Compactification of heterotic and type II

Superstrings on $\mathbb{C}P^2$

1. Solves eq. of motion

2. Preserves space-time SUSY
   - $N=2$ for IIA, IIB
   - $N=1$ for heterotic.

3. What is the four dimensional theory obtained after compactification?
   - Find the spectrum
   - Find interactions
     - Focus on massless fields since they are the only ones relevant at low energy.

We begin with $p$-form fields:
- Scalars, Vectors, 2-form, all RR $p$-form.
  (leaves out the metric).
$\mathcal{C}(b)$: Some $b$-form in $D = 10$.

$\mathcal{C}(b)(x \otimes y) = \sum_{m=0}^{\infty} \phi_m(x) \wedge \omega_m(y)$

Coordinates $\mathcal{C}$, coordinates $\mathcal{C}'$

$\omega_m(y) \equiv 0 \quad m = 1, 2, \ldots, \infty$

A basis of $(b-2)$-forms on $\mathcal{C}'$.

$\phi_m(x)$: Coefficients of expansion which are fins of $x$.

This is the most general expansion in $m$.

Gauge fins:

$S\mathcal{C}(b) = d \wedge (b-1)$

$\wedge (b-1) = \sum_{m=0}^{\infty} \lambda_m(x) \wedge \omega_m(y)$

$d \wedge (b-1) = \sum_{m=0}^{\infty} d\lambda_m(x) \wedge \omega_m(y)$

$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \lambda_m(x) \wedge d\omega_m(b-m-1)(y).$
For reasons to be clear soon, we shall take $\omega^{(k)}_m (y)$ to be eigenfunctions of the Laplacian:

$$\Delta = \left( \star d \star d + d \star d \star d \right)$$

$$\Delta \omega^{(k)}_m (y) = \delta_{m}^{(k)} \omega^{(k)}_m$$

$$\Delta d \omega^{(k)}_m = \left( \star d \star d + d \star d \star d \right) d \omega^{(k)}_m$$

$$= d \star d \star d \omega^{(k)}_m$$

$$= d \left( \star d \star d + d \star d \star d \right) \omega^{(k)}_m = \delta_{m}^{(k)} d \omega^{(k)}_m$$

$\exists \delta_{m}^{(k)} d \omega^{(k)}_m$ is also an eigenfunction of $\Delta$ with the same eigenvalue.

Q. Of the fields $\phi^{(i)}_m (x)$, which fields are massless?

Clue: $\phi^{(i)}_m \exp \lambda$ constant should solve eqs. of motion.
\[ \delta \star d \star \delta c(b) = 0. \]

\[ \Rightarrow \text{Need} \star d \star d \omega_{m}^{(b-9)} = 0. \]

\[ \Rightarrow \text{Has two kinds of solutions:} \]

1. \[ \omega_{m}^{(b-9)} = d \delta c_{m}^{(b-2)-1} \]

2. \[ \omega_{m}^{(b-9)} \text{ harmonic.} \]

\[ \Rightarrow d \omega_{m}^{(b-9)} = 0, \quad \star d \star \omega_{m}^{(b-7)/(b-9)} = 0. \]

Proof:

\[ \int * \omega_{m}^{(b-9)} \wedge * d \star d \omega_{m}^{(b-9)} = 0 \]

\[ \Rightarrow \text{int by parts} \]

\[ \int d \omega_{m}^{(b-9)} \wedge * d \omega_{m}^{(b-9)} = \int |d \omega_{m}^{(b-7)/(b-9)}|^2 \]

\[ \Rightarrow d \omega_{m}^{(b-9)} = 0. \]

Hodge theorem:

\[ \omega_{m}^{(b-9)} = \text{Harmonic form} + d(E) \]
Case I: $\omega^{k-1}_m = dx^m^{(b-2-1)}$

Then constant $\phi_m^{(1)}$

$\Rightarrow \omega^{(b)} = \phi_m^{(1)} \wedge dx^m$

$= d(\phi_m^{(1)} \wedge x^{(b-2-1)})^{(-1)}$

$\Rightarrow$ pure gauge configurations.

Such $\phi_m^{(1)}$'s are pure gauge.

$\omega^{(b-2)}_m$ harmonic.

$\Rightarrow$ $\phi_m^{(1)}$ is massless but not pure gauge.

Thus given a $b$-form $C^{(b)}$ harmonic in the original theory and a $r$ $(b-2)$-form $\omega^{(b-2)}_m(y)$ on $CY_3$, we have a $2$-form field $\phi_m^{(1)}(x)$ in 4-dimensions.
Example: CY3 has:

1. $b_0 = h_{0,0} = 1$ - one zero form.
2. $b_1 = h_{0,1} + h_{1,0} = 0$
3. $b_2 = h_{1,1} + h_{0,2} + h_{2,0} = h_{1,1}$
4. $b_3 = h_{1,2} + h_{2,1} + h_{0,3} + h_{3,0} = 2(h_{1,2} + 1)$
5. $b_4 = b_2 = h_{1,1}$
6. $b_5 = b_1 = 0$
7. $b_6 = b_0 = 0$

IB on CY3:

Zero form $\Phi$ - a scalar, in 4-d
Zero form $C^{(0)}$ - a scalar in 4-d
2-form $B_{\mu\nu} - a$ 2-form, $h_{1,1}$ scalars.
2-form $C_{\mu\nu} - a$ 2-form, $2(h_{1,2} + 1)$ scalars.
4-form $C^{(4)} - a$ four form $C^{(4)}$ in 4-d $\rightarrow h_{1,1}$ 2-forms (- non-dynamical)
\[ 2(h_{1,2} + 1) \] vectors
\[ \rightarrow \] $h_{1,1}$ scalars.
In 4-d we can dualize 2-forms to scalars.

Self-duality of \( C^2 \)

3 scalars dualizing \( 2 \) \( h_{1,1} \) 2-forms are the same as the scalars obtained directly.

Only \( (h_{1,2}+1) \) vectors are independent.

Electric field of \( (h_{1,2}+1) \) of one vector = magnetic field of the other \( (h_{1,2}+1) \) vectors.

 Altogether:

# of scalars: \( 2+2+3 \ h_{1,1} \)

# of vectors: \( (h_{1,2}+1) \).

We still have to determine the scalar fields coming from the metric.
Φ → a scalar.

B_{k\mu} → \mathbb{R} \cdot 2\text{-form}, h_{1,1} \text{ scalars.}

C^{(1)}_k → a vector.

C^{(2)}_{1,1} \text{ 3-form (non-dynamical)}

h_{1,1} \text{ 1-forms}

2(h_{1,2} + 1) \text{ scalars.}

After dualizing 2-forms we get

h_{1,2} + 2 + 2(h_{1,2} + 1) \text{ scalars.}

(h_{1,1} + 1) \text{ vectors.}

Now we turn to the metric.

\[ g_{mn}(x,y) = g^{(4)}(y) + \sum_{k} \left( \mathcal{F}(x) \right)_{k} X_{mn}^{(k)}(y) \]

Some basis of symmetric rank 2 tensors on C^4

\mathcal{F}(x) \rightarrow \text{scalar fields}.\]
Goal: Look for $S(k)$ which are massless.

$\exists$ Constant $S(k)$ should solve linearized eqs.

$\exists$ $g_{\mu\nu} + \sum_k S(k) X(k)_{\mu\nu}(y)$ should be a Ricci flat metric at least up to linear order in $S(k)$.

$\exists$ Kahler deformation $h_{i\bar{j}}$ real.

Complex structure deformation $\phi_{1,2}$ real.

$g_{\mu\nu}(x, y) = \sum_k a(k)_\mu(x) X^k_{\mu\nu}(y)$

Vector fields $\{\}$ some basis of vector fields on $\mathbb{C}^3$.

Q. Under what condition $a(k)(x)$ describes massless vector?

Constant $a(k)$ should solve linearized eqs. of motion.

Substitute $g_{\mu\nu}(y) + \sum_k a(k)^2 X(k)_{\mu\nu}(y)$ into $R_{\mu\nu} = 0$. 
Result: $\nabla_m X^{(i)}_{(k)n} + \nabla_n X^{(i)}_{(k)m} = 0$

$\Rightarrow \gamma^{mn} \in X^{(i)}_{(k)n} \Rightarrow g_{mn}(y)$ must be an isometry of $\text{CY}_3$.

$\text{CY}_3$ has no isometry.

$\Rightarrow$ no massless vector from the metric

$g_{\mu\nu}(x, y) = \eta_{\mu\nu} + \frac{1}{2} \sum_{k} h^{(k)}_{\mu\nu} X^{(k)}_{(y)}$

Coefficients of basis of scalars expansion on $\text{CY}_3$.

Symmetric rank 2 tensor field in 4-d.

Massless field: Constant $h_{\mu\nu}$ should be a 5-th to linearized eq.

Set $h^{(k)}_{\mu\nu}(x)$ constant & substitute into eq. of motion.

$\Rightarrow X^{(k)}_{(y)} = \text{constant}$

$\Rightarrow$ Unique massless $h^{(k)}_{\mu\nu}(x) \rightarrow 4$-d metric.
Summary:

II B.

Scalars: \(2 + 2 + 3 h_{1,1} + h_{1,1} + 2 h_{1,2}\)
\[= 4(h_{1,1} + 1) + 2 h_{1,2}\]

Vectors: \(h_{1,2} + 1\)

metric.

II A

Scalars: \(h_{1,1} + 2 + 2(h_{1,2} + 1) + h_{1,1} + 2 h_{1,2}\)
\[= 4(h_{1,2} + 1) + 2 h_{1,1}\]

Vectors: \(h_{1,1} + 1\)

metric.

Note: II A \(\leftarrow\) II B \(h_{1,1} \leftarrow h_{1,2}\)

\(\times\) Mirror symmetry Conjecture:

II A \(\cap\) M = II B \(\cap\) W

\(h_{1,1}(M) = h_{1,2}(W)\)
\(h_{1,2}(M) = h_{2,1}(W)\)