

Existence of a covariantly constant spinor on CY_3.

$\Rightarrow$ Existence of a 10-d spinor satisfying:

$$D_m \epsilon = 0 \qquad D_\mu \epsilon = 0$$

$\Downarrow$ ∂_m if awe

$\Downarrow$ ∂_μ if $(3+1)$-d space is flat.

Choose the $SU(3)$ singlet component of 4.

$\Rightarrow$ 4 ~~parameter~~ parameter solns.

(Left handed Weyl + right haded Weyl)

$\Downarrow$ $\Downarrow$

2 complex = 4 real.

Heterotic ~~or~~ or type I on CY_3

$\Rightarrow$ $\mathcal{N}=1$ SUSY in $D=4$.

Type IIA / IIB on CY_3.

$\Rightarrow$ $\mathcal{N}=2$ SUSY in $D=4$.