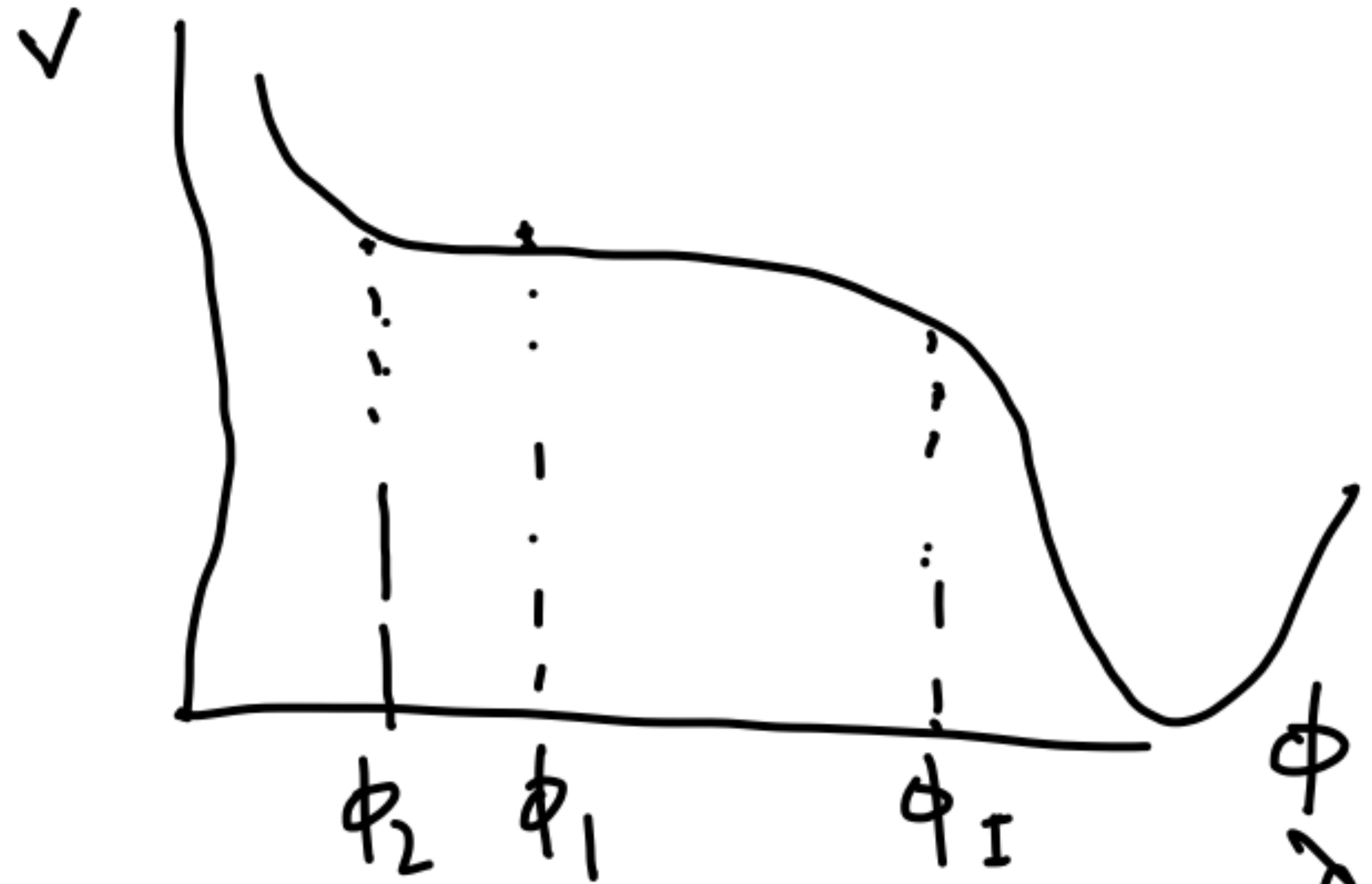

λ_* small

$V'(\phi)$ small

Δt small

$\frac{\dot{\lambda}}{\lambda}$:

$$\boxed{\frac{\lambda_I^{(1)}}{\lambda_I^{(2)}}}$$

$\lambda_{(1)}$.

$\lambda_{(2)}$

2

$\frac{\lambda_{(1)}}{\lambda_{(2)}}$ large

1 MeV.

$$\boxed{10^{16}\ \text{GeV} / 1\ \text{MeV}}$$

Suppose that we have a system in thermal and chemical equilibrium with L conserved charges.

Then the state of the system is completely specified by T, and densities $\tilde{n}_{(\ell)}$ of the conserved charges.

$\Rightarrow$ we can calculate n_i, p_i, ρ_i, s_i for each type of particle.

Route: $\{\tilde{n}_{(\ell)}\}, T \to \tilde{\mu}_{(\ell)} \to \mu_i = \sum_{\ell=1}^{L} c^i_{(\ell)} \tilde{\mu}_{(\ell)}$

$\to n_i, p_i, e_i, s_i$ in terms of μ_i, T.

Application to cosmology:

$$\frac{d}{dt}\left(\tilde{n}_{(\alpha)}\lambda^3\right) = 0 \quad \Rightarrow \quad \tilde{n}_{(\alpha)} = \frac{K_{(\alpha)}}{\lambda^3}$$

$K_{(\alpha)} \rightarrow$ constants fixed by initial condition

At this state T, λ are independent variables.

$\Rightarrow$ ρ_i, p_i, n_i, s_i etc. are known in terms of λ, T.

Now apply energy conservation or equivalently entropy conservation: $\frac{d}{dt}\left(s\lambda^3\right) = 0$ $\quad s\lambda^3 = \text{constant}$

$s = \sum_{i=1}^{K} s_i$

$\Rightarrow$ determines T in terms of λ.

Final step: Use Friedmann eq:

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2\lambda^2} = \frac{8\pi G}{3}\rho = \sum_{i=1}^{K}\rho_i$$

→ Known as fn. of λ.

⇒ Differential eq. for λ

→ can be solved to find $\lambda(t)$.

Ideal gas results:

$$\frac{p}{T} = \frac{1}{V}\ln Q = \mp \frac{g}{2\pi^2}\int_0^\infty k^2\,dk\,\ln\left(1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}\right)$$

$$n = \frac{g}{2\pi^2}\int_0^\infty k^2\,dk\,\frac{e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}{1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}$$

$$e = \frac{g}{2\pi^2}\int_0^\infty k^2\,dk\,\sqrt{k^2+m^2}\,\frac{e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}{1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}$$

For many applications in cosmology, we can assume that conserved charges all vanish.

$$\tilde{n}_{(\ell)} = 0 \Rightarrow \tilde{\mu}_{(\ell)} = 0, \Rightarrow \mu_i = \sum_{\ell=1}^{L} C^i_{(\ell)} \tilde{\mu}_{(\ell)} = 0$$

→ formulae simplify (assume $\mu_i = 0$ for now)

<u>NR limit</u> $T \ll m$

$$\frac{P}{T} = \frac{gm}{4\pi^2} e^{-m/T} \sqrt{\frac{2\pi m}{\beta^3}}$$

$$n = \frac{gm}{4\pi^2} e^{-m/T} \sqrt{\frac{2\pi m}{\beta^3}}, \quad \rho = \frac{gm^2}{4\pi^2} e^{-m/T} \sqrt{\frac{2\pi m}{\beta^3}}$$

exponentially suppressed.

Ultra-relativistic limit: $T \gg m$

Typical $k \gg m \Rightarrow \sqrt{k^2+m^2} \to k$

$$\frac{p}{T} = \mp \frac{g}{2\pi^2} \int_0^\infty \ln\left(1 \mp e^{-\beta k}\right) k^2\, dk$$

$$\underset{\beta k = u}{=} \mp \frac{g}{2\pi^2} \beta^{-3} \underbrace{\int_0^\infty u^2 du \ln\left(1 \mp e^{-u}\right)}_{\text{number}}$$

$$= \frac{g}{2\cancel{\pi^2}} T^3 \underset{\substack{\| \\ \frac{\pi^2}{15}}}{a_B} \frac{\cancel{\pi^2}}{3} \begin{cases} 1 - b \\ 7/8 - f \end{cases}$$

$-\frac{\pi^4}{45}$ for bosons

$\frac{7\pi^4}{360}$ for fermions.

$$p = \frac{g}{6} a_B T^4 \begin{cases} 1 \\ 7/8 \end{cases}$$

$$n = \frac{g}{2\pi^2}\int_0^\infty k^2\,dk\,\frac{e^{-\beta k}}{1 \mp e^{-\beta k}}$$

$$\underset{u=\beta k}{=} \frac{g}{2\pi^2} T^3 \int_0^\infty u^2\,du\,\frac{e^{-u}}{1\mp e^{-u}}$$

number $\rightarrow$ $2\zeta(3)$ for b

$\frac{3}{2}\zeta(3)$ for f

$$\zeta(n) = \sum_{n=1}^{\infty} \frac{1}{n^s} \underset{s=3}{=} 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

$$\rho = \overset{=0}{\mu} n - \frac{\partial}{\partial\beta}(p\beta)$$

$$= -\frac{\partial}{\partial\beta}\left(\frac{g}{6} a_B \beta^{-3}\right)\begin{cases}1\\ 7/8\end{cases}$$

$$= \frac{g}{2} a_B T^4 \begin{cases}1\\ 7/8\end{cases} \Rightarrow \text{For photons } g=2$$

$$\rho = a_B T^4$$

$p = \frac{\rho}{3}$ for f and b

$$s = \frac{1}{T}(p + \rho - \underset{\substack{\| \\ 0}}{\mu n}) = \frac{1}{T}\left(\frac{1}{2} + \frac{1}{6}\right) a_B T^4 \begin{cases} 1 \\ 7/8 \end{cases}$$

$$= \frac{2}{3} a_B T^3 \begin{cases} 1 \\ 7/8 \end{cases}$$

these results are for individual types of particles.

$$p = \sum_{i=1}^{K} p_i = \frac{1}{6} a_B T^4 \sum_i g_i \times \begin{cases} 1 \\ 7/8 \end{cases}$$

g_{eff}.

$$\rho = \frac{1}{2} a_B T^4 g_{eff}, \qquad s = \frac{2}{3} a_B T^3 g_{eff}.$$

We want to calculate the variation of T with λ.

Use $\frac{d}{dt}(s\lambda^3)\Big| = 0$

Suppose that T is such that for every particle species i, either $T \gg m_i$ or $T \ll m_i$

But there is no i s.t. $T \sim m_i$

$T\lambda = \text{constant}$, $T \sim \frac{1}{\lambda}$.

$s = \frac{2}{3} a_B T^3 \times g_{eff}$.

include contribution from particles for which $T \gg m_i$.

This fails when $T \sim m_i$ for some i.

$\rightarrow$ more detailed analysis is needed (we'll see examples later).

For now ask a simpler question:

Two epochs:

(1): $T \gg m_i$ for some i

(2) $T \ll m_i$ for the same i.

$T_{(1)}, \lambda_{(1)}, g_{eff}^{(1)}$ refer to quantities at some time in epoch (1) → earlier

$T_{(2)}, \lambda_{(2)}, g_{eff}^{(2)}$ refer to quantities at some time in epoch (2), → later.

We can still use entropy conservation.

$$T_{(1)}^3 \, g_{eff}^{(1)} \, \lambda_{(1)}^3 = T_{(2)}^3 \, g_{eff}^{(2)} \, \lambda_{(2)}^3$$

$$\Rightarrow \quad T_{(2)} = T_{(1)} \left(\frac{g_{eff}^{(1)}}{g_{eff}^{(2)}} \right)^{1/3} \frac{\lambda_{(1)}}{\lambda_{(2)}}$$

$T_{(2)}$ is larger than expected from $\frac{1}{\lambda}$ behaviour

= entropy has gotten transfered from the massive particle → massless particles

Decoupling (freezout) $\rightarrow$ Thermal freeze out.

Suppose in some epoch, the interaction rate of some particle falls below the expansion rate.

Average rate of collision per unit time

$$< \frac{\dot{\lambda}}{\lambda}$$

The particle falls out of equilibrium

$\Rightarrow$ e_i, p_i etc. are no longer given by the statistical formula.

number density $n \propto \frac{1}{\lambda^3} = \frac{n_F \lambda_F^3}{\lambda^3}$

λ_F : value of λ at freeze out.
n_F : number density at freeze out.

Q. How to calculate e?

→ We need to study how the energy of a free particle change with time.

Homogeneity + isotropy ⇒ assume that the particle starts at $r=0$ and moves radially outwards.

$$d\tau^2 = -ds^2 = dt^2 - a(t)^2 dr^2 \quad (\text{ignore } k).$$

Action of the particle:

$$= \int du \left(-g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{1/2} = \int du \left[\left(\frac{dt}{du}\right)^2 - a(t)^2\left(\frac{dr}{du}\right)^2\right]^{1/2}$$

Eq. for r is obtained by requiring that

$\delta(\text{Action}) = 0$ for arbitrary δr.

$$\delta(\text{Action}) = -\int du \left[\left(\frac{dt}{du}\right)^2 - a(t)^2\left(\frac{dr}{du}\right)^2\right]^{-1/2} \cancel{\frac{1}{2}}\, a(t)^2 \frac{dr}{du}\frac{d}{du}(\delta r) \times \cancel{2}$$

$$= \int du\, \delta r \frac{d}{du}\left[\left\{\left(\frac{dt}{du}\right)^2 - a(t)\left(\frac{dr}{du}\right)^2\right\}^{-1/2} a(t)^2 \frac{dr}{du}\right] = 0$$

$$\left\{\left(\frac{dt}{du}\right)^2 - a(t)^2\left(\frac{dr}{du}\right)^2\right\}^{-1/2} a(t)^2 \frac{dr}{du} = \text{constant}$$

l.h.s. is invariant under $u = f(v)$

Choose $u = \tau$.

$\Rightarrow d\tau^2 = dt^2 - a(t)^2 dr^2 \Rightarrow \{\ \}^{-1/2} = 1$

$\Rightarrow a(t)^2 \frac{dr}{d\tau} = c \Rightarrow \frac{dr}{d\tau} = \frac{c}{a(t)^2}$.

$$\frac{dt}{d\tau} = \left\{1 + a(t)^2\left(\frac{dr}{d\tau}\right)^2\right\}^{1/2} = \left\{1 + \frac{c^2}{a(t)^2}\right\}^{1/2}$$

Energy of the particle in the comoving frame:

$$= m\frac{dt}{d\tau} = m\left\{1+\frac{c^2}{a(t)^2}\right\}^{1/2}$$

$$= \left\{m^2+\frac{c^2m^2}{a(t)^2}\right\}^{1/2}$$

Now suppose that at $t=t_F$, the particle has energy $(m^2+k^2)^{1/2}$

$$\Rightarrow \quad \frac{c^2m^2}{a_F^2} = k^2 \qquad c^2m^2 = k^2a_F^2$$

$\Rightarrow$ Energy at time $t = \left\{m^2+k^2\frac{a_F^2}{a(t)^2}\right\}^{1/2}$

$$= \left\{m^2+\frac{k^2\lambda_F^2}{\lambda^2}\right\}^{1/2}$$