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Review

n_e : no. density of free electrons

n_{1s} : no density of 1s atoms.

$n_{n,l}$: no " " " n,l "

$$n_{n,l} = n_{2s}\,(2l+1)\,\exp\left(-(B_2 - B_n)/T\right)$$

$$\Rightarrow \sum_{\substack{n,l \\ n\geq 2,\, l}} n_{n,l} = n_{2s}\, f(T) \rightarrow \sum_{\substack{n,l \\ n \geq 2}} (2l+1)\exp\left(-(B_2 - B_n)/T\right)$$

$$n_e + n_{1s} + n_{2s} f(T) = .73 \times n_B$$

For $T \ll (B_2 - B_3)$, $f(T) \simeq 4$

$$x = \frac{n_e}{.73\, n_B}, \quad y = \frac{n_{1s}}{.73\, n_B}, \quad n_{2s} = \frac{.73\, n_B (1 - x - y)}{f(T)}.$$

$$\frac{dX}{dt} = -\alpha X^2 (.73\, n_B) + \frac{\beta}{f(T)} (1-X-Y)$$

$$\frac{dY}{dt} = (\Gamma_{2s} + 3\Gamma_{2p} P) \frac{1}{f(T)} (1-X-Y) - \varepsilon Y$$

$\alpha = \sigma v$
↳ e^-p capture x-section into $n\ell$ states for $n \geq 2$

$\beta = \alpha \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_2/T}$, $\varepsilon = (\Gamma_{2s} + 3\Gamma_{2p} \bar{P})\, e^{-\frac{B_1 - B_2}{T}}$

Γ_{2s} : decay rate of $2s \to 1s + \gamma + \gamma$

Γ_{2p} : " " " $2p \to 1s + \gamma$

α, Γ_{2s}, Γ_{2p} can be calculated using microscopic theory or measured.

$P(t)$: The probability that a Lyman α photon produced at time t is not captured by a $1s \to 2p$ transition.

$\bar{P}(t)$: Fraction of thermal photons (near Lyman α) that survive capture up to time t;

Preliminaries: $p(\omega)\,d\omega$ will denote the probability that a photon emitted during $2p \to 1s$ transition has energy in the range $(\omega, \omega + d\omega)$.

$$p(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega - \omega_\alpha)^2 + (\Gamma_{2p}/2)^2}\,,$$

Ex. $\int_{-\infty}^{\infty} p(\omega)\,d\omega = 1$

$\Gamma_p \ll \omega_\alpha$ | Width of dist. Γ_{2p}.

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$\sigma(\omega)$: defined to be the absorption x-section of a photon of energy ω by 1s state to make transition to 2p state.

$$\sigma(\omega) = \frac{3\pi^2}{\omega_\alpha^2} \Gamma_{2p} \cdot \phi(\omega).$$

Suppose a photon of energy ω is produced at time $\underline{t}$.

Total probability that it is absorbed between t' and $t'+dt'$ assuming that it survives till t':

$$\sigma\left(\omega \frac{\lambda(t)}{\lambda(t')}\right) n_{1s}(t') \underset{\overset{\shortparallel}{1}}{c} \, dt'$$

Define $Q_\omega(t')$ to be the probability that a photon of energy ω emitted at time t survives till t'.

$$\frac{dQ_\omega(t')}{dt'} = -Q_\omega(t') \times \sigma\left(\omega \frac{\lambda(t)}{\lambda(t')}\right) n_{1s}(t')$$

B.C. $Q_\omega(t') = 1$ at $t' = t$.

$$\Rightarrow Q_\omega(\infty) = \exp\left[-\int_t^\infty dt' \, \sigma\left(\omega \frac{\lambda(t)}{\lambda(t')}\right) n_{1s}(t')\right]$$

change variables: From t' to $\omega' = \omega \frac{\lambda(t)}{\lambda(t')}$

$$d\omega' = -\omega \frac{\lambda(t)}{\lambda(t')^2} \dot{\lambda}(t') = -\omega' H(t')$$

$$Q_\omega(\infty) = \exp\left[-\int_0^\omega d\omega' \, n_{1s}(t') \, \sigma(\omega') \, \frac{1}{\omega' H(t')}\right]$$

$$\sigma(\omega') \to \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2} \, p(\omega')$$

$$\simeq \exp\left[- \underbrace{n_{1s}(t) \frac{1}{\omega_\alpha H(t)} \times \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2}}_{x(t)} \int_0^\omega d\omega' \, p(\omega')\right]$$

→ assuming that $p(\omega')$ is sharply peaked around ω_α.

$P(t)$: Probability that a photon emitted at t by $2p \to 1s$ transition is not absorbed by $1s \to 2p$ transition

$$= \int_{-\infty}^{\infty} p(\omega) \, d\omega \, Q_\omega(\infty)$$

$$P = \int_{-\infty}^{\infty} d\omega \, p(\omega) \exp\left[-x \int_0^{\omega} p(\omega') \, d\omega'\right]$$

$$p(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega-\omega_\alpha)^2 + (\Gamma_{2p}/2)^2}$$

Change variables to

$$u = \frac{2}{\Gamma_{2p}}(\omega - \omega_\alpha), \quad v = \frac{2}{\Gamma_{2p}}(\omega' - \omega_\alpha)$$

$$P = \frac{1}{\pi} \int_{-\infty}^{\infty} du \, \frac{1}{u^2+1} \exp\left[-\frac{x}{\pi} \int_{-\infty}^{u} \frac{1}{v^2+1} \, dv\right]$$

$-\infty \;\; \cancel{\ } \;\; -\frac{2\omega_\alpha}{\Gamma_{2p}}$ Large.

$\tan^{-1} u + \frac{\pi}{2}$ //

Change variable

$s = \tan^{-1} u$

Ex. $P = F(x(t))$, $F(x) = \frac{1}{x}(1 - e^{-x})$

$$x = \frac{3\pi^2 n_{1s}(t)}{\omega_\alpha^3 H(t)} \Gamma_{2p}$$

$\Rightarrow F(x) \xrightarrow[x \to 0]{} 1$

① If $H(t) = 0 \Rightarrow x = \infty$, $F(\infty) = 0$

② x increases with n_{1s}

$P = F(x)$ decreases with n_{1s}.

$2p \to 1s + \gamma \Rightarrow$ dominates initially.

$2s \to 1s + \gamma + \gamma \Rightarrow$ dominates at late time.

Calculation of $\bar{P}$

$f(\omega)\,d\omega$ = # density of thermal photons in the range $(\omega, \omega+d\omega)$ assuming no absorption by 1s atom.

Probability/time that a 1s atom makes transition to 2p by absorbing a thermal photon (assuming no prior absorption)

$$= \int \sigma(\omega) f(\omega)\, d\omega = \Gamma_0$$

Define Π as the probability / unit time for a thermal photon to cause $1s \to 2p$ transition, taking into account prior absorption effects.

$$\bar{p} = \Pi / \Pi_0$$

$$\Pi_0 = \int_{-\infty}^{\infty} \sigma(\omega) f(\omega) \, d\omega = \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2} \int_{-\infty}^{\infty} \phi(\omega) f(\omega) \, d\omega$$

$$= \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2} f(\omega_\alpha) \int_{-\infty}^{\infty} \phi(\omega) \, d\omega$$

$$\int_{-\infty}^{\infty} \phi(\omega) \, d\omega = 1$$

Define: $q_\omega(t)$: The probability that a thermal photon of energy ω at time t has survived capture from $t' = -\infty$ to t.

$$q_\omega(t) = \exp\left[-\int_{-\infty}^{t} dt'\, n_{1s}(t')\, \sigma\left(\omega \frac{\lambda(t)}{\lambda(t')}\right)\right]$$

change variable $\omega' = \omega \frac{\lambda(t)}{\lambda(t')}$

$$q_\omega(t) = \exp\left[-\frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2\, \omega_\alpha\, H(t)} \int_{\omega}^{\infty} d\omega'\, p(\omega')\right]$$

$x =$

Π : Probability that a thermal photon (keeping track of absorption effects) cause transition from a $1s \to 2p$.

$$= \int_{-\infty}^{\infty} d\omega \, f(\omega) \, q_\omega(t) \, \sigma(\omega)$$

$$= \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2} f(\omega_\alpha) \int_{-\infty}^{\infty} p(\omega) \, d\omega \, \exp\left[-x \int_\omega^\infty p(\omega') d\omega'\right]$$

$$\underset{Ex}{=} \Pi_0 \, F(x(t)) \Rightarrow \bar{P} = \frac{\Pi}{\Pi_0} = F(x(t))$$