Cosmology: History of the Universe.

S. Weinberg, Cosmology.

· Conventions: Units: $\hbar = 1$, $c = 1$, $G = 1$ (not always)

check these

m_{pl}: unit of mass $(\hbar^a c^b G^k)$

$= 1.22 \times 10^{19}\ \text{GeV}/c^2$

$l_{pl} = 1.62 \times 10^{-33}\ \text{cm} \rightarrow$ unit of length

$t_{pl} = \cancel{8.62}\ 5.39 \times 10^{-44}$ sec.

Boltzmann const. $k_B = 8.62 \times 10^{-5}\ \text{eV}/K$.

Sometimes one uses reduced Planck unit

$\hbar = 1,\ c = 1,\ 8\pi G = 1$

Riemann tensor conventions:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\rho}$$

$$R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}, \qquad R = g^{\nu\sigma} R_{\nu\sigma}$$

Einstein's eq: $\quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

Universe is complicated

→ one needs some simplifying assumptions.

Universe is homogeneous and isotropic.

Assumption: There is a prefered choice of time coordinate (cosmic time) such that at any time t, the metric looks the same every where & in all directions.

— initially was taken as an assumption to simplify analysis.

Now we know that it is true when we study average properties over large distances

250×10^6 light-Years

parsec $\simeq 3.26$ ly
$\simeq 3.086 \times 10^{13}$ Km.

What do we mean when we say that the Universe looks the same everywhere and in all directions?

Recall general coordinate trs.

$\tilde{x}^\mu = f^\mu(x), \quad x^\mu = h^\mu(\tilde{x})$

$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{\partial}_\mu h^\rho \, \tilde{\partial}_\nu h^\sigma \, g_{\rho\sigma}(x)$

A transformation is called an isometry if $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x)$.

$d\theta^2 + \sin^2\theta \, d\phi^2, \quad \tilde{\theta} = f(\theta,\phi), \quad \tilde{\phi} = f_1(\theta,\phi)$

$ds^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta} \, d\tilde{\phi}^2$

Homogeneity: Given any two space-time points $(t, \vec{x}_{(1)})$, $(t, \vec{x}_{(2)})$ with same time coordinate, there is an isometry

$$\tilde{t} = t, \quad \tilde{x}^i = f^i(t, \vec{x}), \quad x^i = h^i(t, \vec{\tilde{x}}) \quad \text{s.t.}$$

$$f^i(t, \vec{x}_{(1)}) = x^i_{(2)}.$$

Isotropy: Given a space-time point $(t, \vec{x}_{(0)})$ and another pair of points $(t, \vec{x}_{(1)})$ and $(t, \vec{x}_{(2)})$ at equal geodesic distance from $(t, \vec{x}_{(0)})$, there is an isometry

$$\tilde{t} = t, \quad \tilde{x}^i = \phi^i(t, \vec{x}), \quad x^i = \xi^i(t, \vec{\tilde{x}}) \quad \text{s.t.}$$

$$\phi^i(t, \vec{x}_{(0)}) = x^i_{(0)}, \qquad \phi^i(t, \vec{x}_{(1)}) = x^i_{(2)}.$$

Result: By a suitable choice of coordinate system, the most general homogeneous and isotropic metric may be expressed as:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \underbrace{d\Omega^2}_{d\theta^2 + \sin^2\theta \, d\phi^2} \right)$$

$k = 0, \pm 1$

→ FRW metric

ds_3^2

① $k=0$: $ds_3^2 = dr^2 + r^2 d\Omega^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\tilde{x}^i = x^i + a^i$, $\quad \tilde{x}^i - x^i_{(0)} = R^i{}_j (x^j - x^j_{(0)})$

$R^T R = I$

② $k=1$: $ds_3^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2$ is a metric on a unit 3-sphere.

$$ds_3^2 = dx^2 + dy^2 + dz^2 + dw^2, \quad x^2+y^2+z^2+w^2 = 1$$

$$w = \cos\psi$$
$$z = \sin\psi\cos\theta$$
$$x = \sin\psi\sin\theta\cos\phi, \quad y = \sin\psi\sin\theta\sin\phi$$

Ex. $x^2+y^2+z^2+w^2=1$, $ds_3^2 = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$

$r = \sin\psi$, $d\psi^2 = \frac{dr^2}{1-r^2} \Rightarrow ds_3^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2$

Isometry: $\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\\ \tilde{w}\end{pmatrix} = R\begin{pmatrix}x\\ y\\ z\\ w\end{pmatrix}$ $\quad R^T R = I$

$\to SO(4) \to$ can be used to prove homogeneity and isotropy.

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③ $k=-1$ $\quad ds_3^2 = \dfrac{dr^2}{1+r^2} + r^2 d\Omega^2 \to$ Hyperbolic space

$ds_3^2 = dx^2 + dy^2 + dz^2 - dw^2$, $\quad \underline{w^2 - x^2 - y^2 - z^2} = 1 \mapsto w \overset{?}{=} \pm\sqrt{1+x^2+y^2+z^2}$

$w = \cosh\psi$, $z = \sinh\psi\cos\theta$, $x = \sinh\psi\sin\theta\cos\phi$

$y = \sinh\psi\sin\theta\sin\phi$

$ds_3^2 = d\psi^2 + \sinh^2\psi\, d\Omega^2$, $\quad r = \sinh\psi$

$\Rightarrow ds^2 = \dfrac{dr^2}{1+r^2} + r^2 d\Omega^2$

Isometry : $SO(3,1)$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad R^T \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix} R = \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix}.$$

3+1 dim. Lorentz group.

Present universe : almost flat

How is is consistent with FRW metric?

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$$

t_0 : Today's time. $a_0 \equiv a(t_0)$

$$ds^2 = -dt^2 + a_0^2 \lambda(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$$

$$\lambda(t) = a(t)/a_0 \qquad \lambda(t_0) = 1$$

$$\rho = a_0 r \qquad ds^2 = -dt^2 + \lambda(t)^2 \left(\frac{d\rho^2}{1 - \frac{k\rho^2}{a_0^2}} + \rho^2 d\Omega^2 \right)$$

If $\lambda(t)$ is slowly varying & a_0 is large, then

$$ds^2 \simeq -dt^2 + \underbrace{d\rho^2 + \rho^2 d\Omega^2}_{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$$

Experiment today is consistent with $a_0 = \infty$.

$$ds^2 = -dt^2 + \lambda(t)^2 \left(\frac{d\varrho^2}{1 - k\frac{\varrho^2}{a_0^2}} + \varrho^2 d\Omega^2 \right)$$

$k = 0$ for all practical purpose.

$\lambda(t)$ has been observed to vary with time

Physical significance of $\lambda(t)$.

Let us suppose that light emitted from a point $(t_1, \vec{A})$ reaches a point $(t_0, \vec{B})$.

present time. us

$(t_1, \vec{A})$: source $\qquad (t_2, \vec{B})$: observer

Homogeneity: We can take $\vec{B}$ to be at $r=0$, $\theta = \phi = 0$.

Isotropy: We can take $\vec{A}$ to be at $r = r_1$, $\theta = 0$, $\phi = 0$

Light travels along null geodesic

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

affine parameter. Ex. Show that the path lies along $\theta = 0$, $\phi = 0$.

$ds^2 = 0$ along null geodesic

$$\Rightarrow \; -dt^2 + a(t)^2 \frac{dr^2}{1-kr^2} = 0 \qquad (d\phi = 0,\ d\theta = 0 \text{ along the path})$$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \overset{\cancel{+}}{-} \int_{r_1}^{0} \frac{dr}{\sqrt{1-kr^2}} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

$t_1 < t_0$

Light emitted from $(t_1+\delta t_1, \vec{B})$ reaches $(t_0+\delta t_0, \vec{A})$.

$$\Rightarrow \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}} \quad \Rightarrow \quad \frac{\delta t_0}{a(t_0)} - \frac{\delta t_1}{a(t_1)} = 0.$$

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \Rightarrow \delta t_0 = \frac{\delta t_1}{\lambda(t_1)} \quad \Bigg| \quad \lambda(t) = \frac{a(t)}{a(t_0)}$$

ν_1 : Frequency of light emitted at $(t_1, \vec{B})$

ν_0 : " " " observed at $(t_0, \vec{A})$

$$\frac{1}{\nu_0} = \frac{1}{\nu_1} \frac{1}{\lambda(t_1)} \Rightarrow \boxed{\nu_0 = \nu_1 \lambda(t_1)}$$

Observation: Light from distant sources is red-shifted. $\nu_0 < \nu_1 \Rightarrow \lambda(t_1) < 1$

$\lambda(t_0) = 1 \qquad \lambda(t_1) < 1 \Rightarrow a(t_1) < a(t_0)$

Conclusion: Universe is expanding.

Hidden assumption: source and the observer are at rest in the (t, r, θ, ϕ) coordinate system.

$$\left.\begin{array}{l} (t_1, \vec{B}), (t_1 + \delta t_1, \vec{B}) \\ (t_0, \vec{A}), (t_0 + \delta t_0, \vec{A}) \end{array}\right\} \text{ Comoving object}$$

We'll see later that on the average, the objects are at rest in the (t, r, θ, ϕ) coordinate system.