Compactification of heterotic and type II superstrings on CY_3.

① Solves eq. of motion

② Preserves space-time SUSY

— $\mathcal{N} = 2$ for IIA, IIB

— $\mathcal{N} = 1$ for heterotic.

Q. What is the four dimensional theory obtained after compactification?

① Find the spectrum

② Find interactions

— Focus on massless fields since they are the only ones relevant at low energy.

We begin with p-form fields:

— Scalars, vectors, 2-form, all RR p-form.

(leaves out the metric).

$C^{(p)}$: Some p-form in $D=10$.

$$C^{(p)}(x, y) = \sum_q \phi_m^{(q)}(x) \wedge \omega_m^{(p-q)}(y)$$

Coordinates in $(3+1)$-d

coordinates on CY.

$\omega_m^{(p-q)}(y)$, $m = 1, 2, \cdots \infty$

$\Rightarrow$ A basis of $(p-q)$-forms on CY_3.

$\phi_m^{(q)}(x)$: Co-efficients of expansion which are fn.s of x.

This is the most general expansion.

Gauge trs:

$$\delta C^{(p)} = d\Lambda^{(p-1)}$$

$$\Lambda^{(p-1)} = \sum_q \lambda_m^{(q)}(x) \wedge \omega_m^{(p-q-1)}(y)$$

$$d\Lambda^{(p-1)} = \sum_q d\lambda_m^{(q)}(x) \wedge \omega_m^{(p-q-1)}(y)$$

$$+ \sum_q (-1)^q \lambda_m^{(q)}(x) \wedge d\omega_m^{(p-q-1)}(y).$$

For reasons to be clear soon, we shall take $\omega_m^{(k)}(y)$ to be eigenfunctions of the Laplacian:

$$\Delta = (*d*d + d*d*).$$

$$\Delta \omega_m^{(k)}(y) = \xi_m^{(k)} \, \omega_m^{(k)}.$$

$$\Delta \, d\omega_m^{(k)} = (*d*d + d*d*) \, d\omega_m^{(k)}$$

$$= d*d*d\omega_m^{(k)}$$

$$= d(*d*d + d*d*) \, \omega_m^{(k)} = \xi_m^{(k)} \, d\omega_m^{(k)}.$$

$\Rightarrow$ $d\omega_m^{(k)}$ is also an eigenfunction of Δ with ~~zero~~ same eigenvalue.

Q. Of the fields $\phi_m^{(q)}(x)$, which fields are massless?

Clue: $\phi_m^{(q)}(x) =$ constant should solve linearized eqs. of motion.

$d*$

$\ast d \ast d C^{(p)} = 0.$

$\Rightarrow$ Need $\ast d \ast d \omega_m^{(p-q)}(y) = 0.$

$\rightarrow$ Has two kinds of solutions:

① $\omega_m^{(p-q)} = d\chi_m^{(p-q-1)}$

② $\omega_m^{(p-q)}$ harmonic.

$\Rightarrow d\omega_m^{(p-q)} = 0, \quad \ast d \ast \omega_m^{(p-q)} = 0.$

Proof:

$$\int \ast\omega_m^{(p-q)} \wedge \ast d \ast d\, \omega_m^{(p-q)} = 0$$

int. by parts

$$\int d\omega_m^{(p-q)} \wedge \ast d\, \omega_m^{(p-q)}$$

$$= \int |d\omega_m^{(p-q)}|^2$$

$\Rightarrow d\omega_m^{(p-q)} = 0.$

Hodge theorem:

$\omega_m^{(p-q)} = \text{Harmonic form} + d\,(\cdots)$

Case I: $\omega_m^{(p-q)} = dx_m^{(p-q-1)}$

Then constant $\phi_m^{(q)}$

$\Rightarrow C^{(p)} = \phi_m^{(q)} \wedge dx_m^{(p-q-1)}$

$= d\left(\phi_m^{(q)} \wedge x_m^{(p-q-1)}\right)(-1)^q$

$\Rightarrow$ pure gauge configurations.

Such $\phi_m^{(q)}$'s are pure gauge.

$\omega_m^{(p-q)}$ harmonic.

$\Rightarrow$ ~~$C^{(p)}$~~ $\phi_m^{(q)}(x)$ is massless

but not ~~true~~ pure gauge.

Thus given a p form field ~~(p)~~ $C^{(p)}$ in the original theory and a harmonic $(p-q)$-form $\omega_m^{(p-q)}(y)$ on CY_3, we have a q-form ~~field~~ $\phi_m^{(q)}(x)$ in 4-dimensions.

Example : CY_3 has:

(1) $b_0 = h_{0,0} = 1 \to$ one zero form.

(2) $b_1 = h_{0,1} + h_{1,0} = 0$

(2) $b_2 = h_{1,1} + h_{0,2} + h_{2,0} = h_{1,1}$

(3) $b_3 = h_{1,2} + h_{2,1} + h_{0,3} + h_{3,0} = 2(h_{1,2}+1)$

(4) $b_4 = b_2 = h_{1,1}$

(5) $b_5 = b_1 = 0$

(6) $b_6 = b_0 = 1$

IIB on CY_3:

Zero form $\Phi \to$ a scalar in 4-d

Zero form $C^{(0)} \to$ a scalar in 4-d

2-form $B_{\mu\nu} \to$ a 2-form, $h_{1,1}$ scalars.

2-form $C_{\mu\nu} \to$ a 2-form, $\cancel{2(h_{1,2}+1)}$ $h_{1,1}$ scalars

4-form $C^{(4)}_{\mu\nu\rho\sigma} \to$ a four form $C^{(4)}$ in 4-d

$\to h_{1,1}$ 2-forms. (— nondynamical)

$\to 2(h_{1,2}+1)$ ~~2-forms~~ vectors

$\to$ h_{11} scalars.

In 4-d we can dualize 2-forms to scalars.

Self-duality of $C^{(4)}$

$\Rightarrow$ Scalars dualizing ~~2($h_{2,1}$+~~ h_{11} 2-forms are the same as the scalars obtained directly.

Only $(h_{1,2}+1)$ vectors are independent.
Electric field of $(h_{1,2}+1)$ of the vectors = magnetic field of the other $(h_{1,2}+1)$ vectors.

$\Rightarrow$ Altogether:

# of scalars: $2+2+3h_{1,1}$

# of vectors: $(h_{1,2}+1)$.

We still have to ~~study~~ determine the scalar fields coming from the metric.

IIA

$\Phi \to$ a scalar.

$B_{\mu\nu} \to$ A 2-form, $h_{1,1}$ scalars.

$C^{(1)}_{\mu} \to$ A vector.

$C^{(3)}_{\mu\nu e} \to$ A 3-form (non-dynamical)

$h_{1,1}$ 1-forms

$2(h_{1,2}+1)$ scalars.

After dualizing 2-forms we get

$h_{1,1}+2+2(h_{1,2}+1)$ scalars.

$(h_{1,1}+1)$ vectors.

Now we turn to the metric.

$$g_{mn}(x,y) = g^{CY}_{mn}(y) + \sum_k \xi_{(k)}(x)\, \chi^{(k)}_{mn}(y)$$

Some basis of symmetric rank 2 tensors on CY

$\xi_{(k)}(x) \to$ scalar fields.

Goal: Look for $\xi_{(k)}$ which are massless.

$\Rightarrow$ Constant $\xi_{(k)}$ should solve linearized eqs.

$\Rightarrow$ $g^{CY}_{mn} + \sum_k \xi_{(k)} X^{(k)}_{mn}(y)$ should be a Ricci flat metric at least up to linear order in $\xi_{(k)}$.

$\Rightarrow$ Kahler deformation. $h_{1,1}$ real.

&

Complex structure deformation. $2h_{1,2}$ real.

$$g_{m\mu}(x,y) = \sum_k a_{(k)\mu}(x)\, \mathcal{X}^{(1)}_{(k)m}(y)$$

vector fields on 4-d . some basis ~~of~~ vector fields on CY_3.

Q. Under what condition $a_{(k)\mu}(x)$ describes massless vector?

Constant $a_{(k)\mu}$ should solve linearized eqs. of motion.

Substitute $g^{CY}_{mn}(y) + \cancel{\sum_k}\, \underbrace{a_{(k)\mu}}_{\text{constants}} \cancel{(x)}\, X^{(1)}_{(k)m}(y)$ into $R_{mn} = 0$.

Result: $\nabla_m X^{(1)}_{(k)n} + \nabla_n X^{(1)}_{(k)m} = 0$

$\Rightarrow y^m \to y^m + \epsilon X^{(1)}_{(k)n} g^{mn}_{CY}(y)$ must

be an isometry of CY_3.

CY_3 has no isometry.

$\Rightarrow$ no massless vector from the metric

$$g_{\mu\nu}(x,y) = \eta_{\mu\nu} + \sum_k h^{(x)}_{(k)\mu\nu} X^{(0)}_k(y)$$

Coefficients of expansion $\to$ symmetric rank 2 tensor field in 4-d.

basis of scalars on CY_3.

Massless field: Constant $h_{\mu\nu}$ should be

a soln. to linearized eq.

Set $h_{(k)\mu\nu}(x) =$ constant & substitute

into eq. of motion.

$$\Rightarrow X^{(0)}_k(y) = \text{constant}.$$

$\Rightarrow$ Unique massless $h_{(k)\mu\nu}(x) \to$ 4-d metric

Summary:

II B.

Scalars: $2+2+3h_{1,1}+h_{1,1}+2h_{1,2}$

$= 4(h_{1,1}+1)+2h_{1,2}$

Vectors: $h_{1,2}+1$

metric.

II A

Scalars: $h_{1,1}+2+2(h_{1,2}+1)+h_{1,1}+2h_{1,2}$

$= 4(h_{1,2}+1)+2h_{1,1}$

Vectors: $h_{1,1}+1$

metric.

Note: II A $\leftrightarrow$ II B. $h_{1,1} \leftrightarrow h_{1,2}$

$\Rightarrow$ Mirror symmetry Conjecture:

II A on $\mathcal{M}$ = II B on $\mathcal{W}$.

$h_{1,1}(\mathcal{M}) = h_{1,2}(\mathcal{W})$
$h_{1,2}(\mathcal{M}) = h_{1,1}(\mathcal{W})$