For given λ:

Age $t = H_0^{-1} \int_0^{\lambda} du \left(\Omega_m u^{-1} + \Omega_r u^{-2} + \Omega_\Lambda u^2 \right)^{-1/2}$

Comoving Horizon $r_H = H_0^{-1} \int_0^{\lambda} du \, u^{-1} \left(\Omega_m u^{-1} + \Omega_r u^{-2} + \Omega_\Lambda u^2 \right)^{-1/2}$

Physical horizon: $d_H = r_H \lambda$

① Radiation dominated:

Ω_r dominates

$$t = H_0^{-1} \int_0^{\lambda} du \, \Omega_r^{-1/2} u = H_0^{-1} \Omega_r^{-1/2} \frac{\lambda^2}{2}$$

$$r_H = H_0^{-1} \int_0^{\lambda} du \, u^{-1} \Omega_r^{-1/2} u = H_0^{-1} \Omega_r^{-1/2} \lambda$$

$$d_H = \lambda r_H = H_0^{-1} \Omega_r^{1/2} \lambda^2 = 2t$$

② Matter dominated regime:

$$t = H_0^{-1}\left[\int_0^{\lambda_{eq}} \Omega_r^{-1/2}\, du\, u + \int_{\lambda_{eq}}^{\lambda} du\, \Omega_m^{-1/2} u^{1/2}\right]$$

$$= H_0^{-1}\left[\Omega_r^{-1/2}\,\frac{1}{2}\,\lambda_{eq}^{2} + \Omega_m^{-1/2}\,\frac{2}{3}\left(\lambda^{3/2} - \lambda_{eq}^{3/2}\right)\right]$$

$$\lambda_{eq} = \frac{\Omega_r}{\Omega_m} \Rightarrow \Omega_r = \Omega_m \lambda_{eq} \rightarrow \Omega_m^{-1/2}\,\frac{1}{2}\,\lambda_{eq}^{3/2}$$

For $\lambda >> \lambda_{eq}$, $t \simeq H_0^{-1}\,\frac{2}{3}\,\Omega_m^{-1/2}\,\lambda^{3/2}$

$$r_H = H_0^{-1}\left[\int_0^{\lambda_{eq}} \Omega_r^{-1/2}\, du + \int_{\lambda_{eq}}^{\lambda} du\, \Omega_m^{-1/2} u^{-1/2}\right] = H_0^{-1}\left[\Omega_r^{-1/2}\lambda_{eq} + 2\,\Omega_m^{-1/2}\left(\lambda^{1/2} - \lambda_{eq}^{1/2}\right)\right]$$

$$\simeq 2 H_0^{-1}\,\Omega_m^{-1/2}\,\lambda^{1/2}$$

$$d_H = \lambda r_H = 2 H_0^{-1}\,\Omega_m^{-1/2}\,\lambda^{3/2} \quad \Rightarrow d_H = 3t$$

③ Cosmological constant dominated era:

$$t = t_0 + H_0^{-1} \int_1^{\lambda} du\, \Omega_\Lambda^{-1/2} u^{-1} = t_0 + H_0^{-1} \Omega_\Lambda^{-1/2} \ln \lambda$$

$$r_H = r_H\big|_{t_0} + H_0^{-1} \int_1^{\lambda} du\, \Omega_\Lambda^{-1/2} u^{-2} = r_H\big|_{t_0} + H_0^{-1} \Omega_\Lambda^{-1/2} \left(1 - \frac{1}{\lambda}\right)$$

$$\underset{\text{For large } \lambda}{\simeq} \underbrace{r_H\big|_{t_0} + H_0^{-1} \Omega_\Lambda^{-1/2}}_{\text{calculate numerically.}} \sim H_0^{-1} \times 4.4$$

$$d_H = r_H \lambda \simeq H_0^{-1} \times 4.4 \times \lambda$$

History of temperature.

Today CMB temperature $T_0 \simeq 2.73\,K$

What was it earlier?

Crude analysis: At temp. T, black body radiation energy density KT^4.

At λ, the radiation energy density: $\frac{\Omega_r}{\lambda^4}$.

$$T = \frac{\text{constant}}{\lambda} = \frac{T_0}{\lambda}$$

Planck distribution formula

$$P(\nu)\, d\nu = \frac{8\pi h \nu^3\, d\nu}{e^{h\nu/k_B T} - 1}, \quad h = 2\pi\hbar$$

$\Downarrow$

Energy density of radiation in the frequency range $(\nu, \nu + d\nu)$.

Consider today's photon in the CMB.

ν_0 : frequency of today's photon

$$P(\nu_0)\, d\nu_0 = \frac{8\pi h \nu_0^3\, d\nu_0}{e^{h\nu_0/k_B T_0} - 1}$$

$T_0 \simeq 2.73\,K$

A photon of frequency ν_0 today had frequency $\frac{\nu_0}{\lambda} \equiv \nu$ earlier

Given that today we have black body radiation of temp. T

$\Rightarrow$ when the scale factor was λ, the energy density of photons in the range $(\nu, \nu+d\nu)$ was:

$$\frac{1}{\lambda^4} P(\nu_0)\, d\nu_0 = \frac{1}{\lambda^4} \frac{8\pi h \nu_0^3\, d\nu_0}{e^{h\nu_0/k_B T_0} - 1} \qquad \nu_0 = \lambda \nu$$

$$= \frac{1}{\cancel{\lambda^4}} \frac{8\pi h \cancel{\lambda^4} \nu^3\, d\nu}{e^{h\nu\lambda/k_B T_0} - 1}$$

$\Rightarrow$ black body radiation formula at $T = T_0/\lambda$

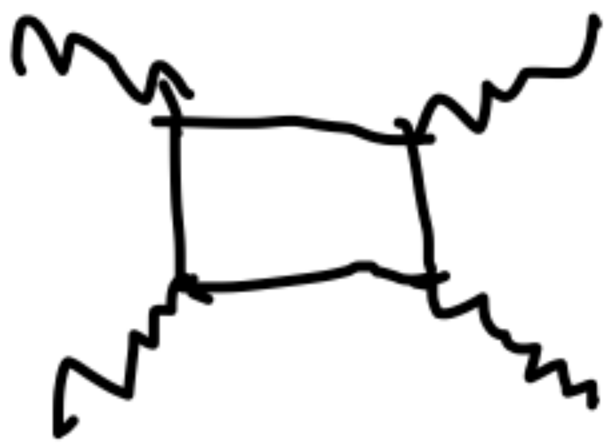
$\frac{1}{e^{h\nu/kT}-1}$ × # of states between ν, $\nu + d\nu$

$4\pi\nu^2 d\nu$ × volume × 2 ← polarization

$$\text{\# of photons/volume} = 4\pi\nu^2 d\nu \times \frac{1}{e^{h\nu/kT}-1} \times 2$$

Each photon has energy $h\nu$

⇒ Energy density in photons in the frequency range $(\nu, \nu + d\nu) = \dfrac{4\pi\nu^2 d\nu \times h\nu \times 2}{e^{h\nu/kT} - 1}$



$T = \frac{T_0}{\lambda}$ $\lambda_{eq} \simeq 10^{-4}$

$T = T_0 \times 10^4 = 2.73 \times 10^4$ K

$k_B T = 8.64 \times 10^{-5}$ eV/K $\times 2.73 \times 10^4$ K $\simeq 1$ eV

At $\lambda = 10^{-5}$, $k_B T \sim 10$ eV

Hydrogen ionizes at 13.6 eV.

$\Rightarrow$ For $\lambda \ll 10^{-5}$, hydrogen would have been ionized, e^- & p instead of H.

For smaller λ, He would have been ionized.

$\lambda = 10^{-4} \Rightarrow k_B T \sim eV$

$\lambda = 10^{-4} \times 10^{-6} \Rightarrow k_B T \sim MeV$

$$\Downarrow$$

$m_e \sim .51 MeV.$ $\quad \gamma + \gamma \to e^+ e^-$ would be possible.

Gas of e^+, e^-, γ + few He, H nuclei.

At very high temp $k_B T \sim 100$ GeV, all the standard model particles would have been present.

The universe started at small $\lambda \simeq 0$

& at high $k_B T > 100$ GeV in thermal

equilibrium.

Upto 100 GeV, microscopic physics is

well understood

$\rightarrow$ we can follow the evolution of

the universe & the various events

that happened.

① Neutrino radiation ② Nucleosynthesis

Physics beyond 100 GeV is not known.

$\Rightarrow$ We do not quite know how the universe was for $k_B T > 100$ GeV.

$\rightarrow$ Most of it is not relevant since the information gets washed out during thermal equilibrium.

However few details survive

$\rightarrow$ could give us clue about the earlier history.

One of the events is the last scattering of the CMB photons.

At $\lambda \sim 10^{-5}$ $k_B T \sim 10\,eV$.

$\lambda > 10^{-5}$, γ energy fell below H. ionization energy. $e^- \; p \rightarrow H$.

Detailed statistical analysis shows that actual H formation occurs around $\lambda \sim 10^{-3}$ (we'll see later)

$\hookrightarrow$ well into the matter dominated era $(\lambda > 10^{-4})$

$\lambda < 10^{-3}$: e^-, p existed as free particles as a result the γ's were scattered often.

$\lambda > 10^{-3}$: H formed, the universe became transparent to the γ's (γ's scattered infrequently).

Today, most of the CMB radiation comes from the $\lambda \sim 10^{-3}$ period.

Last scattering surface:

Suppose we are at $r = 0$, $\theta = \phi = 0$.
CMB photons we see ~~today~~ come from
(r_L, θ, ϕ) for different (θ, ϕ).

$$r_L = \left(r_H \Big|_{\lambda = 1} - r_H \Big|_{\lambda = \lambda_L} \right)$$

Surface at $r = r_L$ is called the last
scattering surface.

Observation.

Last scattering surface was highly uniform.

- Photons that arrive from different directions have almost the same temp. 2.73 K.
- small fluctuation one part in 10^5

  predicted before they were observed.

CMB gives an image of the universe

at $\lambda = \lambda_L \simeq 10^{-3}$

Small fluctuations in T

$\Leftrightarrow$ small fluctuation in the gravitational field.

Suppose there were no fluctuations

$\rightarrow$ completely homogeneous universe.

$\rightarrow$ evolve into a completely homogeneous universe (FRW metric)

$\Rightarrow$ no structure like galaxies & stars

One can reverse this argument.

Can show that in order to explain the structure we see in the universe, there must have been fluctuations of order at least one in 10^5 at $\lambda = 10^{-3}$.

$\rightarrow$ Cobe satellite