$T_{\mu\nu} = p\, g_{\mu\nu} + (p+\rho)\, U_\mu U_\nu$ $\quad U_\mu = (-1, 0, 0, 0)$

$T_{i0} = 0$

$T_{00} = \rho \quad T_{ij} = p\,\delta_{ij}$, is the form in locally inertial frame.

$g_{\mu\nu} = \eta_{\mu\nu}, \quad U^\mu = (1, 0, 0, 0)$

Scalar perturbation:

Variables from $g_{\mu\nu}$: Φ, Ψ, E, B

Variables from $\tilde{T}_{\mu\nu}$: $\tilde{\rho}, \tilde{p}, \tilde{q}, \tilde{\Sigma}$

8 variables, 2 gauge trs. parameters

$$\xi^0, \xi : \xi^i = \partial_i \xi + S_i, \quad \partial_i S_i = 0$$

$\Rightarrow$ 6 gauge invariant variables.

Einstein's eq: $K_{\mu\nu} = 0 \quad K_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi G T_{\mu\nu}$

K_{00}, $\quad K_{0i} = \partial_i K + L_i \, , \; \partial_i L_i = 0$

$$K_{ij} = \delta_{ij} M + \partial_i \partial_j P + (\partial_i U_j + \partial_j U_i) + V^T_{ij}$$

$$\partial_i U_i = 0 , \quad \partial_i V^T_{ij} = 0 , \quad V^T_{ii} = 0$$

$K_{00} = 0$, $\quad K = 0$, $\quad M = 0$, $\quad P = 0$

$\Rightarrow$ 4 equations.

6 variables. $\Rightarrow$ Need two more equations.

Put condition on $T_{\mu\nu} = p\, g_{\mu\nu} + (p+e) U_\mu U_\nu$

$T_\mu{}^\nu = p\, \delta_\mu{}^\nu + (p+e) U_\mu U^\nu$ $\qquad g^{\mu\nu} U_\mu U_\nu = -1$

This form is true for non-relativistic matter and radiation.

→ Also for scalar field theory.

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2} g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi + V(\phi)\right)$$

$$\stackrel{?}{=} p\,g_{\mu\nu} + (p+\rho)U_\mu U_\nu \qquad \left(g^{\mu\nu}U_\mu U_\nu = -1\right)$$

Choose: $\boxed{p = -\frac{1}{2} g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi - V(\phi)}$, $\sqrt{p+\rho}\,U_\mu = \partial_\mu\phi$

$$g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = (p+\rho)g^{\mu\nu}U_\mu U_\nu = -(p+\rho) \Rightarrow \rho = -p - g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi$$

$$\boxed{U_\mu = \partial_\mu\phi\Big/\sqrt{-g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi}} \qquad \boxed{\rho = -\frac{1}{2} g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi + V(\phi)}$$

$$T_\mu{}^\nu = p\,\delta_\mu{}^\nu + (p+e)\,U_\mu U^\nu \qquad \boxed{\Sigma_{ii}=0}$$

$$T_0{}^0 = -(\bar e+\tilde e), \quad T_i{}^0 = q_i, \quad T_i{}^j = (\bar p+\tilde p)\,\delta_{ij} + \Sigma_{ij}$$

$$T_i{}^0 = (p+e)\,U_i U^0 \Rightarrow \boxed{U_i \propto q_i}$$ (first order in perturbation)

$$g_{\mu\nu}U^\mu U^\nu = -1 \Rightarrow g_{00}(U^0)^2 = -1 + 2g_{0i}U^iU^0 + g_{ij}U^iU^j$$

$$-(1-2\Phi)(U^0)^2 = -1 \Rightarrow U^0 = 1+\Phi+\cdots, \quad U_0 = g_{00}U^0 = -1+\Phi+\cdots$$

Comparison

$$T_i{}^0 \Rightarrow (\bar p+\bar e)\,U_i U^0 = q_i$$

$$T_0{}^0:\ -(\bar e+\tilde e) = p + (p+e)(-1) = -e \Rightarrow \boxed{e = \bar e+\tilde e}$$

$$T_i{}^j \Rightarrow p = \bar p+\tilde p, \quad \Sigma_{ij}=0$$

$$(\bar p+\bar e)(+1)\,U^i = q^i$$

$$\boxed{U^i = +q^i/(\bar p+\bar e)}$$

$\rho = \bar{\rho} + \tilde{\rho}$, $p = \bar{p} + \tilde{p}$, $u_i = q_i/(\bar{p} + \bar{\rho})$, $\Sigma_{ij} = 0$

$\Sigma_{ij} = 0 \Rightarrow$ sets the tensor, vector and scalar component of Σ_{ij} to 0.

Evolution of tensor perturbation is source free.

$$\Sigma_{ij} = \Sigma^T_{ij} + (\partial_i \Sigma_j + \partial_j \Sigma_i) + (\partial_i \partial_j \tilde{\Sigma} - \frac{1}{3}\delta_{ij} \vec{\partial}^2 \tilde{\Sigma})$$

$\Rightarrow$ in particular that $\tilde{\Sigma} = 0$.

$\rightarrow$ gets rid of one variable.

We need to get rid of one more variable.

$\rightsquigarrow$ use equation of state

$$p = f(\rho)$$

Assume local thermal equilibrium.

$$\bar{p} + \tilde{p} = f(\bar{\rho} + \tilde{\rho}) \Rightarrow \bar{p} = f(\bar{\rho})$$

$$\Rightarrow \boxed{\tilde{p} = f'(\bar{\rho})\,\tilde{\rho}} \qquad \boxed{\tilde{\Sigma} = 0}$$

$\rightarrow$ reduces the number of gauge invariant variables to 4.

Gauge invariant variables

$$\Phi_B = \Phi - \frac{d}{dt}\left\{\lambda^2\left(\dot{E} - \lambda^{-1}B\right)\right\}$$

$$\Psi_B = \Psi + \lambda^2 H\left(\dot{E} - \lambda^{-1}B\right)$$

$$-\zeta = \Psi + \frac{H}{\partial_t\bar{\rho}}\tilde{\rho}\,, \qquad R = \Psi - \frac{H}{\bar{\rho}+\bar{p}}\tilde{q}$$

$$p_e = \tilde{p} - \frac{\partial_t\bar{p}}{\partial_t\bar{\rho}}\tilde{\rho}\,, \qquad \tilde{\Sigma}$$

$$\boxed{\begin{matrix}\tilde{\Sigma} = 0 \\ p_e = 0\end{matrix}}\,, \qquad \tilde{p} = f'(\bar{\rho})\tilde{\rho}\,, \qquad \bar{p} = f(\bar{\rho}) \Rightarrow \partial_t\bar{p} = f'(\bar{\rho})\partial_t\bar{\rho}$$

$$p_e = \tilde{p} - f'(\bar{\rho})\tilde{\rho} = 0$$

Ex. Einstein's eq $\Rightarrow$ (Momentum space, drop the ^)

$$3H(\dot{\Psi}_B + H\Phi_B) + \frac{\vec{k}^2}{\lambda^2}\Psi_B = 4\pi G \frac{\partial_t \bar{\rho}}{H}(S + \Psi_B)$$

$$\dot{\Psi}_B + H\Phi_B = 4\pi G \frac{\bar{\rho} + \bar{p}}{H}(R - \Psi_B)$$

$$\Psi_B - \Phi_B = 8\pi G \lambda^2 \tilde{\Sigma} = 0$$

$$\dot{S} = -H\frac{p_e}{\bar{\rho} + \bar{p}} - \Pi = -\Pi$$

$$\Pi = -H\frac{\vec{k}^2}{3\lambda^2 H^2}\left[S - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho} + \bar{p})}\frac{\vec{k}^2}{\lambda^2 H^2}\right)\right]$$

Initial conditions:

→ set at the horizon exit.

$$\Phi_B = 0\,, \quad \Psi_B = 0\,,$$

$$\mathcal{R}_I = -\zeta_I = \Psi + \frac{H}{\partial_t \bar{\phi}}\,\tilde{\phi}$$

$\phi = \bar{\phi} + \tilde{\phi}$ scalar field during inflation.

We'll show that $\mathcal{R} = \mathcal{R}_I$, $\zeta = \zeta_I$ at the horizon exit.

Initial conditions:

$\mathcal{R} = \mathcal{R}_I$, $S = S_I$, $\Psi_B = 0$, $\Phi_B = 0$ at the horizon exit. $(S_I = -\mathcal{R}_I)$

$\rightarrow$ We can find $\Phi_B, \Psi_B, \mathcal{R}, S$ at later time (t_{LS}) in terms of $\mathcal{R}_I$ at t_*.

$\langle \Phi_B(\vec{k},t)\, \Phi_B(\vec{k}',t)\rangle$ at $t = t_{LS}$ can be expressed in terms of $\langle \mathcal{R}_I(\vec{k},t)\, \mathcal{R}_I(\vec{k}',t)\rangle$ at t_*.

We need to show that $\mathcal{R} = \mathcal{R}_I$, $\zeta = \zeta_I$ at the horizon exit.

$$\mathcal{R} = \Psi - \frac{\mathcal{H}}{\bar{\rho} + \bar{p}} \tilde{q}, \quad \mathcal{R}_I = \Psi + \frac{\mathcal{H}}{\partial_t \bar{\phi}} \tilde{\phi}$$

$$T_i{}^0 = \partial_i \tilde{q} + Q_i \Rightarrow \text{defines } \tilde{q}. \quad \left| \begin{array}{l} \bar{\rho} = \frac{1}{2}\dot{\phi}^2 + V \\ \bar{p} = \frac{1}{2}\dot{\phi}^2 - V \end{array} \right\} \text{Ex.}$$

For Scalars:

$$T_\mu{}^\nu = \partial_\mu \phi \, \partial^\nu \phi - \delta_\mu{}^\nu \left(\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi + V(\phi) \right)$$

$$T_i{}^0 = \partial_i \phi \, \partial^0 \phi = \partial_i \tilde{\phi} \left(-\partial_t \bar{\phi} \right) \Rightarrow \tilde{q} = -\partial_t \bar{\phi} \, \tilde{\phi}$$

$$\boxed{\Rightarrow \mathcal{R} = \Psi - \frac{\mathcal{H}}{(\partial_t \bar{\phi})^2} \left(-\partial_t \bar{\phi} \, \tilde{\phi} \right) = \Psi + \frac{\mathcal{H}}{\partial_t \bar{\phi}} \tilde{\phi} = \mathcal{R}_I} \quad Q_i = 0$$

$$-\zeta = \Psi + \frac{H}{\partial_t \bar\rho}\tilde\rho, \qquad -\zeta_I = \Psi + \frac{H}{\partial_t \bar\phi}\tilde\phi$$

$$\bar\rho + \tilde\rho = -T_0{}^0 = -\partial_0\phi\,\partial^0\phi + \delta_0^0\left\{\frac{1}{2}g^{00}(\partial_0\phi)^2 + V(\phi)\right\}$$

$$= -\partial_0\phi\left(g^{00}\partial_0\phi\right) + \frac{1}{2}g^{00}(\partial_0\phi)^2 + V(\phi)$$

$$= -\frac{1}{2}g^{00}(\partial_0\phi)^2 + V(\phi)$$

$$\bar\rho = \frac{1}{2}\dot{\bar\phi}^2 + V(\bar\phi), \qquad \tilde\rho \simeq -\partial_0\bar\phi\,\underset{0}{\underbrace{\partial_0\tilde\phi}} + V'(\bar\phi)\tilde\phi$$

$$= V'(\bar\phi)\tilde\phi$$

$$\partial_t\bar\rho = \dot{\bar\phi}\left(\ddot{\bar\phi} + V'(\bar\phi)\right)$$

Eom $$= -3H(\partial_t\bar\phi)^2$$

$$\boxed{\ddot{\bar\phi} + 3H\dot{\bar\phi} + V'(\bar\phi) = 0 \;\Rightarrow\; V'(\bar\phi) = -3H\dot{\bar\phi}}$$

$$-\zeta = \Psi + \frac{H(-3H\partial_t\bar\phi)\tilde\phi}{\{-3H(\partial_t\bar\phi)^2\}}$$

$$= -\zeta_I$$