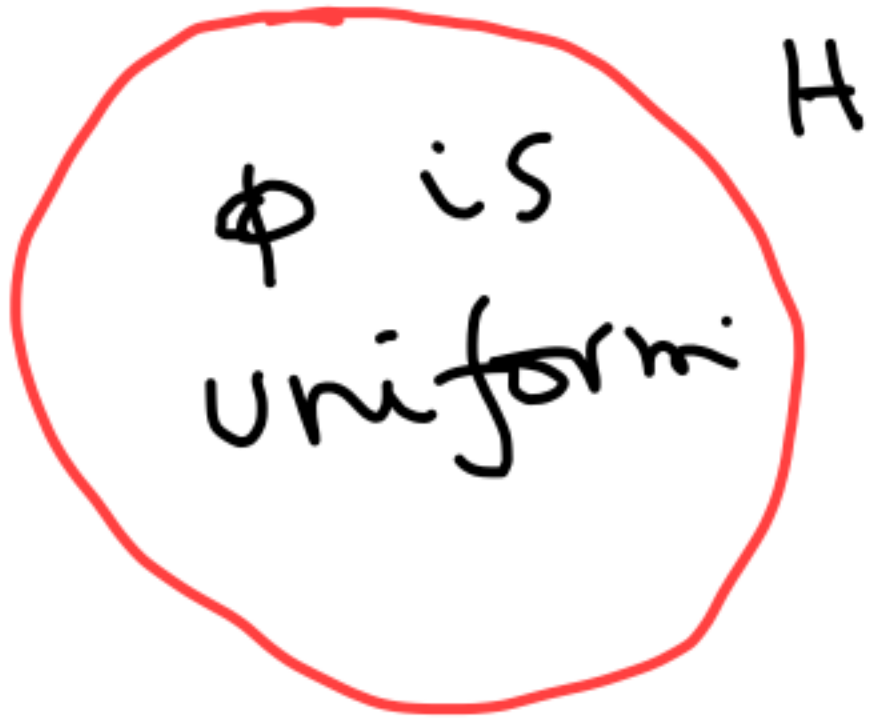
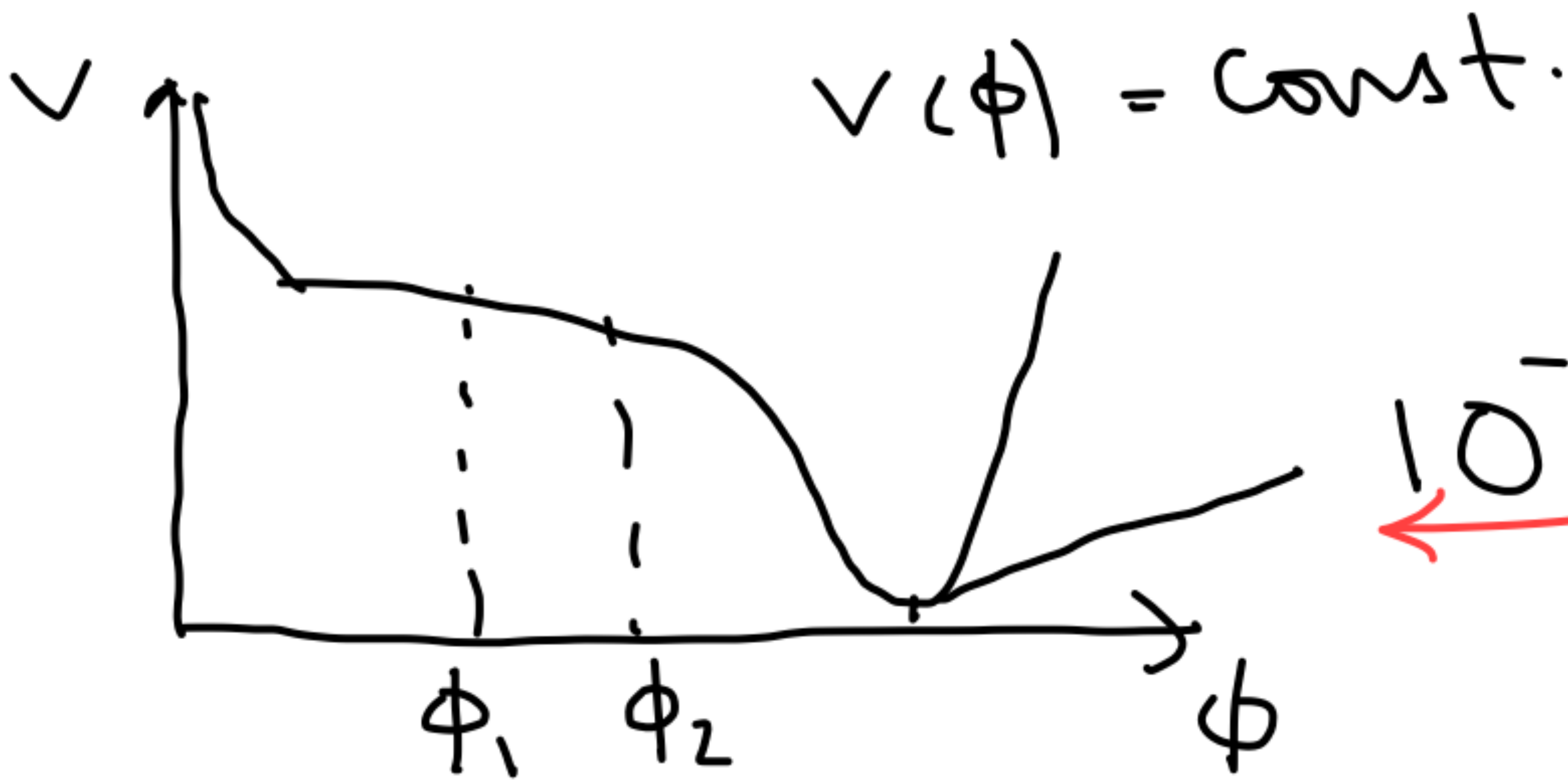
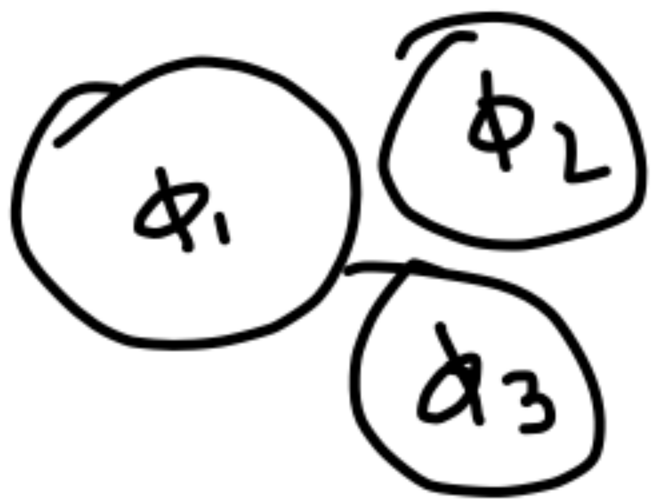



$10^{-120} (M_{Pl})^4$

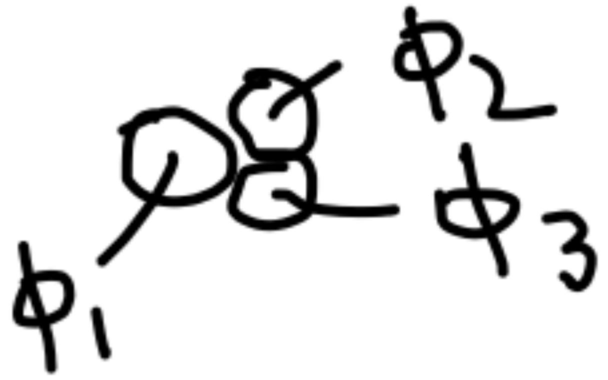



$\frac{1}{\lambda^4}$

$T = 10^{16}\,\text{GeV}$

t

$10^{19}$




$\mathcal{H} = \frac{1}{2}(\vec{\nabla}\phi)^2$

$m^2\phi^2$

$V_0 \sim (10^{16}\,\text{GeV})^4$ → came from GUT.

↳ tensor perturbation

$-\int d^4x\, V_{eff}(\phi_1, \phi_2, \cdots)$

Review of grand canonical ensemble:

$$\mathcal{Q} = \sum_\alpha e^{-\beta E_\alpha + \mu\beta N_\alpha} = \int dE' \sum_{N'} e^{-\beta E' + \mu\beta N' + S(E',N',V)}$$

In large V limit the summand or the integrand is maximized at $N' = N$, $E' = E$:

$$\frac{\partial}{\partial E}\left(-\beta E + \mu\beta N + S(E,N,V)\right) = 0$$

$$\frac{\partial}{\partial N}\left(-\beta E + \mu\beta N + S\right) = 0$$

$$\Rightarrow \quad -\beta + \frac{\partial S}{\partial E} = 0, \quad \mu\beta + \frac{\partial S}{\partial N} = 0,$$

$S = V s$, $E = V e$, $N = V n$

$$\Rightarrow -\beta + \frac{\partial s}{\partial e} = 0$$

$$\mu\beta + \frac{\partial s}{\partial n} = 0$$

$$\mathcal{Q} = \sum_\alpha e^{-\beta E_\alpha + \mu\beta N_\alpha} = \int dE \sum_N e^{-\beta E + \mu\beta N + S}$$

$$\Rightarrow \frac{\partial}{\partial \beta}(\ln \mathcal{Q}) = \langle (-E + \mu N) \rangle = -E + \mu N$$

$$\frac{\partial}{\partial \mu}(\ln \mathcal{Q}) = \beta \; \langle N \rangle = \beta N$$

$$\mathcal{Q} \simeq \Delta E \; \Delta N \; e^{-\beta E + \mu\beta N + S}$$

$$\frac{1}{V} \ln \mathcal{Q} = \frac{1}{V}(\ln \Delta E + \ln \Delta N - \beta E + \mu\beta N + S)$$

$$\xrightarrow[\text{Large } V]{} -\beta e + \mu\beta n + s = p/T$$

$$\Rightarrow s = \frac{p + e - \mu n}{T} \qquad \beta = \frac{1}{T}$$

# Summary

$p = \frac{1}{V} \ln Q \rightarrow p$ as fn. of $\mu, T$

$$e - \mu n = -\frac{1}{V} \frac{\partial \ln Q}{\partial \beta}, \quad \beta n = \frac{1}{V} \frac{\partial \ln Q}{\partial \mu}$$

treat $\beta, \mu$ as independent variables.

$$s = \frac{p + e - \mu n}{T}$$

$$\beta = \frac{\partial s}{\partial e}, \quad \beta \mu = -\frac{\partial s}{\partial n}$$

$e, n$ as independent variables.

Result for ideal gas:

Bose

$$\ln Q = -\frac{V}{8\pi^3} g \int_0^\infty 4\pi k^2 dk \ln\left(1 - e^{-\beta(\epsilon(k) - \mu)}\right)$$

$\epsilon(k) \to \sqrt{k^2 + m^2}$

$g$ : # of states for given $\vec{k}$

1 for scalar, 3 for massive vector

2 for massless vector.

Fermi

$$\ln Q = \frac{V}{8\pi^3} g \int_0^\infty 4\pi k^2 dk \ln\left(1 + e^{\beta(\epsilon(k) - \mu)}\right)$$

## Mixture of many particles:

Assume that the particles are almost free, but have small interactions via which they can exchange energy.

$\Rightarrow$ thermal equilibrium.

$\rightarrow$ a common temperature $T$.

$$Q = \prod_i Q_i \Rightarrow \ln Q = \sum_i \ln Q_i$$

$T_i = T$ $\quad \beta n_i = \frac{1}{V} \frac{\partial \ln Q}{\partial \mu_i} \Rightarrow$ solve for $\mu_i$ in terms of $n_i$

We implicitly assumed that $N_i$ is conserved.

No of particles of type $i$.

In general this may not be the case.

e.g. suppose we have 3 types of particles A, B, C, & we have interactions:

$$A + B \rightleftarrows C + C \implies N_A, N_B, N_C$$

not conserved.

$$\left.\begin{aligned} \tilde{N}_1 &= N_A - N_B \\ \tilde{N}_2 &= N_A + N_B + N_C \end{aligned}\right\} \text{conserved.}$$

Define:

$$Q = \sum_{\alpha} e^{-\beta E_\alpha + \tilde{\mu}_1 \beta \tilde{N}_1 + \tilde{\mu}_2 \beta \tilde{N}_2}$$

$$= \sum_{\alpha} e^{-\beta E_\alpha + \tilde{\mu}_1 \beta (N_A - N_B) + \tilde{\mu}_2 \beta (N_A + N_B + N_C)}$$

$$= \sum_{\alpha} e^{-\beta E_\alpha + \mu_A N_A + \mu_B N_B + \mu_C N_C}$$

$$\mu_A = \tilde{\mu}_1 + \tilde{\mu}_2, \quad \mu_B = \tilde{\mu}_2 - \tilde{\mu}_1, \quad \mu_C = \tilde{\mu}_2$$

$$\ln Q = \ln Q_A + \ln Q_B + \ln Q_C$$

$$\tilde{N}_1 \beta = \frac{\partial}{\partial \tilde{\mu}_1} \ln Q$$

$$\tilde{N}_2 \beta = \frac{\partial}{\partial \tilde{\mu}_2} \ln Q.$$

# General Case

Suppose that we have $K$ types of particles with $L \le K$ conserved charges.

$$\tilde{N}_{(\ell)} = \sum_{i=1}^{K} c^{i}_{(\ell)} N_i$$

$c^{i}_{(\ell)}$ → given numbers.

$N_i$ → # of $i$-type particle

$\tilde{N}_{(\ell)}$: $\ell$-th conserved "charge"

$$Q = \sum_{\alpha} \exp\left[-\beta E_\alpha + \beta \sum_{\ell=1}^{L} \tilde{\mu}_{(\ell)} \tilde{N}_{(\ell)}\right]$$

$$= \sum_{\alpha} \exp\left[-\beta E_\alpha + \beta \sum_{i=1}^{K} \mu_i N_i\right], \quad \mu_i = \sum_{\ell=1}^{L} c^{i}_{(\ell)} \tilde{\mu}_{(\ell)}$$

$$\rightarrow \ln Q = \sum_{i=1}^{K} \ln Q_i, \quad \beta \tilde{N}_{(\ell)} = \frac{\partial}{\partial \tilde{\mu}_{(\ell)}} \ln Q$$

# Apply this to cosmology

Validity of thermodynamics/ stat. mech.
in cosmology.

(1) Need large no. of particles within the Horizon.
→ true in general.

② Thermal equilibrium: Needs large number of energy changing collision per particle per Hubble time
$\rightarrow (1/\dot{\lambda})$.

→ may or may not hold.

If it fails, we say that the particle has fallen out of equilibrium.

Similarly, chemical equilibrium requires large no. of collisions / particle / Hubble time that can change non-conserved charges.

$A, B, C$  $A+B \rightleftarrows C+C$, $N_A, N_B, N_C$ are not conserved.

↙

rate may be so small that within one Hubble time, a given particle has $\ll 1$ collision of this type.

$\Rightarrow$ Treat $N_A, N_B, N_C$ as conserved quantities.

This can change as fn. of time.

(3) Use of ideal gas equation of state

→ Collisions should be rare & interactions small so that most of the time the particles can be regarded as free particle.

— true much of the time but not always.

# Cosmology

Suppose during some epoch in the history of the universe, there are $K$ types of particles and $L$ conserved charges.

$$D_\mu J^\mu_{(l)} = 0 \qquad l = 1, \dots, L$$

$$J^\mu_{(l)} = \sum_{i=1}^{K} C^i_{(l)} J^\mu_i \rightarrow \text{4-currents for density of } i\text{th type of particle.}$$

Introduce $\tilde{\mu}_{(l)}$ for each conserved quantity.

# of variables:

$\lambda(t)$, $T(t)$, $\tilde{\mu}_{(l)}(t)$ $\quad l = 1, \cdots, L$ : $L+2$ variables

We are not counting cosmological const. since it remains constant.

<u>Equation</u>

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2 \lambda^2} = \frac{8\pi G}{3} \rho$$

$\rho \to \sum_i \rho_i$

$$\frac{d}{dt}(\rho a^3) = -3 a^2 \dot{a} p$$

$p = \sum_i p_i$

$p$ fn. of $T$, $\tilde{\mu}_{(l)}$

$$D_\mu J^\mu_{(\alpha)} = 0.$$

Ex. check that for FRW metric, this reduces to

$$\frac{d}{dt}\left(n_{(\alpha)} a^3\right) = 0 \rightarrow L \text{ eqs.}$$

$\bar{n}_{(\alpha)}$ ↳ density of $l$-th charge.

$$\sum_i C^i_{(\alpha)} n_i$$

$n_i$ ↳ given a fn. of $T, \mu_i$

$\mu_i$ ↙ fn. of $\mu_{(\alpha)}$.

Ex. check that as a consequence of these L+2 equations:

$$\frac{d}{dt}(s a^3) = 0 \Rightarrow \text{Entropy conservation.}$$

$s \to \sum_{i=1}^{K} s_i \longrightarrow$ fn. $T, \mu_i \to \sum C^i_{(\alpha)} \tilde{\mu}_{(\alpha)}$