$e^- p \rightleftarrows H$

Full analysis involves

n_e : no density of free electrons (also free protons)

$n_{n,l}$: no. density of n, l states of H atoms.

$\frac{d(n_e \lambda^3)}{dt}$ = rate of production − rate of destruction

$\frac{d(n_{n,l} \lambda^3)}{dt}$ = " "

r.h.s. is a fn. of $n_e, \{n_{n,l}\}$.

$\Rightarrow$ coupled differential eq.

boundary condition: At large T, $n_{n,l} = 0$.

We'll analyze this by making a simplifying assumption:

Three systems:

① Free e^- p system, n_e: no density of free electron.

② n, l states with $n \geq 2$ assumed to be in thermal equilibrium.

$$n_{n,l} = n_{2,s}(2l+1)\, e^{-(B_{2s} - B_{nl})/T}$$

③ 1s states with number density $n_{1,s}$.

$$n_{1s} + \sum_{\substack{n,l \\ n \geq 2}} n_{n,l} + n_e = .73 \times n_B$$

$$n_e + n_{1s} + \sum_{\substack{n,l \\ n\geq 2}} n_{n,l} = n_B \times .73$$

$$\sum_{\substack{n,l \\ n\geq 2}} n_{2s} (2l+1) e^{-(B_{2s}-B_{nl})/T} = n_{2s} f(T)$$

$$B_{2s} - B_{nl} = 13.6\,\mathrm{ev} \left(\frac{1}{4} - \frac{1}{n^2}\right) \geq 13.6 \times \frac{5}{36}.$$

$$e^{-(B_{2s}-B_{nl})/T} \sim e^{-\frac{13.5\times 5}{36} \times \frac{1}{4000\times 8.6\times 10^{-5}}} \sim e^{-5}$$

$T \sim 4000\,K$

small number.

$\Rightarrow$ $f(T)$ gets most of the contribution from 2s and 2p states. $f(T) \simeq 1+3 = 4$

Definitions: $x = \frac{n_e}{.73 n_B}$, $y = \frac{n_{1s}}{.73 n_B}$

$$.73 x n_B = n_e + n_{2s} f(T) + n_{1s}$$

$$\Rightarrow \quad n_{2s} = .73 n_B (1 - x - y) \frac{1}{f(T)}$$

Goal: set up differential equations of the form:

$$\frac{dx}{dt} = f_1(x, y), \qquad \frac{dy}{dt} = f_2(x, y)$$

$$\frac{d}{dt}(n_e\lambda^3) = -\alpha\, n_p\,(n_e\lambda^3) + \beta\, n_{2s}\,\lambda^3$$

$\alpha \to \sigma v \to$ capture x-section of e^- p to form H-atom in state n, l with $n \geq 2$

$\beta \to$ rate of ionization of (n,l) states by thermal photon.

$$\sum_{\substack{n,l \\ n\geq 2}} \beta_{n,l}\, n_{n,l}\, \lambda^3$$

express in terms of $n_{2,S}$

Why not $n=1$? Need γ of energy 13.6 eV.

For $T \sim 4000K$, the actual no. of thermal photons/electron of $E = 13.6$ eV is very small.

Why did we not include $e^- p \to 1s$ transition?

$e^- p \to 1s$ produce a 13.6 eV photon.

This will eventually collide with an 1s atom. $e^{-(B_{1s}-B_{n\ell})/\tau}$

$\to$ Either ionize it, or raise it to a highly excited states.

differs from continuum by a small amount.

There are large no. of thermal photons that can ionize this.

$$\frac{d}{dt}(n_e \lambda^3) = -\alpha n_e^2 \lambda^3 + \beta n_{2s} \lambda^3$$

σv

$\rightarrow$ can be calculated using microscopic theory or can be measured.

How to compute β?

If we had thermal equilibrium (no expansion, $n_e/n_r \rightarrow 0$ so that the system does not affect the bath), then $\frac{d}{dt}(n_e) = 0$

$$\Rightarrow \alpha n_e^2 = \beta n_{2s} \qquad n_{2s}\Big|_{\text{equilibrium}} = n_e^2 e^{B_{2s}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2}$$

$$\Rightarrow \beta = \alpha e^{-B_{2s}/T} \left(\frac{m_e T}{2\pi}\right)^{3/2}$$

$$\frac{d}{dt}\left(n_{1s}\lambda^3\right) = \left(\Gamma_{2s} + 3\,\Gamma_{2p}\,P\right) n_{2s}\lambda^3 - \varepsilon\, n_{1s}\lambda^3$$

Γ_{2s} : Rate of decay of $2s \to 1s + \gamma + \gamma$.

Γ_{2p} : Rate of decay of $2p \to 1s + \gamma$.

P : Probability that the γ from $2p \to 1s$ decay (Lyman α photons) does not excite a $1s$ back to $2p$.

We have <u>not</u> included $n\ell \to 1s$ for $n \geq 3$.

ε : Rate of transition from $1s \to n\ell$ states by thermal photon.

$$\frac{d}{dt}\left(n_{1s}\lambda^3\right) = \left(\Gamma_{2s} + 3P\,\Gamma_{2p}\right) n_{2s}\lambda^3 - \varepsilon\, n_{1s}\lambda^3$$

Γ_{2s}, Γ_{2p} : Decay rates $\rightarrow$ can be computed using microscopic theory or measured.

What about ε?

Try to calculate ε using detailed balance

Assume thermal equilibrium & set $\frac{d}{dt}(n_{1s}) = 0$.

λ = constant.

Requires that the bath is not affected by the e^-, p, H system.

(Can set $P=1$).

Assuming equilibrium:

$$(\Gamma_{2s} + 3\Gamma_{2p})\, n_{2s} = \mathcal{E}_0\, n_{1s}$$

$$\searrow e^{-(B_{1s}-B_{2s})/T}\, n_{1s}$$

$$\Rightarrow \quad \mathcal{E}_0 = (\Gamma_{2s} + 3\Gamma_{2p})\, e^{-(B_{1s}-B_{2s})/T}.$$

How does the actual $\mathcal{E}$ differ from $\mathcal{E}_0$?

Using $\frac{n_\gamma}{n_e} \sim 10^{10}$ and the value of $e^{-(B_{1s}-B_{2s})/T}$

No. of thermal photons present with energy $B_{1s}-B_{2s}$ is small compared to n_e.

Consequence: If a 1s is excited to 2p by a thermal photon, it significantly depletes the thermal population of photons of energy $B_{1s} - B_{2s}$ -

$$\mathcal{E} = (\Gamma_{2s} + 3\bar{P}\,\Gamma_{2p})\, e^{-(B_{1s}-B_{2s})/T}$$

$\bar{P}$ = Fraction of thermal photons of energy $E_{1s} - E_{2s}$ hitting a 1s atom at time t has not gotten absorbed earlier due to $1s \to 2p$ transition.

$$\begin{cases} \frac{d}{dt}\left(n_{1s}\lambda^3\right) = \left(\Gamma_{2s} + 3P\,\Gamma_{2p}\right) n_{2s}\lambda^3 - \mathcal{E}\, n_{1s}\lambda^3 \\ \frac{d}{dt}\left(n_e\lambda^3\right) = \alpha\, n_e^2\lambda^3 - \beta\, n_{2s}\lambda^3 \end{cases}$$

$$n_B = n_{B0}\,\lambda^{-3} = n_{B0}\left(\frac{T}{T_0}\right)^3$$

$$\mathcal{E} = \left(\Gamma_{2s} + 3\bar{P}\,\Gamma_{2p}\right) e^{-(B_{1s}-B_{2s})/T}$$

$$X = \frac{n_e}{.73\, n_B\lambda^3}$$

Divide by $\left(.73\, n_B\lambda^3\right)$

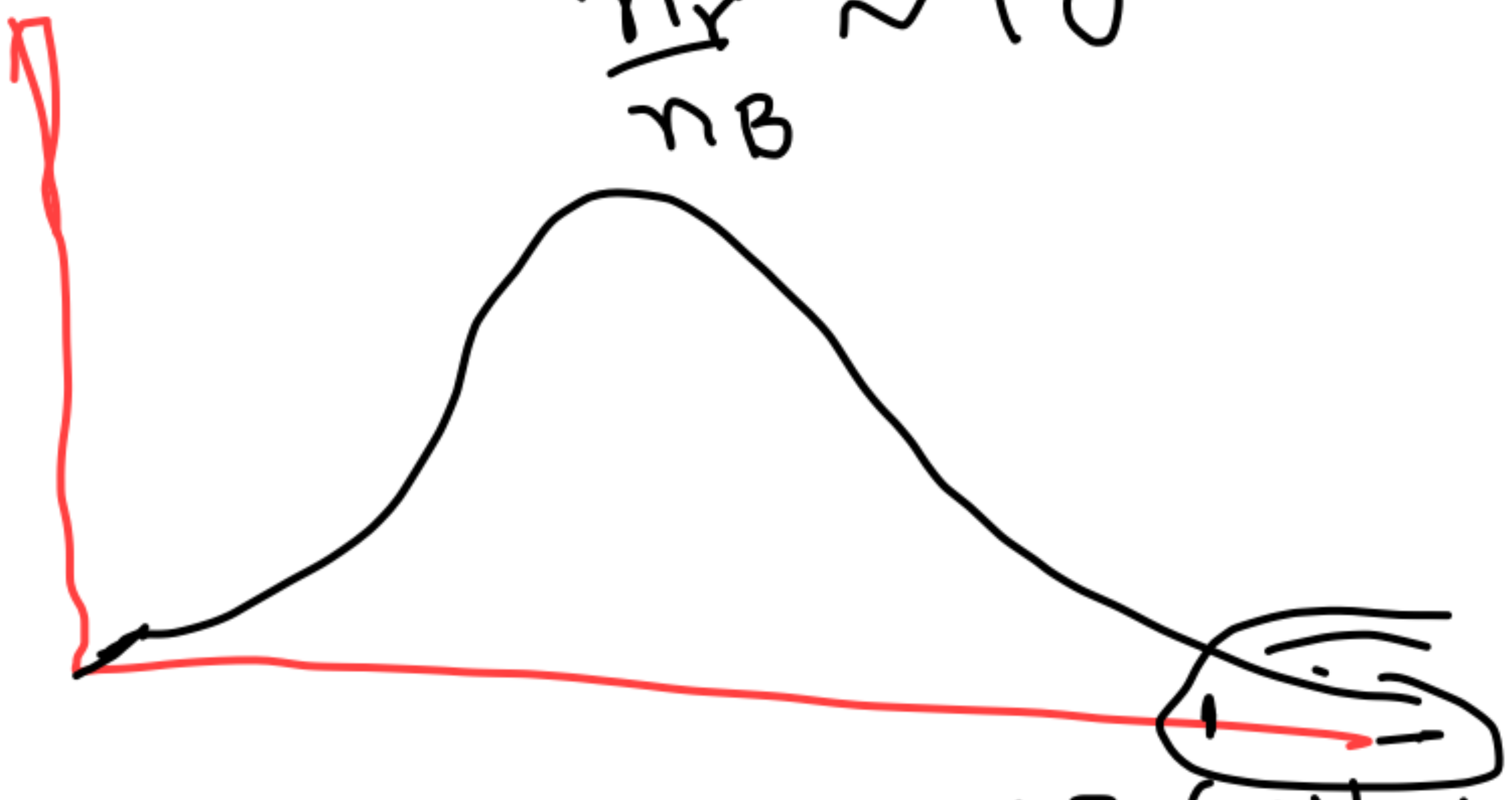
$$Y = \frac{n_{1s}}{.73\, n_B\lambda^3}$$

$$\frac{dX}{dt} = \alpha X^2\left(.73\, n_B\right) - \beta\,\frac{n_{2s}}{.73\, n_B}$$

$$\boxed{\begin{aligned} &n_e + n_{1s} + n_{2s} f(T) = .73\, n_B \\ &\Rightarrow \frac{n_{2s}}{.73\, n_B} = \frac{1-X-Y}{f(T)} \end{aligned}}$$

$$\frac{dY}{dt} = \left(\Gamma_{2s} + 3P\,\Gamma_{2p}\right)\frac{n_{2s}}{.73\, n_B} - \mathcal{E}\, Y$$

$$\frac{n_\gamma}{n_B} \sim 10^{10}$$



$$13.6\,eV \times \frac{3}{4}$$