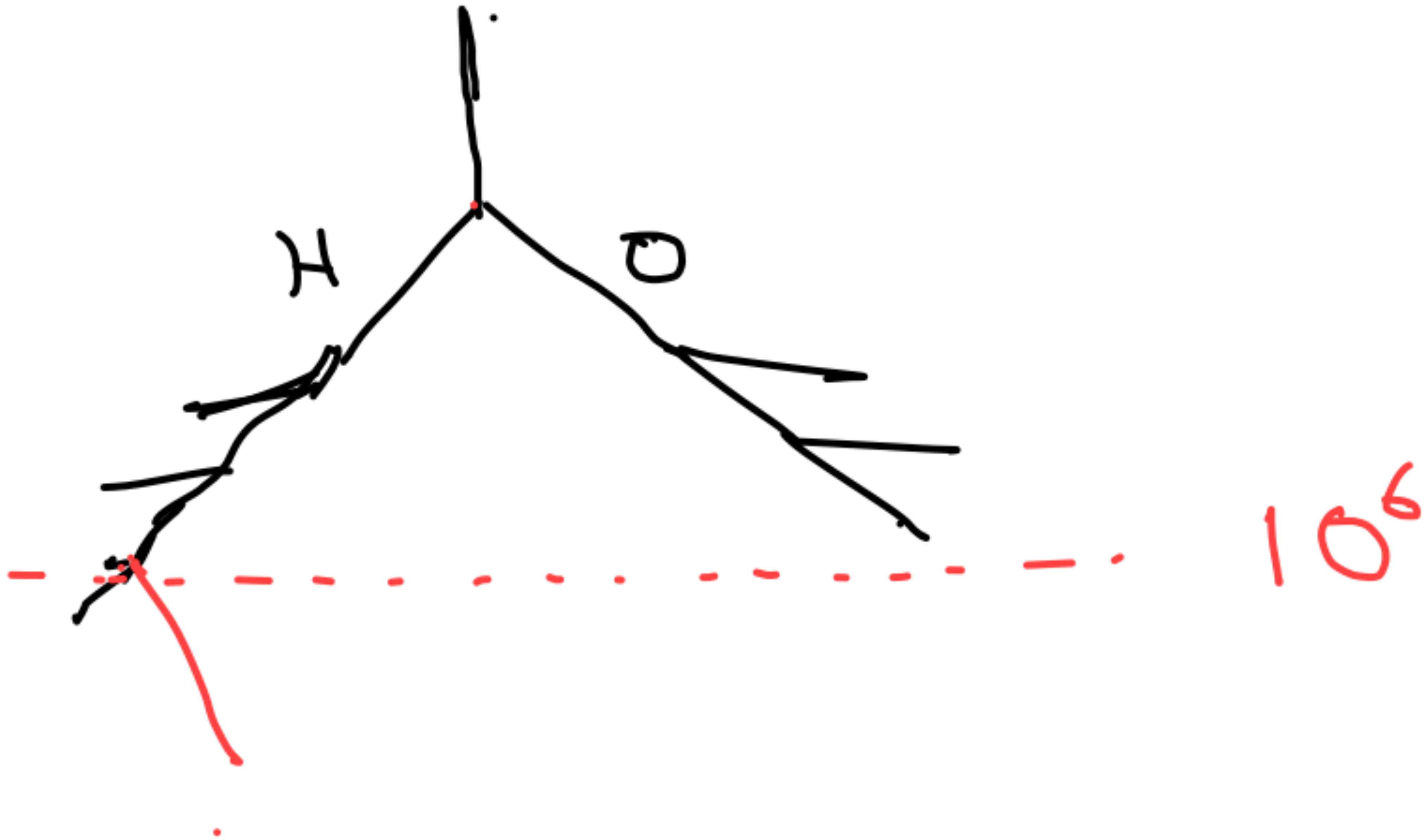
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Recall the general formula (for scalar part)

$$X(\vec{k},t) = f(\vec{k},t) \times \mathcal{R}_I(\vec{k})$$

$\mathcal{R}(\vec{k}, t_*)$

$$X = \left.\frac{\delta T}{T}\right|_{obs.}$$, what is $f(\vec{k})$

Divide the problem into two parts.

① choose a gauge and calculate $\left.\frac{\delta T}{T}\right|_t$ in that gauge.

② Relate $\left.\frac{\delta T}{T}\right|_{obs}$ to $\left.\frac{\delta T}{T}\right|_t$.

Which gauge should we choose?

Newtonian gauge: $E=0$, $B=0$.

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Psi)\,d\vec{x}^2\,\lambda(t)^2$$

for scalar perturbation.

Recall 4 gauge invariant quantities:

$\zeta, \mathcal{R}, \Psi_B, \Phi_B \xrightarrow[\text{eom + bc}]{}$ determined in term of $\mathcal{R}_I = \mathcal{R}(t_*)$

We'll analyze the problem of superhorizon perturbations (point out where extra effects come in for subhorizon pert.)

$$\Psi_B = \Psi + \lambda^2 \mathcal{H}(E - \lambda^{-1} B) = \Psi$$

$$\zeta = -\Psi - \frac{\mathcal{H}}{\partial_t \bar{\rho}}\tilde{\rho} \Rightarrow \Psi_B + \zeta = -\frac{\mathcal{H}}{\partial_t \bar{\rho}}\tilde{\rho}$$

$$\partial_t \bar{\rho} = -3\mathcal{H}(\bar{\rho} + \bar{p}) \Rightarrow \tilde{\rho} = 3(\bar{\rho} + \bar{p})(\Psi_B + \zeta)$$

$\bar{p} = w\bar{\rho} \rightarrow$ eq. of state, $\zeta = -\mathcal{R}$, $\Psi_B = \mathcal{R}\frac{3(1+w)}{5+3w}$

$$\frac{\tilde{\rho}}{\bar{\rho}} = 3(1+w)\left(-\mathcal{R} + \mathcal{R}\frac{3(1+w)}{5+3w}\right) = -\frac{6(1+w)}{5+3w}\mathcal{R}$$

In radiation dominated era $\rho = KT^4$

$$\frac{\tilde{\rho}}{\bar{\rho}} = 4\frac{\tilde{T}}{\bar{T}} = 4\frac{\delta T}{T} = 4\Theta \Rightarrow \Theta = -\frac{3(1+w)}{2(5+3w)}\mathcal{R}$$

$$\Theta = \frac{\tilde{T}}{T}\left(\frac{\delta T}{T}\right) = -\frac{3(1+\omega)}{2(5+3\omega)}\mathcal{R}$$ in radiation dominated era

$$\omega = \frac{1}{3} \Rightarrow \Theta = -\frac{4}{2\times 6}\mathcal{R} = -\frac{1}{3}\mathcal{R}.$$

In radiation dominated era, we have local thermal equilibrium and a thermal distribution.

$$-(1+2\Phi)\,dt^2 + (1-2\Psi)\,d\vec{x}^2$$

$\{q_\alpha\}$: phase space coordinates of a photon (6-dimensional)

δV: small volume element in the phase space in the canonical coordinate system.

$f(\{q_\alpha\}, t)\,\delta V$ = # of photons in phase space volume δV.

$$f(\{q_\alpha\}, t) = \left[\exp\left(\frac{e}{T(1+\Theta)}\right) - 1\right]^{-1}$$

x^i, p^i of a thermal photon.

$$f(\{q_\alpha\}, t) = \left[\exp\left(\frac{e}{T(1+\Theta)}\right) - 1\right]^{-1}$$

e : energy of the photon in locally inertial frame.

In the Newtonian gauge: $e = \sqrt{-g_{00}(p^0)^2}$

$$= \sqrt{(1+2\Phi)(p^0)^2} \simeq (1+\Phi)\, p^0$$

p^μ : 4-momentum in the Newtonian frame.

Ex. $\tilde{\rho}$ defined through $T_0{}^0 = -(\bar{\rho}+\tilde{\rho})$ agrees with Θ defined above via $\frac{\tilde{\rho}}{\bar{\rho}} = 4\Theta$

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We'll try to evolve $f(\{q_\alpha\}, t)$ forward in time.

Ignore collision with electrons & nuclei (okay for superhorizon perturbation but need to be accounted for for subhorizon perturbation).

$f\,\delta V$: no of photons in phase space volume δV

$$\frac{d}{dt}(f\,\delta V) = 0$$

$$\frac{d}{dt}(\delta V) \stackrel{?}{=} 0$$

$$\Rightarrow \frac{df}{dt} = 0 \Rightarrow \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial f}{\partial q_\alpha} \frac{dq_\alpha}{dt} = 0$$

Roughly, we are going to use the coordinate x^i and momenta p^i in the Newtonian gauge as phase space - coordinates.

① $ds^2 = 0$ ② $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$

λ: affine coordinate $\rightarrow$ choose s.t. $\frac{dx^\mu}{d\lambda} = p^\mu$

Null geodesic condition

$$\Rightarrow \quad g_{\mu\nu} p^\mu p^\nu = 0$$

$$\Rightarrow \quad -\underbrace{(1+2\Phi)(p^0)^2}_{e^2} + \lambda^2 (1-2\Psi)\, \vec{p}^{\,2} = 0 \quad \vec{p}^{\,2} \to p_i p_i$$

$$p_i \equiv \frac{e}{\lambda(1-\Psi)}\, \hat{p}_i \qquad \hat{p}_i \hat{p}_i = 1$$

Choose for $\{q_\alpha\}$: x^i for $i = 1, 2, 3$

e, $\hat{p}^i$

Geodesic eq. $(p^\mu = dx^\mu/d\lambda)$

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\, p^\nu p^\rho = 0$$

Ex. For $\mu = 0$

$$\frac{dp^0}{d\lambda} = (1-2\Phi)\, e^2 \left(-\dot{\Phi} - 2\frac{\partial \Phi}{\partial x^i}\frac{\hat{p}_i}{\lambda} + \dot{\Psi} - H \right)$$

$$\frac{dx^0}{d\lambda} = p^0 = e(1-\Phi)$$

$$\Rightarrow \frac{dp^0}{dt} = \frac{1}{p^0}\frac{dp^0}{d\lambda} \underset{\text{Ex}}{\Rightarrow} \frac{de}{dt} = -\frac{\partial \Phi}{\partial x^i} e \frac{\hat{p}_i}{\lambda} + e\dot{\Psi} - eH$$

$$\frac{dp^0}{dt} = \frac{1}{p^0}\frac{dp^0}{d\lambda}$$

$$\|$$

$$\frac{d}{dt}\left\{e(1-\Phi)\right\} = (1-\Phi)\frac{de}{dt} - e\frac{d\Phi}{dt}$$

$$= (1-\Phi)\frac{de}{dt} - e\dot{\Phi} - e\frac{\partial\Phi}{\partial x^i}\frac{dx^i}{dt}$$

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \Big/ \frac{dx^0}{d\lambda} = p^i/p^0 = \cdots$$

Now use:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial e}\frac{de}{dt} + \underbrace{\frac{\partial f}{\partial \hat{p}_i}\frac{d\hat{p}_i}{dt}}_{\text{2nd order in perturbation.}} = 0$$

$$\frac{\partial f}{\partial t} + \cancel{\frac{\partial f}{\partial x^i}\frac{\hat{p}_i}{1-\Phi}} + e\frac{\partial f}{\partial e} \times \left\{ - \cancel{\frac{\partial \Phi}{\partial x^i}\hat{p}_i} + \dot{\psi} - H \right\} = 0$$

In momentum space: $\frac{\partial}{\partial x^i} \to i k_i$

For superhorizon perturbations we can ignore these.

$$\frac{\partial f}{\partial t} + e\frac{\partial f}{\partial e}(\dot{\psi} - H) = 0$$

Q. Can this be reinterpreted as an eq. for θ?

$$\times f^{-1} \Rightarrow \frac{\partial}{\partial t}(f^{-1}) + e\frac{\partial}{\partial e}(f^{-1})(\dot{\psi} - H) = 0$$

$$\frac{\dot{T}}{T} = -H$$

$$f^{-1} = \exp\left(\frac{e}{T(1+\theta)}\right) - 1$$

$$\exp\left(\frac{e}{T(1+\theta)}\right)\left[\left\{-\cancel{\frac{e}{T^2(1+\theta)}\dot{T}} - \frac{e}{T}\frac{\dot{\theta}}{(1+\theta)^2}\right\} + \frac{e}{T(1+\theta)}(\dot{\psi} - \cancel{H})\right] = 0$$

$$\Rightarrow \frac{e}{T}(-\dot{\theta} + \dot{\psi}) = 0$$

$$\dot{\Theta} - \dot{\Psi} = 0 \Rightarrow \Theta = \Psi + C \rightarrow \text{constant.}$$

In radiation dominated era:

$$\Theta = -\frac{1}{3} R, \quad \Psi = \frac{3(1+w)}{5+3w} R = \frac{4}{6} R = \frac{2}{3} \cdot R$$

$$C = \Theta - \Psi = -\frac{1}{3} R - \frac{2}{3} R = -R$$

Matter dominated era: $\Psi = \frac{3}{5} R$

$$\Theta = \Psi + C = \frac{3}{5} R - R = -\frac{2}{5} R.$$

$4 \; \frac{\delta T}{T}$

Q. How to relate this to $\left.\frac{\delta T}{T}\right|_{obs.}$