An aside: Type II on K3.

K3 has: $h_{0,0}=1$, $h_{1,0}=h_{0,1}=0$

$h_{0,2}=1$, $h_{1,1}=20$, $h_{2,0}=1$.

Rest determined by Poincare duality.

Ex. Compute the spectrum of IIA, IIB on K3.

Note: Dual of 2-form = 2-form in 6d.

3 form = Vector in 6d

4-form = Scalar in 6d.

Ex. Massless Spectrum of IIA on K3
= Massless Spectrum of heterotic on $T^4$.

at generic point in the moduli space.

$E_8\times E_8$, $SO(32) \to U(1)^{16}$

$\mathcal{N}=4$ SUSY $\Rightarrow$ determines low energy effective action $\Rightarrow$ Identical for the two

Conjecture: IIA string theory on K3
= Heterotic string theory on $T^4$.

II A / II B on $CY_3$:

We have determined the spectrum of massless bosonic fields in $D=4$.

Q. What is the low energy effective field theory describing the dynamics of these fields?

$\rightsquigarrow$ Can be obtained in principle from the 10-d effective action.

General structure: Each term contains $\leq 2$ derivative

↳ may lose some derivatives as derivative along $CY_3$.

Indirect approach: ~~Ex~~ Constrain the structure of the action ~~for~~ using $N=2$ space-time SUSY.

We focus on IIB on $CY_3$ for definiteness:

- 1 metric.
- $h_{1,2}+1$ vectors.
- $(4h_{1,1}+4+2h_{1,2})$ scalars.

Q1. What is the potential for these scalar fields?

Potential: Terms without derivatives.

$$-\int d^4x \sqrt{\det g}\ V(\vec{\Phi})$$

└ set of scalars.

⇝ Obstruction to switching on constant values of scalar fields beyond linearized approximation.

Absence of obstruction
⇒ Vanishing of potential.

① $h_{1,1}+2h_{1,2}$ scalars from metric:

Ricci flat metric exists for finite deformations of Kahler & Complex structure.

$\Rightarrow$ no obstruction to switching on constant background values of these scalars.

$\Rightarrow$ no potential.

2 Scalars from $p$-forms:

$$c^{(p)} = \underset{\text{constant}}{\underset{\not\uparrow}{\phi_m}} \, \omega_m^{(p)}(y)$$

$dc^{(p)} = 0$ for arbitrary $\phi_m$.

~~Since~~ Since eqs. of motion involve $dc^{(p)}$, when all $dc^{(p)}$ vanish, the eqs. of motion continue to be satisfied.

$\Rightarrow$ no obstruction to switching on constant $\phi_m$ $\Rightarrow$ no potential.

③ Scalars from dualizing 2-form:

$dB = * d\phi$

$\phi$ = constant $\Rightarrow * d\phi = 0 \Rightarrow dB = 0$.

$\Rightarrow$ no effect on eq. of motion

$\Rightarrow$ no potential for $\phi$.

④ Dilation $\Phi$:

The 10-d the eqs. of motion continue to be satisfied for $\Phi$ = constant.

$\Rightarrow$ No potential.

Conclusion: None of the scalars have any potential and we can give them constant vev.

$\Rightarrow$ Moduli fields.

$\Rightarrow (4h_{1,1} + 4 + 2h_{1,2})$ parameter family of vacua.

This causes phenomenological problem.

- long range force from exchange of strictly massless fields.

- time dependence of the scalars in the early universe

→ potential conflict with early universe cosmology.

Known as moduli fixing problem.

For now we proceed to study them as they are.

~~General~~ Fields:

$\phi^\alpha \quad (1 \leq \alpha \leq 4h_{1,1} + 4 + 2h_{1,2})$

$A_\mu^{(i)} \quad : \quad F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}$

$g_{\mu\nu}$.

General form of 2-derivative action without potential:

$$\int d^4x \sqrt{-\det g} \left[ f(\vec{\Phi}) R - h_{\alpha\beta}(\vec{\Phi}) \partial_\mu \phi^\alpha \partial_\nu \phi^\beta g^{\mu\nu} \right.$$

$$- K_{ij}(\vec{\Phi}) F^{(i)}_{\mu\nu} F^{(j)\mu\nu}$$

$$\left. - L_{ij}(\vec{\Phi}) F^{(i)}_{\mu\nu} (*F^{(j)})^{\mu\nu} \right]$$

$$\Downarrow$$

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^{(j)}_{\rho\sigma} \left(\sqrt{-\det g}\right)^{-1}$$

$f$, $h_{\alpha\beta}$, $K_{ij}$, $L_{ij}$ are arbitrary functions of $\vec{\Phi}$.

We shall now constrain the structure of these terms using $\mathcal{N}=2$ SUSY.

Note $f(\vec{\Phi})$ can be removed by $g_{\mu\nu} \to (f(\vec{\Phi}))^{-1} g_{\mu\nu}$, but we shall keep it.

Massless supermultiplets of $\mathcal{N}=2$ supergravity:

Gravity multiplet: metric, 1 vector, Gravitino, 1 ~~Dirac~~ Majorana fermion.

Vector multiplet: 1 vector, 2 scalars. 1 Dirac fermion.

Hyper multiplet: 4 scalars, 1 Dirac fermion

We have $h_{1,2}+1$ vectors
↳ part of gravity multiplet.

$\Rightarrow$ $h_{1,2}$ vector multiplets.

Takes $2h_{1,2}$ scalars. (Complex structure moduli)

Left-overs: $4(h_{1,1}+1)$ scalars.

$(h_{1,1}+1)$ hypermultiplet.

Complexified Kahler, scalars from $C^{(2)}, C^{(4)}, C^{(0)}$, dilaton, $B_{\mu\nu}$

For IIA on $CY_3$ we have similar results with $h_{1,1} \Leftrightarrow h_{1,2}$.

$\vec{\Phi}_V$: scalars in vector multiplet

$\vec{\Phi}_H$: scalars in hyper multiplet.

General structure of the bosonic part of $N=2$ sugra action:

$$\int d^4x \sqrt{-\det g}\,\Big[ f(\vec{\Phi}_V)\,R - G^V_{AB}(\vec{\Phi}_V)\,\partial_\mu \phi^A_V\,\partial^\mu \phi^B_V$$

$$- \frac{1}{4}\left\{ K_{IJ}(\vec{\Phi}_V)\, F^{+(I)}_{\mu\nu}\, F^{+(J)}_{\mu\nu} + h.c. \right\}$$

$$- \frac{1}{2} G^H_{st}(\vec{\Phi}_H)\,\partial_\mu \phi^s_H\;\partial^\mu \phi^t_H \Big]$$

$$F^{+(I)}_{\mu\nu} = \frac{1}{2}\left(F^{(I)}_{\mu\nu} + i * F^{(I)}_{\mu\nu}\right)$$

Note: No coupling between hyper and vector multiplet fields except through gravity.

$\mathcal{N}=2$ SUSY determines $f(\vec{\Phi}_v)$, $G^v_{AB}(\vec{\Phi}_v)$, $K_{IJ}(\vec{\Phi}_v)$ in terms of a single complex analytic function $F$ of $h_{1,2}+1$ variables $Z^0, \dots Z^{h_{1,2}}$ satisfying:

$$F(\lambda \vec{Z}) = \lambda^2 F(\vec{Z})$$

$F(\vec{Z}) \to$ Prepotential

The action is written in terms of $N+1$ complex scalars $Z^0, \dots Z^{h_{1,2}}$ with a gauge symmetry:

$$Z^I(x) \to \lambda(x) Z^I(x)$$

Fixing this gauge $\Rightarrow h_{1,2}$ complex scalars

Define:

$$F_I \equiv \frac{\partial F}{\partial Z^I}, \quad F_{IJ} = \frac{\partial^2 F}{\partial Z^I \partial Z^J}, \quad I, J = 0, \dots h_{1,2}$$

graviphoton

$$N_{IJ} = \frac{1}{4}(F_{IJ} + \overline{F}_{IJ}), \quad \phi_v^A = \frac{Z^A}{Z^0}$$

Then $K_{IJ}(\vec{\Phi}_V) = \frac{i}{4} F_{IJ} - N_{IK} Z^K N_{JL} Z^L / (Z^{K'} N_{K'L'} Z^{L'})$

$$f(\vec{\Phi}_V) = \frac{i}{2}\left(Z^I \bar{F}_I - \bar{Z}^I F_I\right)$$

It remains to give $G^V_{AB}(\vec{\Phi}_V)$.

We shall give this in a convenient choice of gauge (D-gauge)

$$-i\left(Z^I \bar{F}_I - \bar{Z}^I F_I\right) = 1$$

$\downarrow$

allows us to find $\{Z^I\}$ in terms of $\{\phi^A\}$.

$$G^V_{AB}\, \partial_\mu \phi^A_V\, \partial^\mu \phi^B_V = \mathcal{M}_{I\bar{J}}\, \partial_\mu Z^I \partial^\mu \bar{Z}^{\bar{J}}$$

$$\mathcal{M}_{I\bar{J}} = N_{IJ} + N_{IK} \bar{Z}^K N_{JL} Z^L$$

$G^H_{st}(\vec{\Phi}_H)$ : describes a $4(h_{1,1}+1)$ dimensional quaternionic metric.

$\Downarrow$

Holonomy group $Sp(1) \times Sp(h_{1,1})$

$Sp(n)$: $2n \times 2n$ symplectic matrix.

$$\Omega \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \Omega^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$$Sp(1) \times Sp(h_{1,1}) \subset SO(4(h_{1,1}+1))$$

Our task: For IIB on $CY_3$, find

1. The prepotential $F(\vec{Z})$
2. The quaternionic metric $G^H_{st}$.

— Obtained by comparing terms ~~obtained~~ in the general $\mathcal{N}=2$ sugra action with ~~the~~ what we get from compactification of ~~[illegible]~~ 10-d sugra on $CY_3$.

Result for $F(\vec{Z})$ (prepotential)

On given $CY_3$ choose a fixed basis of $2h_{1,2}+2$ 3-cycles $\{A^I\}$, $\{B_J\}$ satisfying:

$A^I \cup A^J = 0 = B_I \cup B_J$, $A^I \cup B_J = \delta^I{}_J$

Suppose for given complex structure, $\Omega$ denotes the (3,0) form representing the unique element of $\mathcal{H}^{(3,0)}$. (defined up to overall multiplicative constant)

Define: $Z^I \equiv \int_{A^I} \Omega$

$\Downarrow$

$h_{1,2}+1$ complex variables $Z^I$

$\swarrow$

Can be use to label the complex structure.

Changing complex structure

$\to$ changes $\Omega \to$ changes $Z^I$.

$F_I(\vec{Z}) = \int_{B_I} \Omega$

$\to$ defines $F_I(Z)$

$\frac{\partial F_I}{\partial Z^J} = \frac{\partial F_J}{\partial Z^I} \Rightarrow F_I = \partial_I F$ locally.

Note: Under $\Omega \to \lambda\Omega$

$Z^I \to \lambda Z^I, \quad F_I \to \lambda F_I$

$\Rightarrow \quad F \to \lambda^2 F$

$\Rightarrow \quad F(\lambda Z) = \lambda^2 F(Z)$ as required.

This determines $F(Z)$ for a given $CY_3$.

We still have to determine $G^H_{st}$.

$\to$ The quaternionic metric on the hypermultiplet moduli space.

For IIA on $CY_3$ the $h_{1,1}$ vector multiplet scalars are associated with Kahler moduli space.

$\omega_1, \cdots \omega_{h_{1,1}}$: A fixed basis of $H^{1,1}$

$\to$ linearly independent closed 2-forms.

$$d_{abc} = \int_{CY_3} \omega_a \wedge \omega_b \wedge \omega_c \qquad a, b, c = 1, \cdots h_{1,1}.$$

Result

$$F(W^0, W^1, \dots W^{h_{1,1}})$$

$$= -\frac{i}{6} d_{abc} \frac{W^a W^b W^c}{W^0}$$

$$a, b, c = 1, \dots h_{1,1}.$$

This determines the prep-tential in both IIA and IIB on $CY_3$ in the large volume, weak coupling limit.

↓ Only 2-derivative terms

↓ string tree level.

Now return to IIB:

We need to determine $G^{H}_{5+}(\vec{\Phi}_H)$

Final algorithm (C-map):

(i) Consider IIA on same $CY_3$.

→ Its vector multiplet action is known from the prep-tential:

$$F(W^0, \dots W^{h_{1,1}}) = -\frac{i}{6} d_{abc} \frac{W^a W^b W^c}{W^0}$$

$\Rightarrow$ allows us to write down the action involving ~~vector~~ $h_{1,1}$ vector multiplets and gravity.

$\Rightarrow$ Metric $g_{\mu\nu}$,

$h_{1,1}+1$ vectors $A_\mu^i$.

$2h_{1,1}$ scalars.

(2) Compactify on a circle of radius $R$ labelled by $y$.

Metric $\Rightarrow$

metric $g_{\alpha\beta}$, 1 vector $g_{y\alpha}$, scalar $g_{yy}$.

Vector:

$h_{1,1}+1$ vectors $A_\alpha^i$, $h_{1,1}+1$ scalar $A_y^i$.

Scalars $\Rightarrow$ $2h_{1,1}$ scalars.

$\Rightarrow$ 1 metric, $\underbrace{h_{1,1}+2 \text{ vectors}}_{\Downarrow}$, $3h_{1,1}+2$ scalars.

dualize to scalars.

$$dA^{(i)} = * d\phi^{(i)}$$

$$\Rightarrow h_{1,1} + 2 + 3h_{1,1} + 2 = 4(h_{1,1} + 1)$$

scalars.

3-d action:

$$\int \sqrt{-\det g^{(3)}} \left[ R^{(3)} - \frac{1}{2} \tilde{G}^{V}_{st} (\vec{\Phi}^{V}) \partial_\alpha \Phi^{s}_{M} \partial^{\alpha} \Phi^{t}_{M} \right.$$

$\Downarrow$

identified as $G^{V}_{st}$

of hypermultiplet metric

for IIB on $CY_3$.

(Related to the fact that

IIA on $CY_3 \times S^1$ = IIB on $CY_3 \times S^1$)

$\longleftrightarrow$

exchanges vector & hypermultiplets

One can show that $G^{V}_{st}$ is

quaternionic metric.

Corrections to the two derivative action $\Rightarrow$ Correction to $F$, $G^{V}_{\alpha\beta}$

① $\alpha'$ corrections (higher derivative corrections) $\rightarrow$ Can arise from $\partial_m$ contracted by $g^{mn}$.

~~Sca~~ Overall scaling: A Kahler deformation.

Large $CY_3$ $\quad g_{mn} \rightarrow \lambda g_{mn}$
$\lambda$ large.

$\Rightarrow$ Higher derivative corrections vanish.

$\Rightarrow$ Higher derivative corrections to $F$ and $G^{V}$ must involve the Kahler moduli.

IIB: Kahler moduli $\rightarrow$ Hyper multiplet

$\Rightarrow$ Cannot affect $F$
$\quad$ no $\alpha'$ correction.

IIA: Kahler moduli $\rightarrow$ Vector multiplet

$\Rightarrow$ $G^{V}$ does not receive $\alpha'$ correction.

What about string loop corrections?

Must involve $e^{\Phi}$ since $e^{\langle \phi \rangle}$ is string coupling.

In $\Phi \to -\infty$ limit the corrections vanish.

Both in IIA & IIB, $\Phi$ is in hypermultiplet.

$\Rightarrow$ No ~~loo~~ string loop correction to F although $G^{V}$ could be corrected.

~~IIA~~ In IIA on $CY_3$, F receives $\alpha'$ corrections.

However these are constrained.

Complex scalar fields: $\frac{Z^I}{Z^0} \leftrightarrow$ components of $(B + iJ)$ along $\omega_m^{(2)}$'s.

$B \to B + \text{const.} \Rightarrow$ exact shift symmetry even of higher derivative terms since $\mathcal{H}$ vanish.

$\Rightarrow$ F should not depend on B.

$\Rightarrow$ Action should not depend on

$Re(Z^I/Z^0)$.

$\to$ Strong constraint on F.

$\to$ Rules out perturbative $\alpha'$ corrections.

We can have corrections non-perturbative in $\alpha'$, e.g.

$$e^{i(B+iJ)_m} \quad \text{component along } \omega_m.$$

$$\sim e^{-\text{size of } CY_3 + i \times \text{comp. of } B.}$$

$\downarrow$

Contribution from world-sheet instantons..