Current density of matter ~ 6(protons + neutrons)/m^3

If the current energy density came only from ordinary matter, then it would be $6p/m^3$

In actual practice 5% of the energy density is ordinary matter

$$6 \times \frac{5}{100} /m^3$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$\frac{d}{dt}(\rho a^3) = -3a^2\dot{a}p$$

ρ, p, a

2 eqs.

Need a third eq.

→ Eq of State (reln. between p, ρ)

① Radiation: $p_r = \frac{\rho_r}{3}$

② Matter: $p_m = 0$

③ Cosmological constant $p_\Lambda = -\rho_\Lambda$

Unknowns: $a, \rho_r, p_r, \rho_m, p_m, \cancel{\rho_\Lambda}, p_\Lambda$

Eqs: Friedmann, Energy con., ~~3~~ 2 eqs. of state

If we have n different sources of $T_{\mu\nu}$, then

# of unknowns $= 1 + 2n$

# of eqs. $= 2 + n$

Cosmological const. $\to$ p_Λ, ρ_Λ are constants, not variables.

$$D^\mu T_{\mu\nu} = 0$$
$$\parallel$$
$$T^m_{\mu\nu} + T^r_{\mu\nu} + T^\Lambda_{\mu\nu}$$

$$D^\mu T^\Lambda_{\mu\nu} = 0 \qquad T^\Lambda_{\mu\nu} \propto g_{\mu\nu} \qquad D^\mu g_{\mu\nu} = 0$$

$$T^\Lambda_{\mu\nu} = -8\pi G \Lambda g_{\mu\nu}$$

$$D^\mu (T^m_{\mu\nu} + T^r_{\mu\nu}) = 0 \quad \text{holds separately.}$$

$$\frac{d}{dt}(\rho_\Lambda a^3) = \underbrace{-3 p_\Lambda a^2 \dot{a}} \Rightarrow$$
$$\parallel \qquad\qquad\qquad\qquad$$
$$\rho_\Lambda 3 a^2 \dot{a} = 3\rho_\Lambda a^2 \dot{a}$$

| 6 eqs., 7 unknowns

suppose we postulate that $D^\mu T^\Lambda_{\mu\nu} = 0$ separately.

$$\frac{d}{dt}(\rho_\Lambda a^3) = -3 p_\Lambda a^2 \dot{a}$$
$$= 3 \rho_\Lambda a^2 \dot{a}$$

$$\Rightarrow \frac{d}{dt}(\rho_\Lambda) = 0 \Rightarrow \rho_\Lambda = \text{constant.}$$

Once we assume that $\rho_\Lambda = \text{const.}$ then $D^\mu T^\Lambda_{\mu\nu} = 0$ follows from $D^\mu g_{\mu\nu} = 0$

We need a new equation when both matter and radiation are present.

Need additional input when both are present and are of the same order.

For now we make the assumption that $T^m_{\mu\nu}$ and $T^r_{\mu\nu}$ are separately conserved.

Physically $\Rightarrow$ The interaction between matter and radiation other that gravitational can be ignored. $D^\mu T^m_{\mu\nu} = 0$, $D^\mu T^r_{\mu\nu} = 0$.

$$\frac{d}{dt}\left(\rho_m a^3\right) = -3 p_m a^2 \dot{a} = 0$$

$$\frac{d}{dt}\left(\rho_r a^3\right) = -3 p_r a^2 \dot{a} = -\rho_r a^2 \dot{a}$$

today's density

$$\Rightarrow \quad \rho_m(t) = \frac{c}{a(t)^3} = \frac{c}{\{a_0 \lambda(t)\}^3} = \frac{\rho_{m0}}{\lambda(t)^3}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\rho_r a^4\right) = 0 \Rightarrow \rho_r = \frac{\tilde{c}}{a(t)^4} = \frac{\rho_{r0}}{\lambda(t)^4}$$

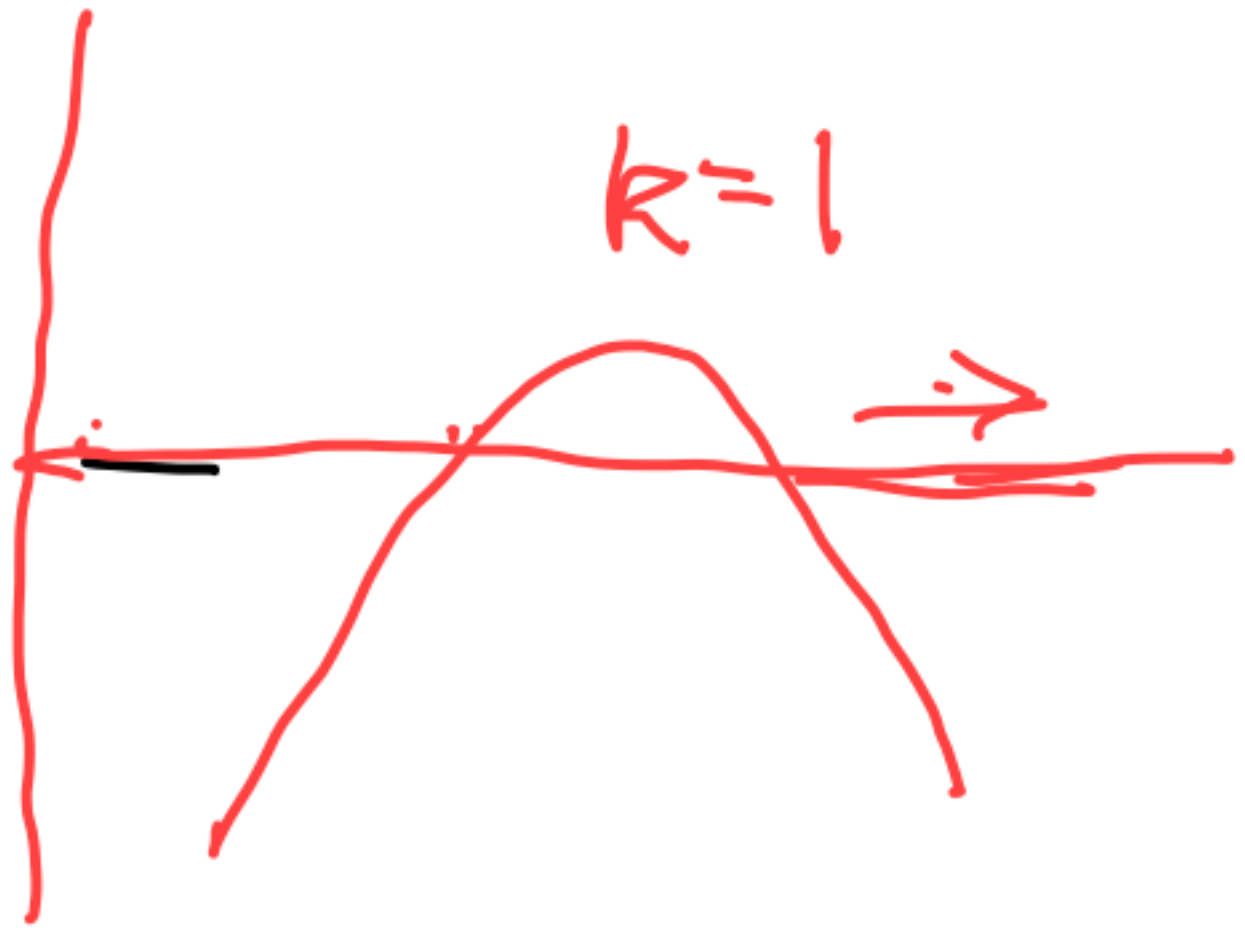
$$\frac{d}{dt}\left(\rho_\Lambda\right) = 0 \qquad \rho_\Lambda = \rho_{\Lambda 0}$$

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_r + \rho_m + \rho_\Lambda\right) \quad \Rightarrow \quad \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2 \lambda^2} = \frac{8\pi G}{3}\left(\frac{\rho_{r0}}{\lambda^4} + \frac{\rho_{m0}}{\lambda^3} + \rho_{\Lambda 0}\right)$$

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2\lambda^2} = \frac{8\pi G}{3}\left(\frac{\rho_{r0}}{\lambda^4} + \frac{\rho_{m0}}{\lambda^3} + \rho_{\Lambda 0}\right)$$

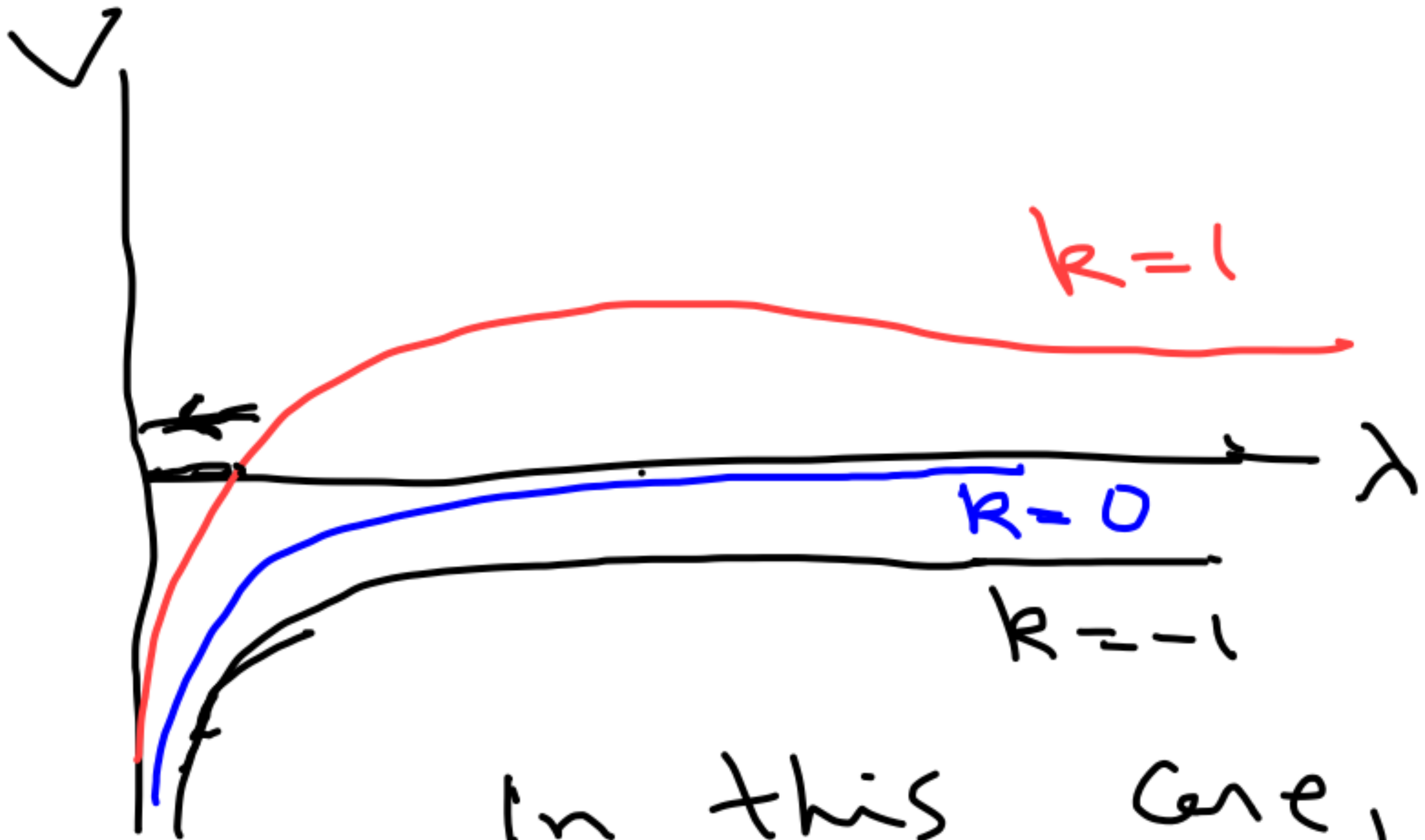
$$\Rightarrow \quad \frac{1}{2}\dot{\lambda}^2 + V(\lambda) = 0 \qquad\qquad \dot{\lambda} = \pm\sqrt{V(\lambda)}$$

$$V(\lambda) = \frac{k}{2a_0^2} - \frac{4\pi G}{3}\left(\frac{\rho_{r0}}{\lambda^2} + \frac{\rho_{m0}}{\lambda} + \underline{\underline{\rho_{\Lambda 0}\lambda^2}}\right)$$



>25 years ago, ρ_Λ was thought to be 0.

$$V(\lambda) = \frac{k}{2a_0^2} - \frac{4\pi G}{3}\left(\frac{\rho_{m0}}{\lambda} + \frac{\rho_{r0}}{\lambda^2}\right)$$



$k = 0, 1$, λ increases forever.

In this case, $V(\lambda)$ increases monotonically $\Rightarrow$ λ will decelerate.

Normally we take $t=0$ as the time of the big bang.

② Late time:

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2 \lambda^2} = \frac{8\pi G}{3}\left(\frac{\rho_{m0}}{\lambda^3} + \frac{\rho_{r0}}{\lambda^4} + \rho_{\Lambda 0}\right)$$

λ large $\Rightarrow$ ρ_Λ dominates

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = \frac{8\pi G}{3}\rho_{\Lambda 0} \Rightarrow \frac{\dot{\lambda}}{\lambda} = \sqrt{\frac{8\pi G}{3}\rho_{\Lambda 0}}$$

$$\lambda = K \exp\left(\sqrt{\frac{8\pi G}{3}\rho_{\Lambda 0}}\; t\right).$$

$\Rightarrow$ exponential expansion.

Matter dominated universe

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = \frac{8\pi G}{3}\,\frac{\rho_{m0}}{\lambda^3}$$

$$\lambda\left(\dot{\lambda}\right)^2 = \frac{8\pi G}{3}\rho_{m0}$$

$$\lambda^{1/2}\dot{\lambda} = \sqrt{\frac{8\pi G}{3}\rho_{m0}}$$

$$\Downarrow$$

$$\frac{2}{3}\frac{d}{dt}\left(\lambda^{3/2}\right)$$

$$\Rightarrow \quad \lambda^{3/2} = \frac{3}{2}\sqrt{\frac{8\pi G}{3}\rho_{m0}}\,(t+K)$$

$$\lambda = \left\{\frac{3}{2}\sqrt{\frac{8\pi G}{3}\rho_{m0}}\right\}^{2/3}(t+K)^{2/3}$$

Today's values of energy densities:

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2\lambda^2} = \frac{8\pi G}{3}\left(\frac{\rho_{r0}}{\lambda^4} + \frac{\rho_{m0}}{\lambda^3} + \rho_{\Lambda 0}\right)$$

At $t = t_0$, $\lambda = 1$, $\dot{\lambda} = H_0$

$$H_0^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}\left(\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0}\right)$$

$\Leftarrow \rho_c \sim 9\times 10^{-30}$ gm/c.c.

$$\frac{k}{a_0^2} = \frac{8\pi G}{3}\left(\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0} - \frac{3H_0^2}{8\pi G}\right)$$

$H_0 = 70\,\text{Km/s/}\underline{\text{Mpc}}$ $\quad 3.086\times 10^{13}\times 10^{6}\,\text{Km}$

$$\frac{k}{a_0^2} = \frac{8\pi G}{3}\left(\rho_{m0}+\rho_{r0}+\rho_{\Lambda 0}-\rho_c\right) \qquad \rho_c = \frac{3H_0^2}{8\pi G}$$

$$= \underbrace{\frac{8\pi G}{3}\rho_c}_{H_0^2}\left(\frac{\rho_{m0}}{\rho_c}+\frac{\rho_{r0}}{\rho_c}+\frac{\rho_{\Lambda 0}}{\rho_c}-1\right)$$

$$\frac{\rho_{m0}}{\rho_c} \Downarrow \Omega_m \qquad \frac{\rho_{r0}}{\rho_c} = \Omega_r \qquad \frac{\rho_{\Lambda 0}}{\rho_c} = \Omega_\Lambda$$

$$\rho_{r0} = K\, T^4, \quad T \rightarrow 2.73\,K \quad \Rightarrow \quad \Omega_r = 5\times 10^{-5} \ll 1$$

$$K = \frac{8\pi^5 k_B^4}{16\, h^3 c^3}$$

$$h = 2\pi\hbar$$

$$\Omega_m = \frac{\rho_{m0}}{\rho_c}$$

Examine observed matter:

galaxies + dust $\sim$ $\Omega_m \simeq .04$

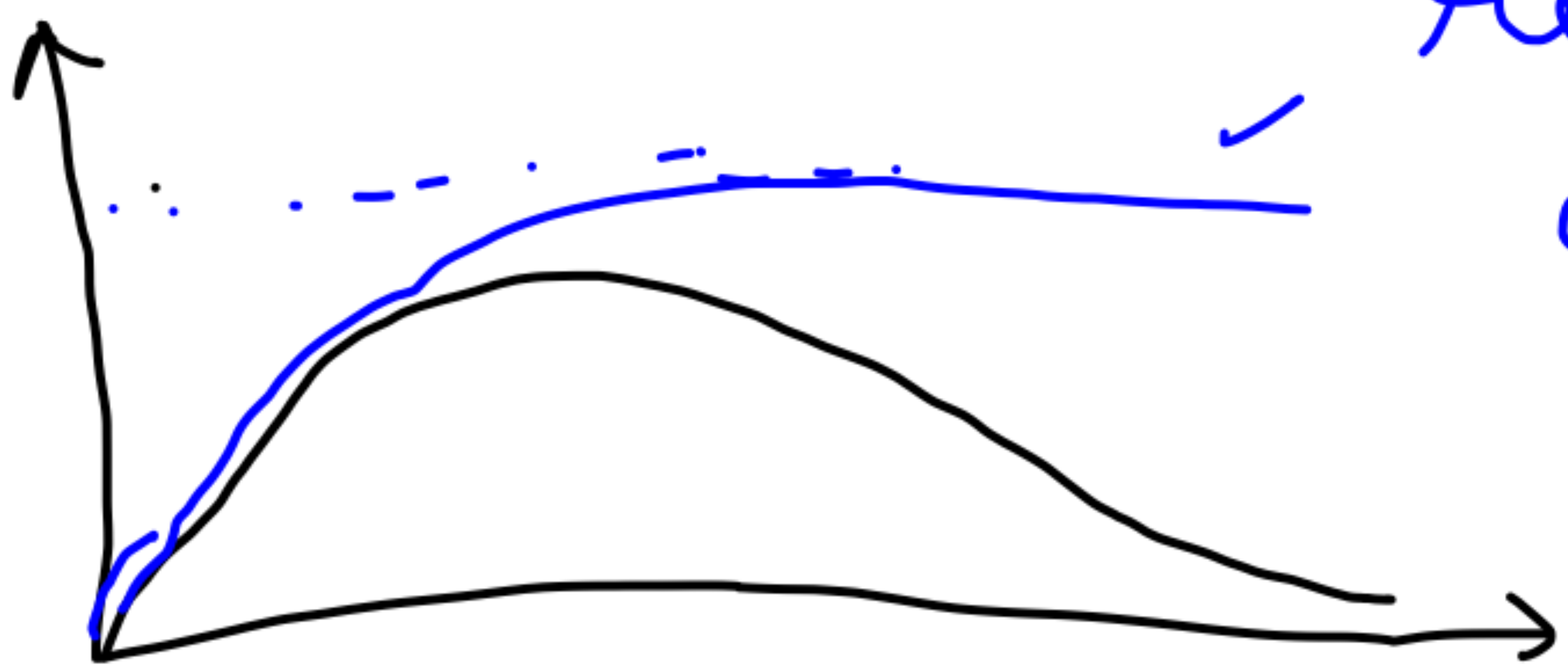
We know that this is not all the matter.

Rotation curve of the stars:

Look at stars near the edge of a galaxy.

mass of the star. $\frac{m v^2}{r} = \frac{GMm}{r^2}$ — mass of the star. | $v = \sqrt{\frac{GM}{r}}$

Mass in the galaxy inside radius r $\longrightarrow$ distance from center



region at the edge where there are very few visible stars.

$\Rightarrow$ suggests that the galaxy contain some invisible matter.

= dark matter.

$\Omega_m \sim .3$

Candidates: New kinds of elementary particles.

$\Omega_\Lambda \sim .7$

→ primordials black holes.

⋮

$\Omega_m + \Omega_\Lambda + \Omega_r = 1$

within error bar

$$-\frac{1}{2}\int d^4x \sqrt{-\det g}\,\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V(\phi)\right)$$

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\left(T^{r}_{\mu\nu} + T^{m}_{\mu\nu} + T^{\phi}_{\mu\nu}\right)$$

$$\Downarrow$$

$$\left(\dots + \underline{\underline{\tilde{V}(\phi)}}\, g_{\mu\nu}\right)$$

Quintessence