Question on energy dependence

$$(f_{n\ell\vec{m}}, f_{n'\ell'\vec{m}'}) = \delta_{nn'}\delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'}$$

Do we need factors of $2\omega_{n\ell}$? (Analog of $2E$)

Recall: $(h,f) = \int_\Sigma \cdots (h^* \partial_0 f - \partial_0 h^* f)$

Brings down $\omega_{n\ell}$ factor

$$[a_{n\ell\vec{m}}, a^\dagger_{n'\ell'\vec{m}'}] = \delta_{nn'}\delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'}$$

Conformal field theory (CFT_d)

$$\left\langle \prod_i O_i(x_i) \right\rangle_{e^{2\Omega(x)} g_{\mu\nu}} = \left\langle \prod_i O_i'(x_i) \right\rangle_{g_{\mu\nu}} \times \exp\left(A[g_{\mu\nu}, \Omega] \right)$$

Weyl "symmetry"

Drops out in normalized correlators.

Now consider a general coordinate trs:

scalars

$$\left\langle \prod_i O_i(x_i) \right\rangle_{g_{\mu\nu}} = \left\langle \prod_i O_i(x_i'') \right\rangle_{g''} = \left\langle \prod_i O_i'(x_i'') \right\rangle_{g_{\mu\nu}}$$

$$g''_{\mu\nu}(x'') = \frac{\partial x^\rho}{\partial x''^\mu} \frac{\partial x^\sigma}{\partial x''^\nu} g_{\rho\sigma}(x)$$

Suppose that $g''_{\mu\nu}(x) = e^{2\Omega(x)} g_{\mu\nu}(x)$ for some Ω.

$$\left\langle \prod_i O_i(x_i) \right\rangle_g = \left\langle \prod_i O'_i(x''_i) \right\rangle_g.$$

$\rightarrow$ a genuine symmmetry (conformal trs.)

coordinate trs. that changes the metric at most by an overall scaling.

If O_i are tensors that we have to transform them both by g.c.t. and by Weyl trs.

Example: Independent conformal trs. in flat space-time:

① Translation: $x^\mu \quad x^\mu + a^\mu$.

② Lorentz trs. $x^\mu \to \Lambda^\mu{}_\nu x^\nu$.

③ Scaling $x^\mu \to \lambda x^\mu$.

④ special conformal: $x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2 b \cdot x + b^2 x^2}$

— generates $SO(d,2)$ group.

$SO(d,2)$ isometry of AdS_{d+1} induces a scaling of r and a coordinate trs. on the boundary.

↳ conformal trs.

For large r, the metric along the boundary: $r^2(-dt^2 + d\Omega_{d-1}^2)$

AdS_{d+1}/CFT_d correspondence

$\boxed{Z = e^{W}}$

A theory of quantum gravity on AdS_{d+1}

$\equiv$ CFT_d living on the boundary of AdS_{d+1}.

AdS_{d+1} admits different coordinate systems

$\rightsquigarrow$ different boundaries.

Global coordinates $\rightarrow$ Boundary is $\mathbb{R} \times S^{d-1}$

Poincare coordinates $\rightarrow$ Boundary is Minkowski space.

We need a map between quantities in the two theories.

Which object in AdS is mapped to which object in CFT?

(1) The Hilbert spaces are isomorphic.

Recall: In AdS_{d+1} in weak coupling limit, a scalar field produces ∞ no. of states:

$$a^\dagger_{n\ell\vec{m}}|0\rangle, \quad a^\dagger_{n_1\ell_1\vec{m}_1} a^\dagger_{n_2\ell_2\vec{m}_2}|0\rangle, \; \cdots$$

CFT_d on $\mathbb{R}\times S^{d-1}$ must also have these states.

(2) The boundary correlation fn.s in AdS_{d+1} must agree with CFT correlation fn.s.

$$\lim_{r_i\to\infty}\left(\prod_i r_i^{\Delta_i}\right)\left\langle \prod_i \phi_i(r_i, t_i, \vec{\theta}_i)\right\rangle_{AdS_{d+1}} = \left\langle \prod_i O_i(t_i, \vec{\theta}_i)\right\rangle_{CFT_d}$$

different scalar fields with different masses.

Apparent puzzle: In CFT_d there is state-operator correspondence.

$\forall$ local operator $\Leftrightarrow$ a state in CFT_d on $\mathbb{R}\times S^{d-1}$

$a^\dagger_{0\,0\,\vec{0}}|0\rangle \leftrightarrow$ primary operator $\mathcal{O}$ dual to ϕ
$\quad n\,\ell\,\vec{m}$

$a^\dagger_{n\ell\vec{m}}|0\rangle \leftrightarrow \partial_{\mu_1}\cdots\partial_{\mu_k}\mathcal{O} \leftrightarrow \partial_{\mu_1}\cdots\partial_{\mu_k}\phi$

AdS_{d+1} $\qquad$ CFT_d $\qquad$ $t, \vec{\Omega}$ $\qquad$ AdS_{d+1}

Radial derivatives of ϕ are not independent $\sim r^{\Delta_i}$

$a^\dagger_{n_1\ell_1\vec{m}_1}\, a^\dagger_{n_2\ell_2\vec{m}_2}|0\rangle \leftrightarrow$ "multi-trace" operator $\Big\}$ Product of the single particle ops.

Approximate – valid only in the weak coupling limit.

Valid also for higher spin fields.

① Bulk theory has $g_{\mu\nu}$. $\not\exists$ has radial components

$\Leftrightarrow$ Dual operator is $T_{\mu\nu}$ $\rightarrow$ only along the boundary.

② CFT_d always has $T_{\mu\nu}$ as an operator.

$\Leftrightarrow$ there must be a dual spin 2 field

$\Rightarrow$ Bulk must contain gravity.

AdS_{d+1}/CFT_d is a general framework.

There are a few examples where both sides are known.

Type IIB string theory on $AdS_5 \times S^5$ $\longleftrightarrow$ $\mathcal{N}=4$ SUSY $SU(N)$ Yang-Mills theory in $d=4$

Not proven but many tests.

$\Downarrow$

Two parameters

string coupling g_s

radius of AdS_5: R

$\Downarrow$

Yang-Mills coupling const. g_{YM}

N

In different coordinate system AdS_{d+1} has different boundaries.

e.g. Poincaré coordinate:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2\right)$$

Boundary: $z \to 0$. metric $ds_B^2 \propto -dt^2 + \sum_{i=1}^{d-1} dx_i^2$

Minkowski

IR cut-off in gravity vs. UV cut-off in CFT

Take AdS_{d+1} in global coordinates:

$$ds^2 = -(R^2 + r^2)\,dt^2 + \frac{dr^2}{1 + r^2/R^2} + r^2\, d\Omega_{d-1}^2$$

Boundary at $r = R/\delta \rightarrow$ small no.

$$ds_B^2 = \frac{R^2}{\delta^2}\left(-\underbrace{dt^2}_{R} + \underbrace{d\Omega_{d-1}^2}_{S^{d-1}}\right)$$

Can be removed by Weyl trs.

Gravity theory has some intrinsic UV cut-off e.g. l_s in string theory.

$\hookrightarrow$ string scale.

Since gravity correlators $\rightarrow$ CFT correlators, this automatically gives a cut-off in CFT_d.

Weyl trs. scales this cut-off to $\underbrace{l_s \delta / R}_{\text{cut-off in CFT}}$

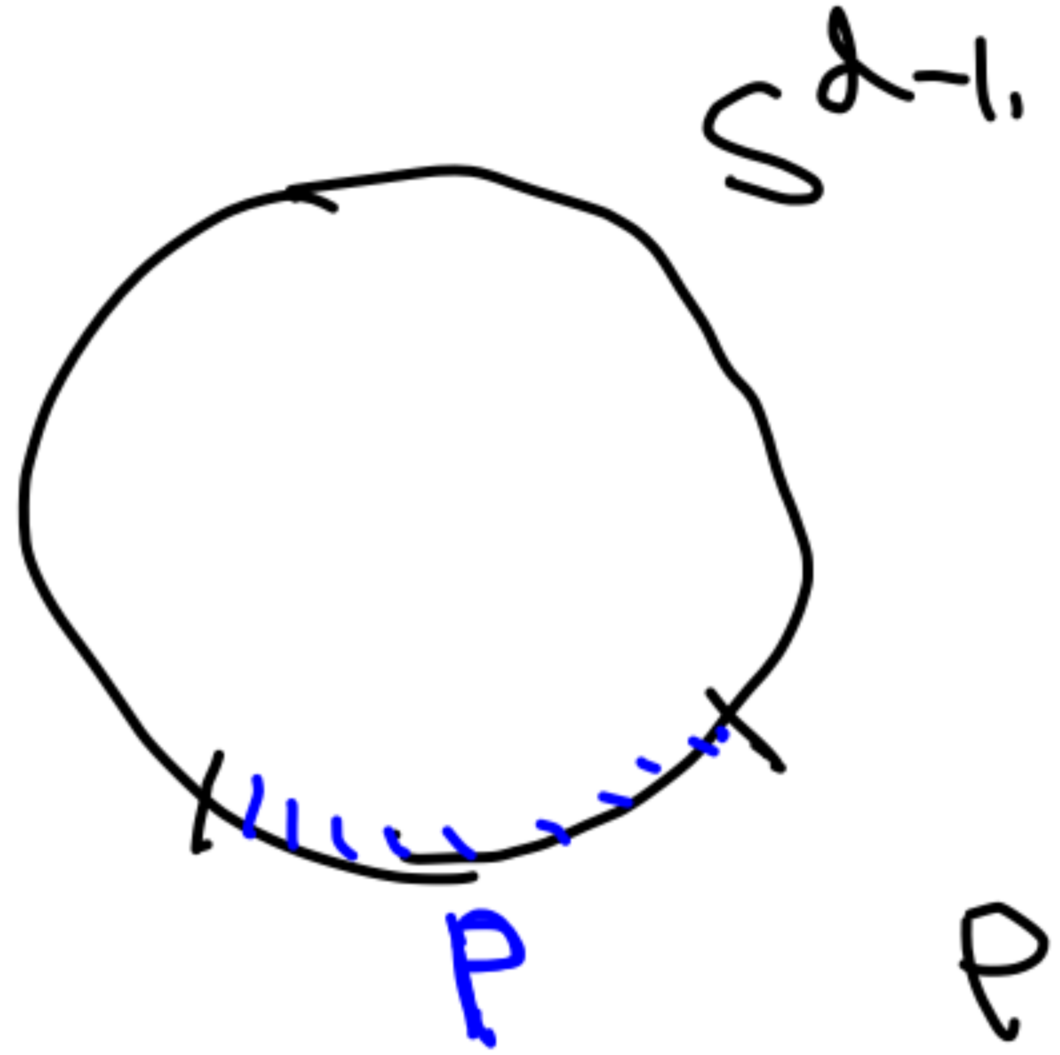
IR cut-off in gravity

$\Rightarrow$ UV cut-off in the CFT.

Entanglement entropy

Consider a CFT on $\mathbb{R} \times S^{d-1}$.

Take a spatial slice $t=0 \Rightarrow S^{d-1}$.



Take the vacuum state of this CFT.

In this state dof inside P are entangled with the dof outside P.

Entanglement entropy $\mathcal{E}(P)$.

Q. Which quantity in AdS_{d+1} computes $\mathcal{E}(P)$?

A. Ryu-Takayanagi formula!

$t = 0$ slice of AdS_{d+1}

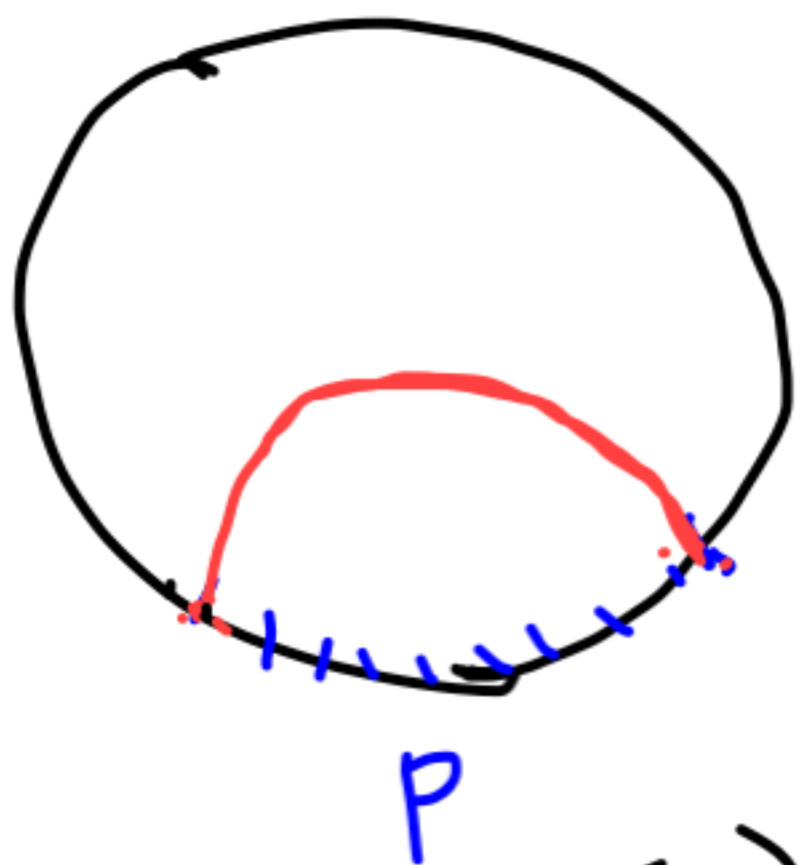
① Find the "surface" inside AdS_{d+1} whose boundary matches P & whose area is minimum subject to this constraint.

② $\mathcal{E}(P) = \text{Area}(S)/(4G_N)$ → Newton's constant.

$\mathcal{E}(P)$ → CFT_d

$$ds^2 = -(R^2+r^2)\,dt^2 + \frac{dr^2}{1+r^2/R^2} + r^2\, d\Omega_{d-1}^2$$

Metric along S^{d-1} is large close to the boundary $\Rightarrow$ large surface area.



The minimal area surface has to approach the boundary of AdS_{d+1} near the edges

$\rightarrow$ gives divergent term as $\delta \to 0$

$$\mathrm{Area}(S)/4G_N \sim \mathrm{Area}(\partial P)/(\delta/R)^{d-2} + \text{corrections}.$$

In CFT one finds that $\mathcal{E}(P)$ is UV divergent.

$\rightarrow$ depends on UV cut-off.

$(l_s R/\delta)$.

$\Downarrow$

same dependence on δ as $\dfrac{\text{Area}(S)}{4G_N}$

— consistency checks.

Other checks

① In CFT_2, $\mathcal{E}(P)$ has been computed explicitly $\rightarrow$ agrees with R-T formula.

② In any Quantum theory, entanglement entropy satisfies certain inequalities.

— Strong subadditivity formula:

$$\mathcal{E}(A \cup B) + \mathcal{E}(A \cap B) \leq \mathcal{E}(A) + \mathcal{E}(B)$$

follows from R-T formula geometrically

Proof in AdS_{d+1} is simple.

Proof in quantum theory is complicated.

→ see refs. given in last year's lecture.

Goal of the rest of the lectures:

– Bulk reconstruction.

Given CFT correlators, how to construct gravity correlators $\langle \prod_i \phi_i(x_i) \rangle \rightarrow$ approximate

→ how local physics (approximate) emerges in the bulk.