Ideal gas results

$$\underset{\overset{\shortparallel}{\frac{p}{T}}}{\frac{1}{V}\ln \mathcal{Q}} = \overset{\text{boson}}{\underset{\text{fermion}}{\mp}} \frac{g}{8\pi^3}\int_0^\infty 4\pi k^2\, dk\, \ln\left\{1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}\right\}$$

Ex: $$n = \frac{1}{\beta}\frac{\partial}{\partial\mu}\left(\frac{1}{V}\ln \mathcal{Q}\right) = \frac{g}{2\pi^2}\int_0^\infty k^2\, dk\, \frac{e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}{1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}$$

$$\rho = \mu n - \frac{\partial}{\partial\beta}\left(\frac{1}{V}\ln \mathcal{Q}\right) = \frac{g}{2\pi^2}\int_0^\infty k^2\, dk\, \sqrt{k^2+m^2}\, \frac{e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}{1 \mp e^{-\beta\left(\sqrt{k^2+m^2}-\mu\right)}}$$

Later we'll study:

1. non-relativistic limit: $T \ll m$
2. Relativistic limit: $T \gg m$

Back to cosmology

If there are L conserved charges then we introduce L chemical potentials $\tilde{\mu}_{(l)}$.

Variables: λ, T, $\tilde{\mu}_{(l)}$

Equations: $\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{a_0^2 \lambda^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\sum_{i=1}^{k}\rho_i$

$\frac{d}{dt}(\rho\lambda^3) + 3p\lambda^2\dot{\lambda} = 0$, $\frac{d}{dt}(\tilde{n}_{(l)}a^3) = 0$ for $l = 1, \ldots L$

$\Rightarrow$ $L+2$ eqs.

We need to express $\rho, p, \tilde{n}_{(l)}$ in terms of $\tilde{\mu}_{(l)}, T$.

We know n_i, p_i, e_i in terms of μ_i, T using the expressions given earlier.

Then we use $e = \sum_{i=1}^{K} e_i$, $p = \sum_{i=1}^{K} p_i$

$$\mu_i = \sum_{\ell} C^i_{(\ell)} \tilde{\mu}_{(\ell)}$$

$$\tilde{n}_{(\ell)} = \sum_i C^i_{(\ell)} n_i$$

A consequence of these eqs: Entropy conservation

$$\frac{d}{dt}\left(s\lambda^3\right)=0$$

$$\rho=\sum_{i=1}^{k}\rho_i,\qquad p=\sum_{i=1}^{k}p_i$$

__Proof:__ $s=\sum_{i=1}^{k}s_i$

$$\frac{d}{dt}\left(s\lambda^3\right)=\sum_{i=1}^{k}\left(\frac{ds_i}{dt}\lambda^3+3s_i\lambda^2\dot{\lambda}\right)$$

$$\frac{1}{T}\left(p_i+\rho_i-\mu_i n_i\right)$$

$$\frac{\partial s_i}{\partial \rho_i}\frac{d\rho_i}{dt}+\frac{\partial s_i}{\partial n_i}\frac{dn_i}{dt}=\frac{1}{T}\frac{d\rho_i}{dt}-\frac{\mu_i}{T}\frac{dn_i}{dt}$$

$$\frac{d}{dt}\left(s\lambda^3\right)=\boxed{\frac{1}{T}\frac{d\rho}{dt}\lambda^3}-\frac{\lambda^3}{T}\sum_i\mu_i\frac{dn_i}{dt}+\boxed{\frac{3}{T}\lambda^2\dot{\lambda}\left(p+\rho\right)}-3\lambda^2\dot{\lambda}\sum_{i=1}^{k}\mu_i n_i$$

$$\frac{1}{T}\frac{d}{dt}\left(\rho\lambda^3\right)+\frac{3}{T}\lambda^2\dot{\lambda}p=0$$

$$\frac{d}{dt}\left(s\lambda^3\right) = -\frac{\lambda^3}{T}\sum_{i=1}^{K} \mu_i \frac{dn_i}{dt} - \frac{3}{T}\lambda^2\dot{\lambda}\sum_{i=1}^{K} \mu_i n_i$$

Use $\mu_i = \sum_{l=1}^{L} c^i_{(l)}\,\tilde{\mu}_{(l)}$, $\quad \tilde{n}_{(l)} = \sum_{i=1}^{K} c^i_{(l)}\, n_i$

$$\frac{d}{dt}\left(s\lambda^3\right) = -\frac{1}{T}\sum_{i=1}^{K}\sum_{l=1}^{L}\left\{\lambda^3 c^i_{(l)}\,\tilde{\mu}_{(l)}\frac{dn_i}{dt} + 3\lambda^2\dot{\lambda}\, c^i_{(l)}\,\tilde{\mu}_{(l)}\, n_i\right\}$$

$$= -\frac{1}{T}\sum_{l=1}^{L}\left(\lambda^3\tilde{\mu}_{(l)}\frac{d\tilde{n}_{(l)}}{dt} + 3\lambda^2\dot{\lambda}\,\tilde{\mu}_{(l)}\,\tilde{n}_{(l)}\right)$$

$$= -\frac{1}{T}\sum_{l=1}^{L}\tilde{\mu}_{(l)}\frac{d}{dt}\left(\lambda^3\tilde{n}_{(l)}\right) = 0$$

$$D_\mu J^\mu_{(\ell)} = 0$$

In the comoving coordinates:

$$J^0_{(\ell)} = \tilde{n}_{(\ell)}, \quad J^k_{(\ell)} = 0 \qquad k = r, \theta, \phi,$$

Some further useful results.

① Particle and anti-particle always carry opposite values of a conserved charge.

$$\tilde{n}_{(\ell)} = \sum_{i=1}^{k} C_{(\ell)}^{i} n_i = \sum_{\alpha=1}^{k/2} C_{(\ell)}^{\alpha} (n_\alpha - \bar{n}_\alpha)$$

particle anti-particle

$$C_{(\ell)}^{\bar{\alpha}} = - C_{(\ell)}^{\alpha}$$

$$\mu_\alpha = \sum_{\ell=1}^{L} C_{(\ell)}^{\alpha} \tilde{\mu}_{(\ell)}, \quad \bar{\mu}_\alpha = \sum_{\ell=1}^{L} C_{(\ell)}^{\bar{\alpha}} \tilde{\mu}_{(\ell)} = - \sum_{\ell=1}^{L} C_{(\ell)}^{\alpha} \tilde{\mu}_{(\ell)}$$

$$= -\mu_\alpha.$$

$\Rightarrow$ chemical potentials carried by particles & anti-particles are negatives of each other.

Corollary: If a particle is its own antiparticle then $\mu_\alpha = 0$.

→ e.g. photons have $\mu = 0$.

(2) If for a given system we know the $\tilde{n}_{(\alpha)}$'s, then we can find $\tilde{\mu}_{(\alpha)}$'s.

Procedure: $\tilde{n}_{(\alpha)} = \sum_{i=1}^{k} C^{i}_{(\alpha)} n_i$ → given in terms of μ_i and T

Then use $\mu_i = \sum_{\alpha} C^{i}_{(\alpha)} \tilde{\mu}_{(\alpha)}$

⇒ $\tilde{n}_{(\alpha)}$ in terms of $\tilde{\mu}_{(\alpha)}$ for fixed T → L eqs. in L unknowns, unique soln.

Now suppose that for some system $\tilde{n}_{(\ell)} = 0$.

<u>Claim:</u> A soln for $\tilde{\mu}_{(\ell)}$'s is $\tilde{\mu}_{(\ell)} = 0$, for all ℓ for all ℓ.

<u>Proof:</u> If $\tilde{\mu}_{(\ell)} = 0$ then $\mu_\alpha = \sum_\ell C^\alpha_{(\ell)} \tilde{\mu}_{(\ell)} = 0$

$$\bar{\mu}_\alpha = -\mu_\alpha = 0$$

$$n_\alpha = \frac{g}{2\pi^2} \int_0^\infty dk\, k^2 \frac{e^{-\beta\sqrt{k^2+m_\alpha^2}}}{1 \mp e^{-\beta\sqrt{k^2+m_\alpha^2}}}$$

$$\bar{n}_\alpha = \frac{g}{2\pi^2} \int_0^\infty dk\, k^2 \frac{e^{-\beta\sqrt{k^2+m_\alpha^2}}}{1 \mp e^{-\beta\sqrt{k^2+m_\alpha^2}}} = n_\alpha$$

$\Rightarrow \tilde{n}_{(\ell)} = \sum_\alpha C^\alpha_{(\ell)} (n_\alpha - \bar{n}_\alpha) = 0$ eqs. are solved.

Importance of this result in cosmology:

$\rightarrow$ This is a very good approximation for most studies.

$$\frac{\#\text{ of photons}}{\#\text{ of baryons}}$$

$\#$ of photons $= 10^8/\text{meter}^3$

$\#$ of baryons ≈ 1 proton$/\text{meter}^3$

Clearly this approximation is not useful when we are studying processes that involve these excess baryons.

$\Rightarrow$ e.g. ratio of He/H in the universe.

Two limits

③ Non-relativistic limit: $T \ll m$, $\beta \gg \frac{1}{m}$

$$\frac{p}{T} = \frac{1}{V}\ln Q = \mp \frac{g}{2\pi^2}\int_0^\infty dk\, k^2 \ln\left(1 \mp e^{-\beta\sqrt{k^2+m^2}}\right)$$

$$\simeq \frac{g}{2\pi^2}\int_0^\infty dk\, k^2\, e^{-\beta\sqrt{k^2+m^2}}$$

$$\simeq \frac{g}{2\pi^2}\int_0^\infty dk\, k^2\, e^{-\beta m - \frac{\beta k^2}{2m}}$$

$$\simeq \frac{g m}{\underset{4}{\cancel{2}}\pi^2}\, e^{-\beta m}\sqrt{\frac{2\pi m}{\beta^3}} \rightarrow \text{small due to } e^{-\beta m} = e^{-m/T} \text{ factor}$$

Ex check

$$n = \frac{g}{2\pi^2} \int dk\, k^2 \frac{e^{-\beta\sqrt{k^2+m^2}}}{1 - e^{-\beta\sqrt{k^2+m^2}}}$$

$$= 0$$

$$\simeq \frac{gm}{4\cancel{2}\pi^2} e^{-\beta m} \sqrt{\frac{2\pi m}{\beta^3}}$$

$$\rho \simeq \frac{g m^2}{4\cancel{2}\pi^2} e^{-\beta n} \sqrt{\frac{2 m \pi}{\beta^3}}$$

Number density falls of exponentially as T falls.

Physically In thermal equilibrium:

$\alpha\ \bar{\alpha} \rightleftarrows$ radiation.

At some point when the density is low enough, the number of time a α encounters an $\bar{\alpha}$ in one Hubble time is $<< 1$.

$\Rightarrow \alpha$ ~~falls~~ out of equilibrium

$\rightarrow$ freeze out. After freeze out $n_\alpha \lambda^3$ is conserved.

$$n_\alpha = n_F \left(\frac{\lambda_F}{\lambda}\right)^3$$

This can be used to calculate relic density.

For example if dark matter is a collection of heavy particles then it may be the relic density after freeze out.

Freeze out may happen also when the particles are relativistic.

Application: We'll use that to calculate the density of neutrinos in the universe ~~today~~.

not detected yet.