Scalar perturbation

$\Phi_B, \Psi_B, S, \mathcal{R}.$

→ All 2-point correlation functions of scalar fluctuations can be expressed in terms of $\langle \mathcal{R}_I(\vec{k}, t_*) | \mathcal{R}_I(\vec{k}', t_*) \rangle$

t_* : horizon exit time $(k^2 = \lambda^2 H^2)$

$X(\vec{k}, t)$: Some scalar observable (like $\frac{\delta T}{T}$)

$$X(\vec{k}, t) = f(\vec{k}, t)\, \mathcal{R}_I.$$

$f(k,t)$, $k = |\vec{k}|$ due to rotational inv.

simple for $t < t_{reentry}$

complicated for $t > t_{reentry}$

can be computed.

Use same symbol for original and Fourier trs. variable (no. ^)

– drop t from the argument but understood to be present.

$$\chi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \chi(\vec{k}) = f(k)\, \mathcal{R}_I(\vec{k}, t_*)$$

$\vec{x} \to (r, \theta, \phi)$

US: $r = 0$

CMB: $r = r_{LS}$.

$$\vec{k} = k\left(\sin\theta_{\vec{k}}\cos\phi_{\vec{k}},\ \sin\theta_{\vec{k}}\sin\phi_{\vec{k}},\ \cos\theta_{\vec{k}}\right)$$

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (-i)^l\, Y_{lm}(\theta,\phi)\, Y_{lm}(\theta_{\vec{k}}, \phi_{\vec{k}})^*\, j_l(kr)$$

$$X(r,\theta,\phi) = \sum_{l,m} X_{lm} Y_{lm}(\theta,\phi)$$

$$X_{lm} = \int \frac{d^3k}{(2\pi)^3} f(k) \mathcal{R}_I(\vec{k},t_*)$$

$$4\pi(-i)^l Y_{lm}(\theta_{\vec{k}},\phi_{\vec{k}})^* j_l(kr)$$

$$\langle X_{lm}(t) X_{l'm'}(t)^* \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3}$$

$$f(k) f(k')^* 16\pi^2 (-i)^{l-l'} Y_{lm}(\theta_{\vec{k}},\phi_{\vec{k}}) Y_{l'm'}(\theta'_{\vec{k}},\phi'_{\vec{k}})^*$$

$$j_l(kr) j_{l'}(k'r) \langle \mathcal{R}_I(\vec{k},t_*) \mathcal{R}_I(\vec{k}',t_*)^* \rangle$$

$$= (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}') \underset{P_R(k)}{\text{"}} \mathcal{R}_I(-\vec{k}',t_*)$$

$$\langle X_{\ell m}(t)\, X_{\ell' m'}(t)^* \rangle \qquad \boxed{r = r_{LS}}$$

$$= \frac{2}{\pi}\int k^2\, dk\, \sin\theta_{\vec{k}}\, d\theta_{\vec{k}}\, d\phi_{\vec{k}}\, |f(k)|^2\, \mathcal{P}_R(k)$$

$$(i)^{\ell-\ell'}\, j_\ell(kr)\, j_{\ell'}(kr)\, Y_{\ell m}(\theta_{\vec{k}}, \phi_{\vec{k}})\, Y^*_{\ell' m'}(\theta_{\vec{k}}, \phi_{\vec{k}})$$

$$= \frac{2}{\pi}\, \delta_{\ell\ell'}\, \delta_{mm'} \int k^2\, dk\, |f(k)|^2\, j_\ell(kr)^2\, \mathcal{P}_R(k)$$

$$= C_\ell^{XX}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$C_\ell^{XX} = \frac{2}{\pi}\int k^2\, dk\, |f(k)|^2\, j_\ell(kr)^2\, \mathcal{P}_R(k)$$

↳ experimentally measured.

$$\langle X_{\ell m}(r)\, X_{\ell' m'}(r)^{*} \rangle \Big|_{t=t_{ls}}$$

$$= \int \sin\theta\, d\theta\, d\phi\ \sin\theta'\, d\theta'\, d\phi'$$

$$Y_{\ell m}(\theta,\phi)^{*}\, Y_{\ell' m'}(\theta',\phi')\ \langle X(r,\theta,\phi)\, X(r,\theta',\phi') \rangle$$

$\langle\ \rangle$: Average over different orientation of the coordinate system.

C_ℓ^{XX} can be measured.

$$C_\ell^{xx} = \frac{2}{\pi}\int_0^\infty dk\, k^2\, |f(k)|^2\, j_\ell(kr)^2\, P_R(k)$$

$$P_R(k) \Downarrow \frac{2\pi^2}{k^3}\Delta_s^2(k) \rightarrow \text{slowly varying fn. of } k$$

For perturbations that are superhorizon at t_{LS}, should give slowly varying $f(k)$.

$j_\ell(kr)$ varies rapidly with k for large ℓ with sharp peak around $kr \sim \ell$, $k \sim \frac{\ell}{r}$.

$$C_\ell^{xx} \underset{\text{large } \ell}{\simeq} 4\pi \left|f\left(\frac{\ell}{r}\right)\right|^2 \Delta_s^2\left(\frac{\ell}{r}\right) \underbrace{\int_0^\infty \frac{dk}{k}\left(j_\ell(kr)\right)^2}_{=\frac{1}{2\ell(\ell+1)}}$$

$$\ell(\ell+1)\, C_\ell \simeq 2\pi \underbrace{\left| f\left(\frac{\ell}{r}\right)\right|^2 \Delta_s^2\left(\frac{\ell}{r}\right)}_{\text{slowly varying fn. of } \ell}$$

provide $\dfrac{\ell}{r} < \lambda \mathcal{H}\big|_{t_{ls}}$ (condition that the perturbation remains superhorizon at t_{ls}).

Q. What range of l corresponds to $l/r < \lambda H$?

r determined from: $\displaystyle\int_{t_{LS}}^{t_0} \frac{dt}{\lambda(t)} = r$

$\lambda(t)^2 dr^2 = dt^2$ along the null geodesic.

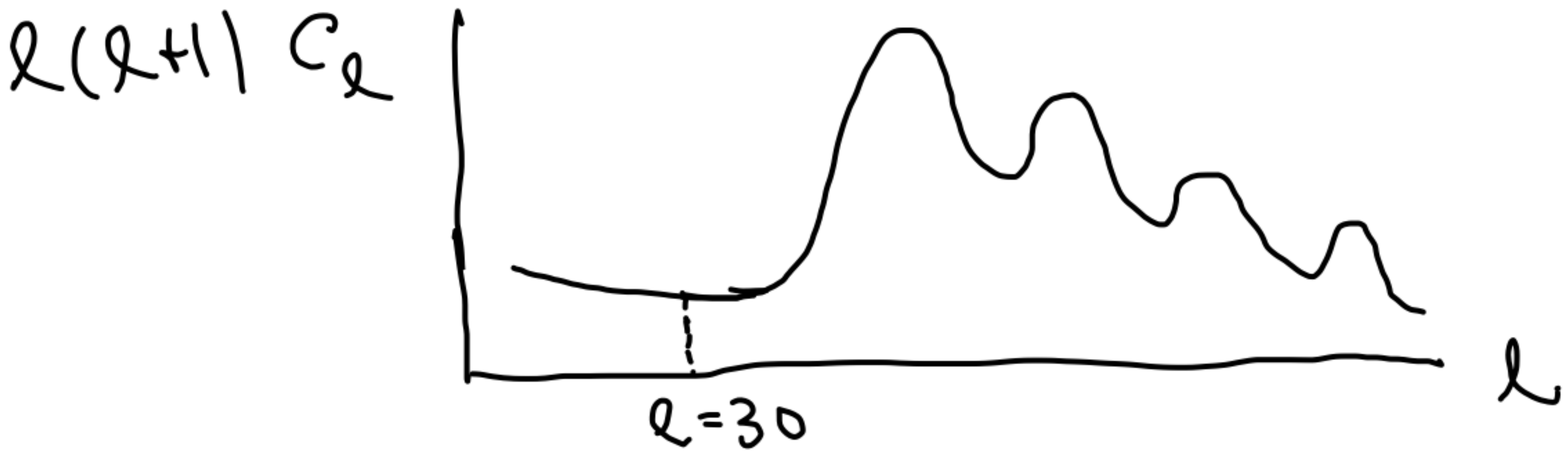
Recall that at $t = t_{LS}$, the apparent horizon $\sim \lambda^{-1} H^{-1}$ spans $2°$ angle in the sky ~~today~~.

$$\frac{\lambda^{-1} H^{-1}}{2\pi r} = \frac{2°}{360°} \Rightarrow \lambda H = \frac{180}{2\pi r}, \quad \lambda^{-1} H^{-1} = \frac{2\pi r}{180} \simeq \frac{r}{30}$$

$$\frac{l}{r} < \frac{30}{r} \Rightarrow l < 30.$$

$\ell(\ell+1)C_\ell$ should be almost flat for ℓ large but $\lesssim 30$.

$\ell=1$ (dipole) is not observable since it is used to determine the local velocity of earth through the CMB.



To explain the oscillations, we need to go back to the original equations, and study them without making the $k^2 << \lambda^{+2} H^2$ assumption.

$$① \quad 3\mathcal{H}(\dot{\Psi}_B + \mathcal{H}\Phi_B) + \frac{\vec{k}^2}{\lambda^2}\Psi_B = 4\pi G \frac{\partial_t \bar{\rho}}{\mathcal{H}}(S + \Psi_B)$$

$$② \quad \dot{\Psi}_B + \mathcal{H}\Phi_B = 4\pi G \frac{\bar{\rho} + \bar{p}}{\mathcal{H}}(\mathcal{R} - \Psi_B)$$

$$③ \quad \Psi_B - \Phi_B = 0 \qquad (\tilde{\Sigma} = 0)$$

$$④ \quad \dot{S} = -\Pi, \quad \Pi = -\frac{\vec{k}^2}{3\lambda^2 H^2}\left\{S - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho} + \bar{p})}\frac{\vec{k}^2}{\lambda^2 H^2}\right)\right\}$$

③ $\Rightarrow \Psi_B = \Phi_B$, ① $\Rightarrow S$ in terms of $\Phi_B, \dot{\Phi}_B$ ② $\Rightarrow \mathcal{R}$ in terms of Φ_B,

↳ substitute into ④ Φ_B

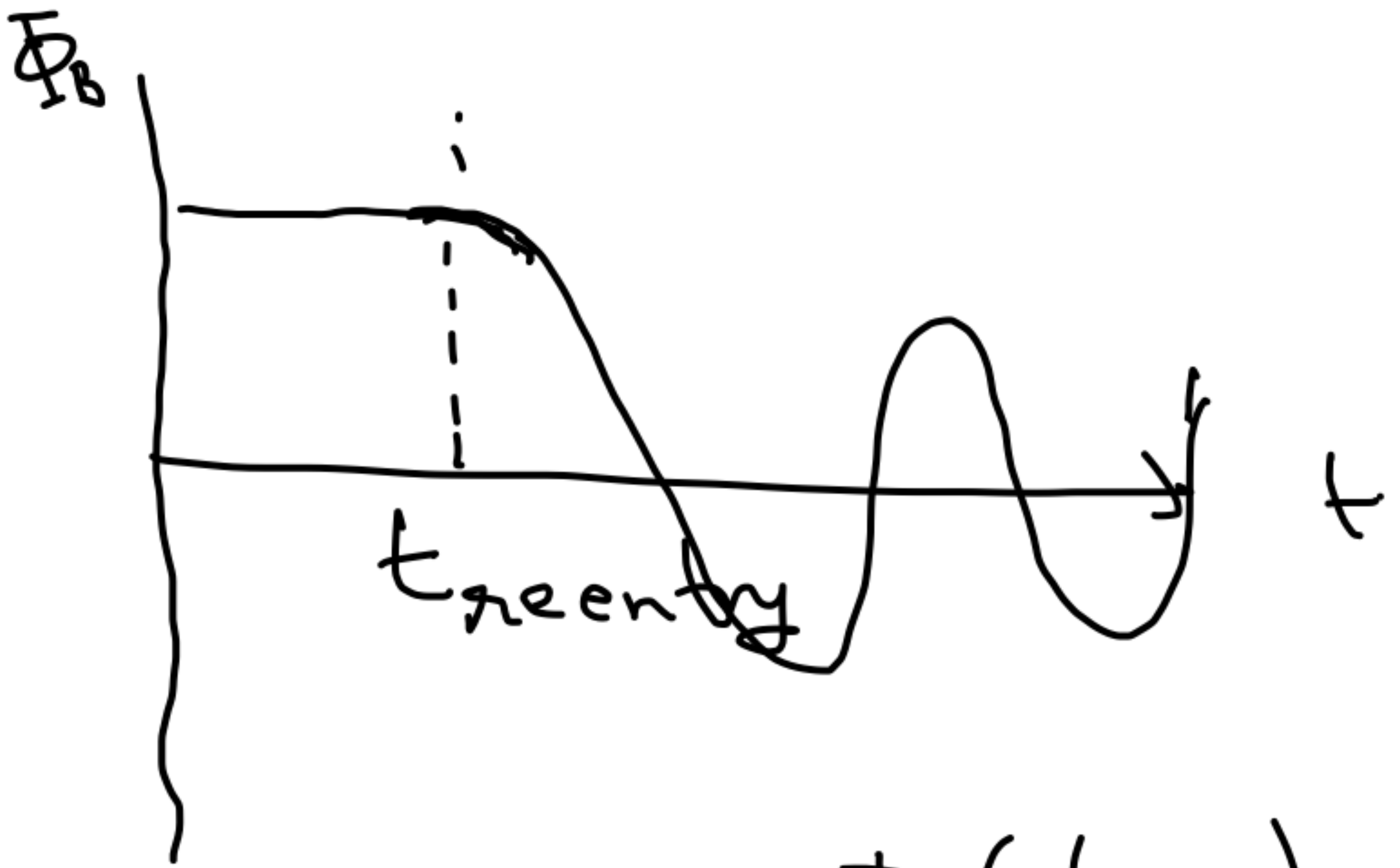
Upshot: An equation for Φ_B of the form

$$\ddot{\Phi}_B + A\dot{\Phi}_B + B\,\Phi_B = 0$$

$\rightarrow$ damped oscillator.

For superhorizon perturbation effectively the last term is small and $\dot{\Phi}_B \to 0$ at late time $\Rightarrow$ $\Phi_B \to$ constant.

For subhorizon perturbations, use b.c. that at $t = t_{\text{entry}}$, $\Phi_B = 0$, $\Phi_B = c\,\mathcal{R}_I$ constant.

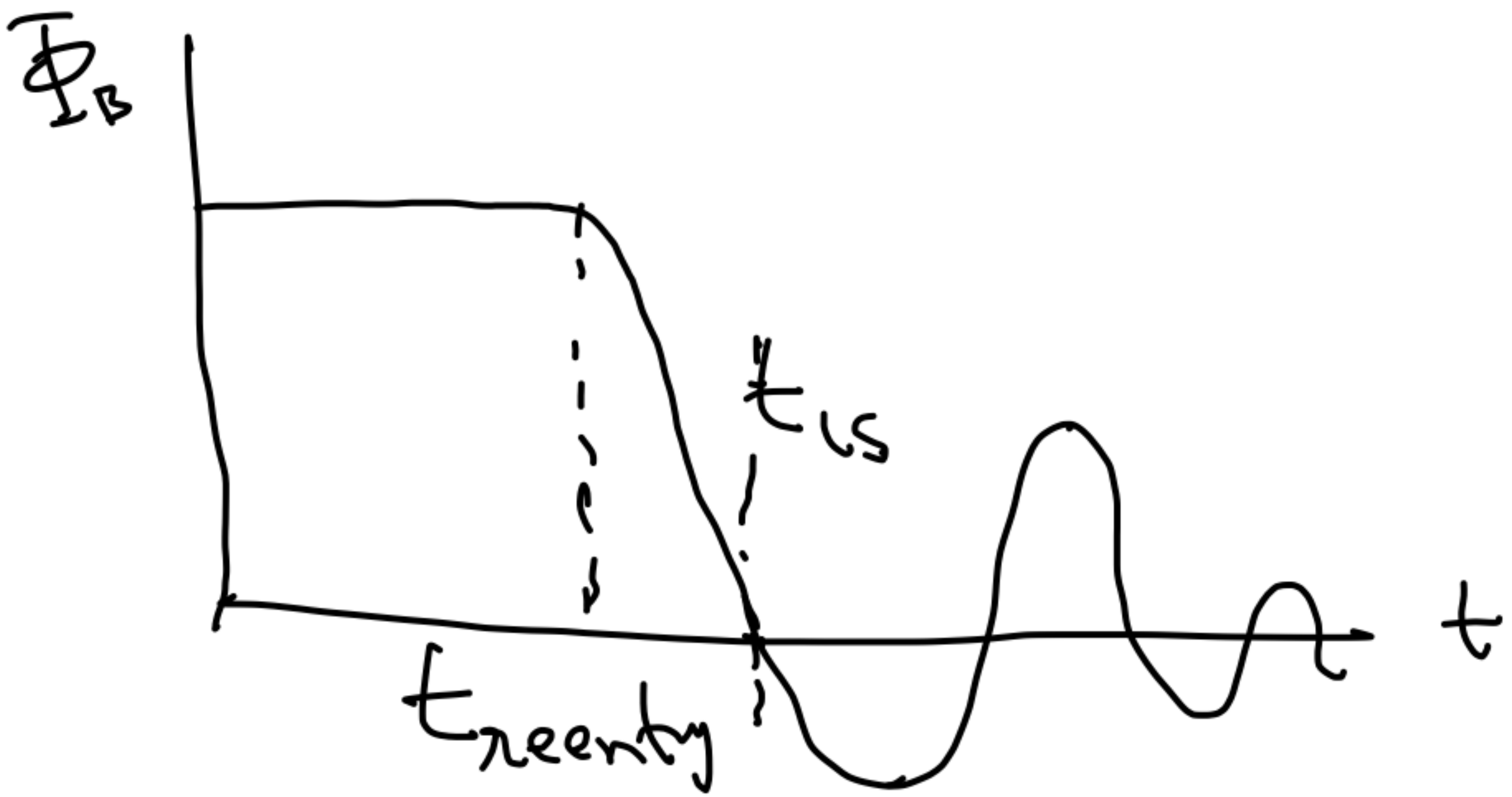


Effect of $\Phi(t_{LS})$ t_{LS} is fixed.

Suppose $t_{reentry} = t_{LS}$ $\Rightarrow$ Φ_B is at its maximum at t_{LS}.

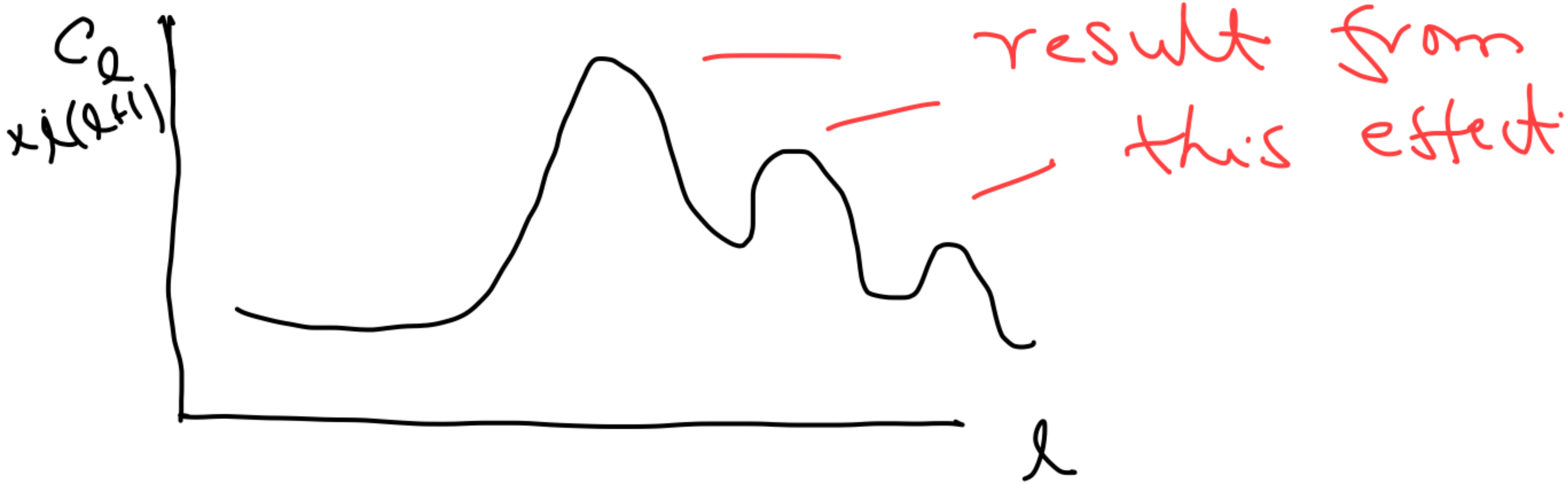
$t_{reentry}$ depends on k. $(k = \lambda H)$

Suppose k is such that



$\Rightarrow$ $\Phi_B(t_{is})$ oscillates as fn of k.

$\psi_B, S, \mathcal{R}$ also oscillate as fn. of k.



In the equation, $f(k, t_{ls})$ oscillates as fn. of k.

$f\left(\frac{\ell}{r}, t_{ls}\right) \rightarrow$ oscillates as fn. of ℓ.