$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right)$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

Comoving objects: At rest in (t,θ,ϕ,r) coordinate system.

$(t,\vec{A})$, $(t,\vec{B})$

$\vec{A} \to (r_1, 0, 0)$

$\vec{B} \to (0,0,0)$ — r, θ, ϕ

$$a(t)\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}}$$

= physical distance between $\vec{A},\vec{B}$ at time t.

Comoving distance: Distance between $\vec{A},\vec{B}$ measured in today's metric.

$$a(t_0)\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}} = \text{fixed.}$$

Physical distance $= \lambda(t) \times$ comoving distance.

Some definitions.

$$\lambda(t) = \frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \cdots$$

H_0 = Hubble constant $\simeq 70\,\mathrm{km/sec/Mpc}$

$\mathrm{Mpc} = 10^6 \times 3.26$ l.y.

q_0 = deceleration parameter

It was thought earlier that the expansion is slowing down.

We know today that $q_0 < 0$

$\rightarrow$ Universe is undergoing accelerated expansion.

Red-shift parameter: $z = \frac{a(t_0)}{a(t_1)} - 1 = \frac{1}{\lambda(t_1)} - 1$

Today: $z = 0$

A distant object having red-shift z means that the light we receive today from that object has redshift $(1+z)$.

Suppose that light emitted from a source at $(r_1, 0, 0)$ at time t_1 reaches us today (t_0)
$(r = 0, \theta = 0, \phi = 0)$

$$\int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = a_0 \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = d_c$$
(comoving distance of the object from us)

$\Rightarrow z$ as a fn. of d_c if we know $\lambda(t)$.

Knowing $\lambda(t)$, we can find z as fn. of d_c.

If we know z as a fn. of d_c, we can find $\lambda(t)$ as fn. of t.

Ex. Suppose we know that

$$\underline{\underline{z}} = a \, \underline{\underline{d_c}} + b \, d_c^2 + c \, d_c^3$$

Ex. Find $\underline{H_0}$, $\underline{q_0}$ etc. in terms of $a, b, c, \ldots$

Time evolution of $a(t)$:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

Ex. Check that: $\dot{a} = \frac{da}{dt}$, $\ddot{a} = \frac{d^2 a}{dt^2}$, ...

$$R_{00} = -\frac{3\ddot{a}}{a}, \quad R_{i0} = 0, \quad R_{ij} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}\right) g_{ij}$$

$$R = g^{00} R_{00} + g^{ij} R_{ij} = \frac{3\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}\right)$$

$$= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \, T_{\mu\nu}.$$

$$T_{i0} = 0, \quad T_{ij} \propto g_{ij}$$

$$T_{00} = \rho(t), \quad T_{ij} = p(t)\, g_{ij}$$

Ex. 00 : $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho(t)$ → Friedmann equation.

$i0$: $0 = 0$

ij : $\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$

Ex. By combining $\frac{d}{dt}$(Friedmann eq) with the second eq. we can eliminate $\ddot{a}$ and get

$$\boxed{\frac{d}{dt}(\rho a^3) = -3a^2\dot{a}p}$$

Ex. check that this follows from $D^\mu T_{\mu\nu} = 0$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$D^{\mu}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = 0$$ as a consequence of Bianchi identity.

$$\Rightarrow D^{\mu}(T_{\mu\nu}) = 0$$

$$\boxed{\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho} \qquad \boxed{\frac{d}{dt}(\rho a^3) = -3p a^2 \dot{a}}$$

Physical interpretation of ρ and p:

$T_{00} = \rho$, $T^{00} = \rho \Rightarrow \rho =$ energy density.

$T_{i0} = 0 \Rightarrow$ Momentum density $= 0$

Energy flux $= 0$

$\Rightarrow$ On the average the objects producing $T_{\mu\nu}$ must be at rest in the comoving coordinate system.

$$T_{ij} = p\, g_{ij}$$

Use some coordinate system in which

$$g_{ij} = \delta_{ij}$$

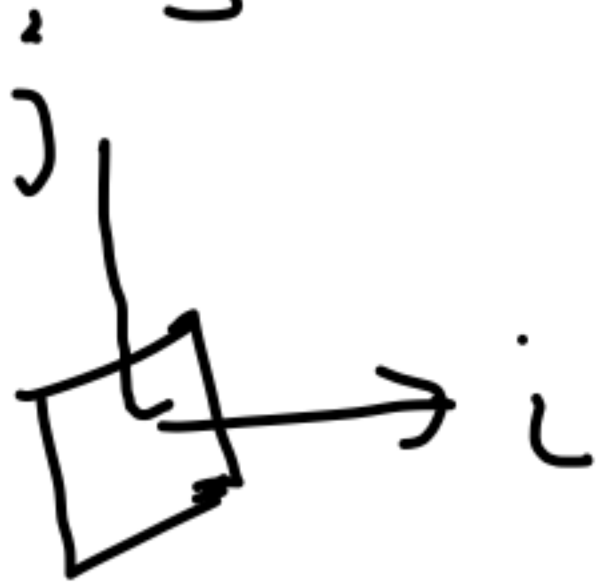
$$ds^2 = -dt^2 + a(t)^2 \left(\frac{d\rho^2}{1-k\rho^2/a_0^2} + \rho^2 d\Omega^2 \right)$$

$$T_{ij} = p\,\delta_{ij}$$

$$\Downarrow$$

Flux of jth component of momentum along i-th direction.

$T_{ij} = 0$, for $i \neq j$.



P_j and P_i are uncorrelated.

$\Rightarrow$ Total P_j flux along i should vanish.

If $i = j \Rightarrow$ objects moving from left to right and right to left both give +ve p_i flux. $\Rightarrow T_{ii} \neq 0$.

i independence of T_{ii} is a consequence of isotropy.

Physical interpretation of p.

Total momentum of i the fluid incident δA on this wall in unit time:

Momentum transfer to the wall / unit time $= \frac{1}{2} T_{ii}\, \delta A \times 2 = T_{ii}\, \delta A = p\, \delta A$

= Force.

Force $= p\, \delta A$

Force/area $= p$

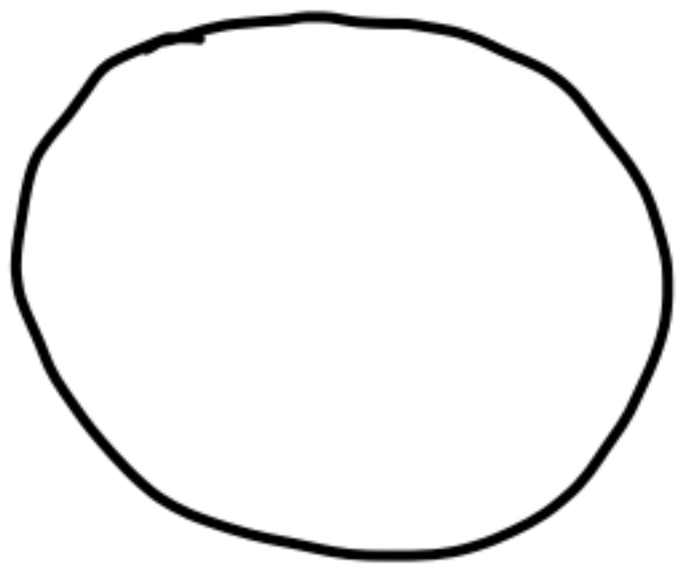
pressure

$$\left.\begin{aligned} &\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \\ &\frac{d}{dt}\left(\rho a^3\right) = -3a^2 \dot a\, p \end{aligned}\right\}$$

$\rho(t)$, $p(t)$, $a(t)$

$\rightarrow$ 3 unknowns

2 equations



— comoving volume.

$$\frac{d}{dt}\left(\frac{4}{3}\pi a^3 \rho\right) = -4\pi a^2 \frac{da}{dt} p$$

$$0 = dQ = dU + p\,dV$$

We need one more equation.

Third equation comes from
equation of state

→ Relation between p and ρ.

— provided by thermodynamics.

Examples: ① Radiation $\quad p = \frac{\rho}{3}$

3rd equation.

② Non-relativistic matter.

$v << c$

Approximate this by ideal gas.

$\rho = n\, m$

m — mass of the particles

n — number density of particles

\# of protons+neutrons per unit volume.

6 / cubic meter

$p = n\, k_B T$

$k_B T$ — kinetic energy of the particle $\ll m$.

$\frac{1}{2} m v^2 \ll mc^2$

$$\Rightarrow \frac{p}{\rho} = \frac{k_B T}{m} \ll 1$$

$\Rightarrow$ eq. of state for non-relativistic matter: $p = 0$.

③ Dark energy / cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - 8\pi G \underbrace{\Lambda}_{\text{constant}} g_{\mu\nu}$$

$$= 8\pi G \big(T^{m,r}_{\mu\nu} + \underbrace{T^{\Lambda}_{\mu\nu}}_{= -\Lambda g_{\mu\nu}}\big)$$

$$\boxed{p_\Lambda = -\rho_\Lambda}$$

$$T_{00} = -\Lambda g_{00} = \Lambda \Rightarrow \rho_\Lambda = \Lambda$$

$$\underbrace{T_{ij}}_{= p_\Lambda g_{ij}} = -\Lambda g_{ij} \Rightarrow p_\Lambda = -\Lambda$$