Photon freezeout at 10^5 K.

Matter-Radiation equality:

$$\frac{1}{2} \underset{\pi^2/15}{\underset{\shortparallel}{a_B}} T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \right) = \Omega_m \frac{3 H_0^2}{8\pi G} \left(\frac{T}{T_0}\right)^3$$

$H_0 = 100h$ km/s/Mpc.

$\Rightarrow$ Ex. $T = 6.6 \times 10^4 \mathrm{K} \; \Omega_m h^2$

$\simeq 10^4$ K $\quad$ $\Omega_m \to .3$, $h \to .7$

$\Downarrow$

$10^4 \times 8.6 \times 10^{-5}$ eV $\simeq .86$ eV.

Recombination: Ions → atoms.

$He^{++} + e^- \rightarrow He^+ + \gamma$ binding energy $4 \times 13.6\,eV = 54.4\,eV$.

$He^+ + e^- \rightarrow He$: binding energy $\simeq 24\,eV$.

$\boxed{p + e^- \rightarrow H}$: binding energy $\boxed{13.6\,eV}$.

→ remove certain fraction of free electrons.

$X_n = .13, \quad X_p = 1 - X_n \simeq .87$

$.13$: fraction of protons in He

$.74$ fraction of protons that are free.

$\frac{.13}{.74}$ fraction of electrons get bound

Consider a temperature T when the He atoms have already formed.

Important particles: p, e^-, γ, various states of H-atom (n, ℓ) states.

n_e : no. density of free electrons

n_p : no density of free protons.

$n_{n,\ell}$: no density of H atoms in n,ℓ state

$n_e = n_p$ due to charge neutrality.

Conserved charges

Total # electrons $= \left(n_e + \sum_{n,l} n_{n,l}\right) \lambda^3$

Total # protons $= \left(n_p + \sum_{n,l} n_{n,l}\right) \lambda^3$

Assume thermal equilibrium.

In grand canonical ensemble:

$$\tilde{\mu}_1 \left(n_e + \sum_{n,l} n_{n,l} \right) + \tilde{\mu}_2 \left(n_p + \sum_{n,l} n_{n,l} \right)$$

$$= \mu_e n_e + \mu_p n_p + \sum_{n,l} \mu_{n,l} n_{n,l}$$

$$\Rightarrow \mu_e = \tilde{\mu}_1, \quad \mu_p = \tilde{\mu}_2, \quad \mu_{n,l} = \tilde{\mu}_1 + \tilde{\mu}_2 = \mu_e + \mu_p$$

$$n_\alpha = g_\alpha \, e^{-(m_\alpha - \mu_\alpha)/T} \left(\frac{m_\alpha T}{2\pi}\right)^{3/2}, \quad \alpha = e^-, p, (n,l).$$

degeneracy

$$g_e = 2, \quad g_p = 2, \quad g_{n,l} = 2 \times 2 \times (2l+1).$$

$$n_e = 2 \, e^{-(m_e - \mu_e)/T} \left(\frac{m_e T}{2\pi}\right)^{3/2}$$

$$n_p = 2 \, e^{-(m_p - \mu_p)/T} \left(\frac{m_p T}{2\pi}\right)^{3/2}$$

binding energy

$$n_{n,l} = 2 \times 2 \times (2l+1) \times e^{-(m_e + m_p - B_{n,l} - \mu_e - \mu_p)/T} \times \left(\frac{m_p T}{2\pi}\right)^{3/2}$$

$$= (2l+1) \, n_e n_p \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{B_{n,l}/T}$$

Use $n_e = n_p$

$$n_{n,l} = (2l+1)\, n_p^2\, e^{B_{n,l}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2}$$

$$\frac{n_{n,l}}{n_{1,s}} = (2l+1)\, e^{(B_{n,l} - B_{1s})/T} = (2l+1)\, e^{-\frac{13.6\,eV}{T} \times \left(1 - \frac{1}{n^2}\right)}$$

$\frac{3}{4}$ for $n=2$

$$B_{n,l} = \frac{13.6}{n^2}$$

$B_{1s} = 13.6\ eV.$

$$T \sim 3000K \sim 3 \times 10^3 \times 8.65 \times 10^{-5}$$
$$= 2.4 \times 10^{-1}\ K$$

$$e^{-\frac{13.6\,eV}{T}} \sim e^{-13.6 \times \frac{10}{2.4}} \sim e^{-60} \ll 1.$$

$\therefore$

Define: $X = \dfrac{n_p}{n_p + n_{1s}}$ $\qquad 1 - X = \dfrac{n_{1s}}{n_p + n_{1s}}$

$$n_{n,l} = (2l+1)\, e^{B_{n,l}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \times \underbrace{n_e n_p}_{n_p^2}$$

$$1 - X = e^{B_{1s}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \frac{n_p^2}{n_p + n_{1s}}$$

$$= e^{B_{1s}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2} X^2 \underbrace{\left(n_p + n_{1s}\right)}_{.74 \times \frac{\Omega_B}{m_p} \frac{3 H_0^2}{8\pi G} \left(\frac{T}{T_0}\right)^3}$$

$$1 - X = X^2 S$$

13.6 eV

$$S = 1.75 \times 10^{-22} \left(\frac{T}{1K}\right)^{3/2} \Omega_B h^2 \, e^{157894 K/T}$$

analog of ϵ in nucleosynthesis.

For $\Omega_m h^2 = 0.2$ $\quad (\Omega_m = .3, \; h = .7)$

$h^2 = .5$

T (in K)	X
4500	.998
4000	.900
3800	.634
3600	.290
3400	.094
3200	.024

The actual process forms out of thermal equilibrium & hence is slower.

Thermal equilibrium ⇒

$e^- p \to H + \gamma$ with certain frequency.

$\gamma H \to e^- + p$ | why is density of H so low at $T \sim 13.6\,eV$.

As T falls: For $\gamma + H \to e^- + p$, the γ needs to have energy $\sim B_{n,l}$.

If $T \ll B_{n,l}$ then the fraction of γ with this energy $\propto e^{-\frac{B_{n,l}}{T}}$

Consider the process

$$e^- p \to H_{1s} + \gamma$$

has energy ~ 13.6 eV.

How many such photons are present in the bath?

$T \sim 3000K$

$$\exp(-13.6/T) \sim e^{-60} \sim 10^{-20}$$

No. of such photons in the bath $\sim n_r \times 10^{-20}$

# of such photons produced in $e^- p$ binding $\sim 10^{-10} n_r$

The 13.6 eV photon cannot lose energy very fast.

→ cannot lose energy efficiently by scattering with electrons.

→ can lose energy due to redshift. but that also slow.

This photon will eventually hit an H which has formed in 1S state and either kick the electron out or excite it to some $|n,l\rangle$ state.

Conclusion: We cannot populate 1S state by direct electron capture.

→ Mostly populated by $2p \to 1S + \gamma$

Another process: $2s \to 1S + \gamma + \gamma$
(slow)

Full treatment of $e^- p$ recombination is done numerically.

$$\frac{d}{dt}(n_{n,l} x^3) = \text{rate of production} - \text{rate of annihilation}$$

$$\frac{d}{dt}(n_e x^3), \quad \frac{d}{dt}(n_p x^3) \rightarrow \text{not independent eqs.}$$

arxiv: astro-ph/9909275

→ uses 300 energy levels.

We'll use a simpler version

Regard this as a combination of 3 systems:

① Free e^-, p pair:

② All states in level (n, ℓ) for $n \geq 2$

→ assumed to be in thermal equilibrium

③ H-atom in the 1s state:

Justification for treating $|n,l\rangle$ states with $n \geq 2$ in thermal equilibrium:

Energy difference $|E_{n,l} - E_{n',l'}| < \frac{13.6\,eV}{4}$

γ that is produced during such transitions has $E < \frac{13.6\,eV}{4}$.

Fraction of γ in the bath with such E:

$\exp\left(-\frac{13.6\,eV}{4T}\right)$ $\quad T \sim 3000K$ $\quad e^{-13.6\,eV/(4T)} \gg 10^{-10}$