Some formal definitions & results.

Consider a linear map from some Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$.

$\alpha, \beta \in \mathcal{H}_1 \qquad \alpha \to A \in \mathcal{H}_2, \quad \beta \to B$ in $\mathcal{H}_2$.

The map will be called an isometry if for every pair α, β, $(\alpha, \beta) = (A, B)$.

Some obvious properties.

① $\dim \mathcal{H}_1 \leq \dim \mathcal{H}_2$.

② If $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$ then the map is called unitary.

③ Composition of two isometries is an isometry.

We'll take $\mathcal{H}_1$ and $\mathcal{H}_2$ to be collection of qudits.

$i_s, j_s : 1, \dots d$

$\mathcal{H}_1 : |i_1 \cdots i_k\rangle$ $\quad$ $\mathcal{H}_2 : |j_1 \cdots j_n\rangle$, $\mathcal{H}^k \to \mathcal{H}^n$

$k \leq n$.

$\mathcal{H}$: Hilbert space of one qudit

Element of $\mathcal{H}_1$: $\sum a_{i_1 \cdots i_k} |i_1 \cdots i_k\rangle$

" " $H_2 = \sum b_{j_1 \cdots j_n} |j_1 \cdots j_n\rangle$

<u>Isometry:</u>

$$\sum a_{i_1 \cdots i_k} |i_1 \cdots i_k\rangle \to \sum_{j_1 \cdots j_n} \sum_{i_1 \cdots i_k} \boxed{M_{j_1 \cdots j_n, i_1 \cdots i_k}} \; a_{i_1 \cdots i_k} |j_1 \cdots j_n\rangle$$

$d^n \times d^k$ matrix.

$$\langle \alpha, \beta \rangle = (A, B) \Rightarrow \sum_{j_1 \cdots j_n} M^*_{j_1 \cdots j_n, i_1 \cdots i_k} M_{j_1 \cdots j_n, l_1 \cdots l_k} = \delta_{i_1 l_1} \cdots \delta_{i_k l_k}$$

$\Rightarrow M^\dagger M = I$

(order is important).

Operator pushing:

Take any operator O acting on $\mathcal{H}_1$.

$$MO = O'M \qquad O' = MOM^\dagger$$

$$O'M = MOM^\dagger M = MO$$

$M_{j_1 \cdots j_n, i_1 \cdots i_k}$: isometry from $\mathcal{H}^k \to \mathcal{H}^n$.

Define $N_{j_1 \cdots j_n j_{n+1}, i_1 \cdots i_{k-1}} = \frac{1}{\sqrt{d}} M_{j_1 \cdots j_n, i_1 \cdots i_{k-1} j_{n+1}}$

a map from $\mathcal{H}^{k-1} \to \mathcal{H}^{n+1}$

claim: N is an isometry.

Proof: $\sum_{j_1 \cdots j_{n+1}} N^*_{j_1 \cdots j_{n+1}, i_1 \cdots i_{k-1}} N_{j_1 \cdots j_{n+1}, l_1 \cdots l_{k-1}}$

$= \frac{1}{d} \sum_{j_1 \cdots j_{n+1}} M^*_{j_1 \cdots j_n, i_1 \cdots i_{k-1} j_{n+1}} M_{j_1 \cdots j_n, l_1 \cdots l_{k-1} j_{n+1}}$

$= \frac{1}{d} \sum_{j_{n+1}} \delta_{i_1 l_1} \cdots \delta_{i_{k-1} l_{k-1}} \delta_{j_{n+1} j_{n+1}} = \frac{1}{d} \times d \, \delta_{i_1 l_1} \cdots \delta_{i_{k-1} l_{k-1}}$

Perfect tensors provide isometry.

$T_{i_1 \cdots i_{2n}}$ any balanced bipartition gives a unitary matrix

$$M_{i_1 \cdots i_n, i_{n+1} \cdots i_{2n}} = T_{i_1 \cdots i_{2n}}$$

$$\shortparallel$$

$$N_{i_1 \cdots i_n i_{n+1}, i_{n+2} \cdots i_{2n}} = P_{i_1 \cdots i_{n+2}, i_{n+3} \cdots i_{2n}}$$

$$\mathcal{H}^{n-1} \to \mathcal{H}^{n+1} \qquad \mathcal{H}^{n-2} \to \mathcal{H}^{n+2}, \cdots$$

Claim: Tensor network code is an isometry.

The code:

$$\langle a_1 \cdots a_n | A_1 \cdots A_k \rangle = \prod_{\text{bulk points}} T_{i_1 \cdots i_6}$$

with all internal indices contracted.

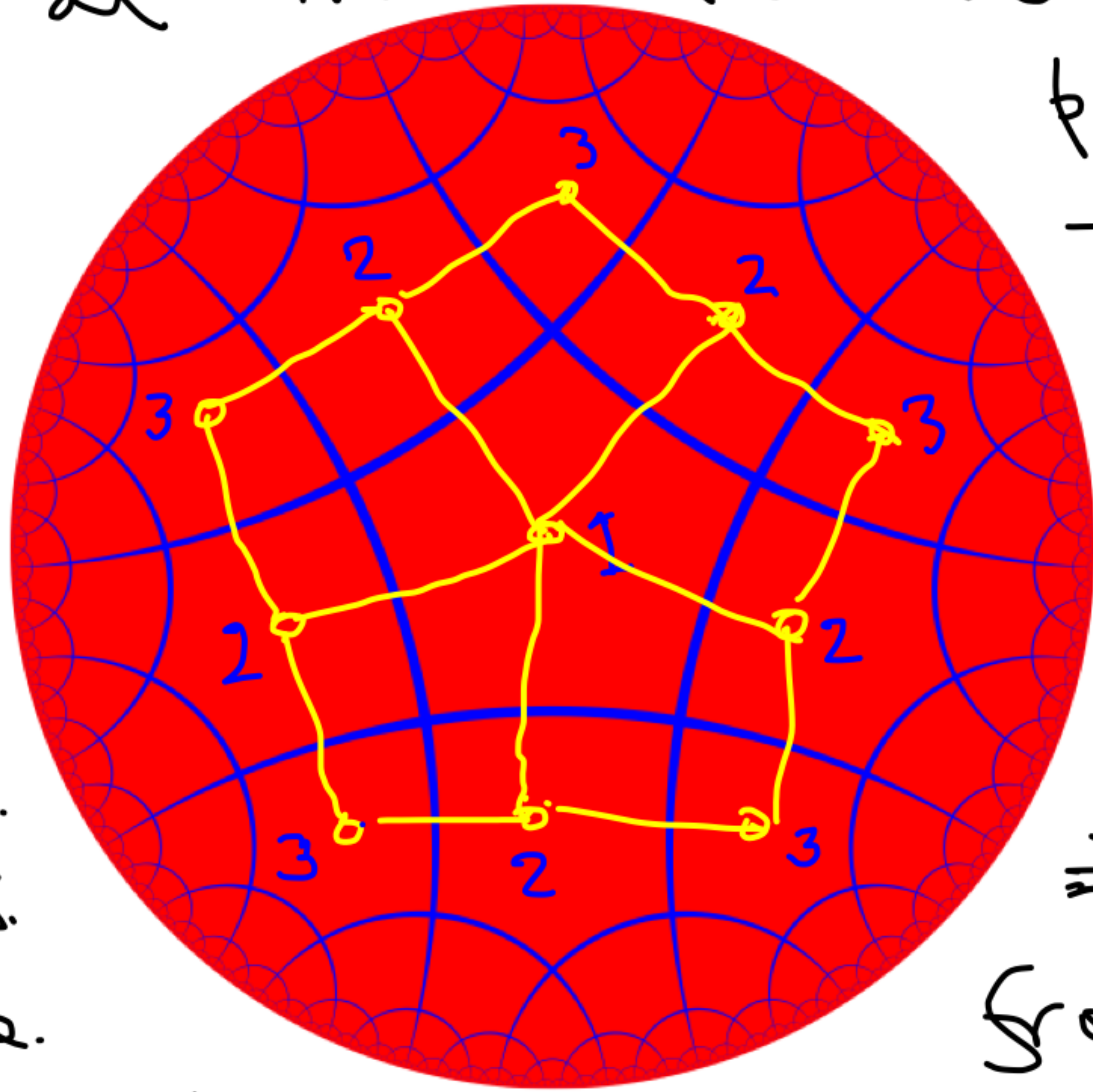
$a_1 \cdots a_n$: physical qubits, boundary

$A_1 \cdots A_k$: logical qubits, bulk

We'll indicate the proof.

In a given layer, each vertex gets input from at most two vertices in the previous layer.

$\Rightarrow$ We can interpret the perfect tensor of a vertex as an isometry whose inputs come from the previous layers and the bulk.



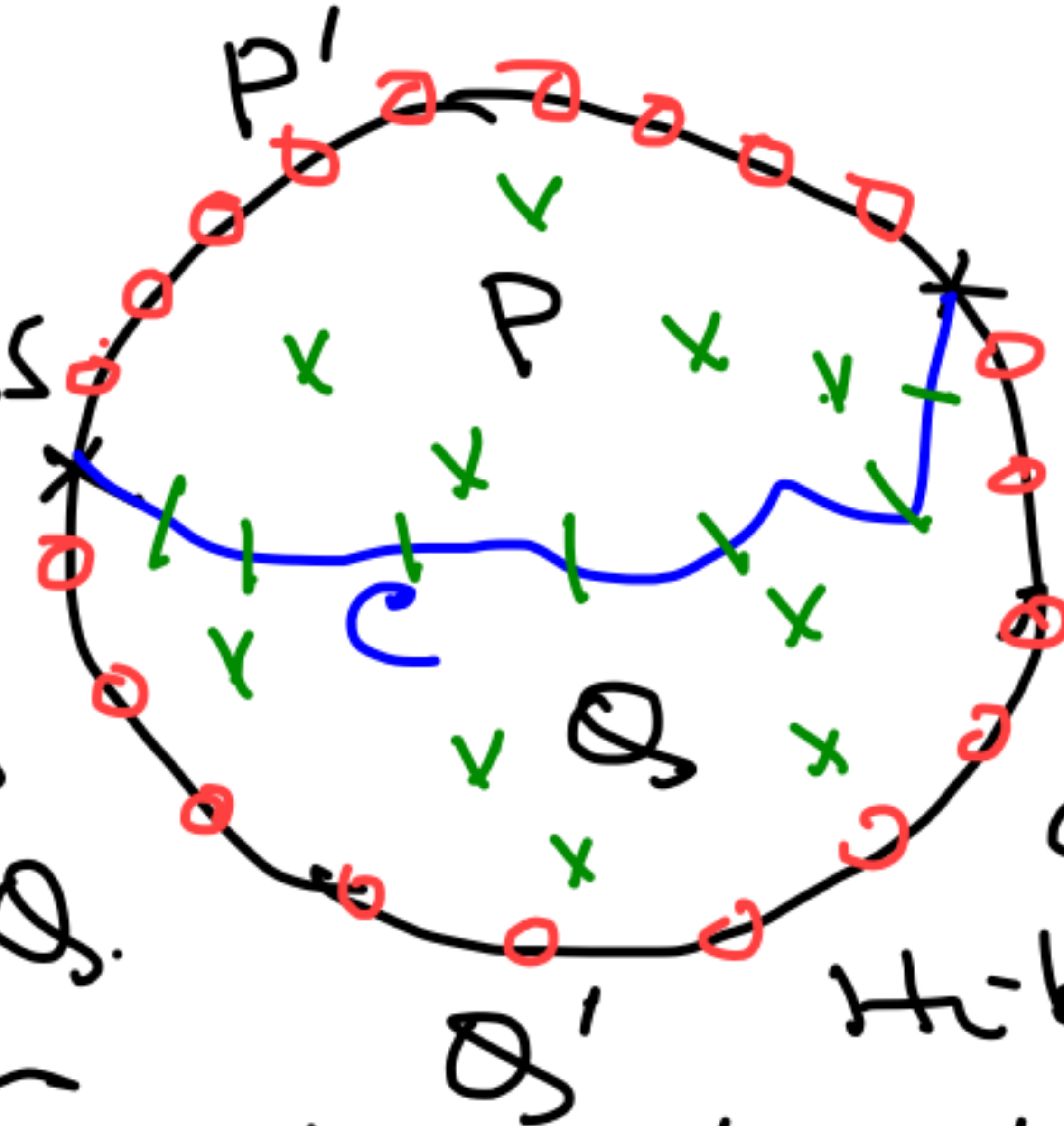
We have a composition of many isometries.

$\Rightarrow$ the map from bulk $\to$ boundary is an isometry.

RT formula

A: labels the basis of bulk qubits in P

B: labels the basis of bulk qubits in Q.



n_c: # of links cut by C $\to$ "area" of C

a: labels the basis of Hilbert space of qubits in P'

b: labels the basis of Hilbert space of qubits in Q'

i: labels the basis of Hilbert space of qubits in links cut by C $\to$ 2^{n_c} dim.

Take contracted product of the perfect tensor inside:

P: M_{iAa} | Q: N_{iBb}

$$\langle ab | AB \rangle \propto \sum_i M_{iAa} N_{iBb}$$

Consider a bulk state $\sum_A P_A |A\rangle \sum_B Q_B |B\rangle$

will be mapped to

$|i\rangle$

$$\propto \sum_{a,b} |a\rangle \otimes |b\rangle \sum_{A,B} \langle ab | AB \rangle P_A Q_B$$

$$= \sum_{a,b} |a\rangle \otimes |b\rangle \sum_i \sum_{A,B} M_{iAa} N_{iBb} P_A Q_B$$

$$= \sum_i \underbrace{\sum_a \sum_A M_{iAa} |a\rangle P_A}_{|P_i\rangle} \underbrace{\sum_b \sum_B N_{iBb} |b\rangle Q_B}_{|Q_i\rangle}$$

$$= \sum_i |P_i\rangle |Q_i\rangle$$

The representation of $\sum_A P_A |A\rangle \sum_B Q_B |B\rangle$ on the boundary is.

$$\sum_i |P_i\rangle \; |Q_i\rangle$$

$$\rho(P') \propto \sum_{i,j} |P_i\rangle\langle P_j| \; \langle Q_j|Q_i\rangle$$

$$\sum_i |P_i\rangle \sum_b |b\rangle\langle b|Q_i\rangle$$

$$\sum_i |P_i\rangle \sum_b \langle b|Q_i\rangle |b\rangle \sum_j \langle P_j| \sum_{b'} \langle b'| \; \langle Q_j|b'\rangle$$

$$\sum_b \langle Q_j|b\rangle\langle b|Q_i\rangle$$
$$= \langle Q_j|Q_i\rangle$$

$$\rho(P') \propto \sum_{i,j} |P_i\rangle\langle P_j| \quad \langle Q_j | Q_i \rangle$$

i, j : run over 2^{n_c} labels.

$$\Rightarrow \quad \mathrm{Rank}(\rho) \leq 2^{n_c}$$

$$\Rightarrow \quad S(\rho) \leq \ln \mathrm{rank}(\rho) = \underset{\text{area}}{\underset{\Downarrow}{n_c}} (\ln 2).$$

$$\rho = \begin{pmatrix} \rho_1 & & & & \\ & \ddots & & & \\ & & \rho_r & & \\ & & & 0 & \\ & & & & 0 \ldots \end{pmatrix}$$

$$\sum \rho_i = I$$

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$$

$$= -\sum_i \rho_i \ln \rho_i$$ is maximized when $\rho_i = \frac{1}{r} \; \forall i$

$$S(\rho) = -\frac{1}{r} \ln \frac{1}{r} \times r = \ln r.$$

$$S(\rho) \leq \underset{\substack{\downarrow \\ \text{area}(c)}}{n_c} \underset{\searrow\ \frac{1}{4G_N}}{\ln 2} \Rightarrow S(\rho) \leq \frac{A_c}{4G_N}$$

If C_0 is the cut with minimal "area" with boundaries fixed at the boundaries of ρ':

then $S(\rho) \leq n_{c_0} \ln 2 = \dfrac{A_{c_0}}{4G_N}$

$$\rho \propto \sum_{i,j} |P_i\rangle\langle P_j| \; \langle Q_j|Q_i\rangle$$

$S(\rho) \leq \ln(\text{rank}\,\rho)$ is saturated if

ρ has all eigenvalues same.

$\rho \propto$ Identity matrix inside the subspace.

$\rightarrow$ can be achieved if

$\langle P_i|P_j\rangle \propto \delta_{ij}$ and $\langle Q_j|Q_i\rangle \propto \delta_{ij}$

$\rho \propto \sum_i |P_i\rangle\langle P_i| \rightarrow$ same eigenvalues.

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$$|P_i\rangle = \sum_A M_{iAa} P_A |a\rangle, \quad |Q_i\rangle = \sum_B N_{iBb} Q_B |b\rangle$$

claim: $\langle P_i|P_j\rangle \propto \delta_{ij}$ if M_{iAa} gives an isometry from $(iA) \to a$

Consider the state $|i\rangle \otimes \underbrace{P_A |A\rangle}_{|\tilde{P}\rangle}$

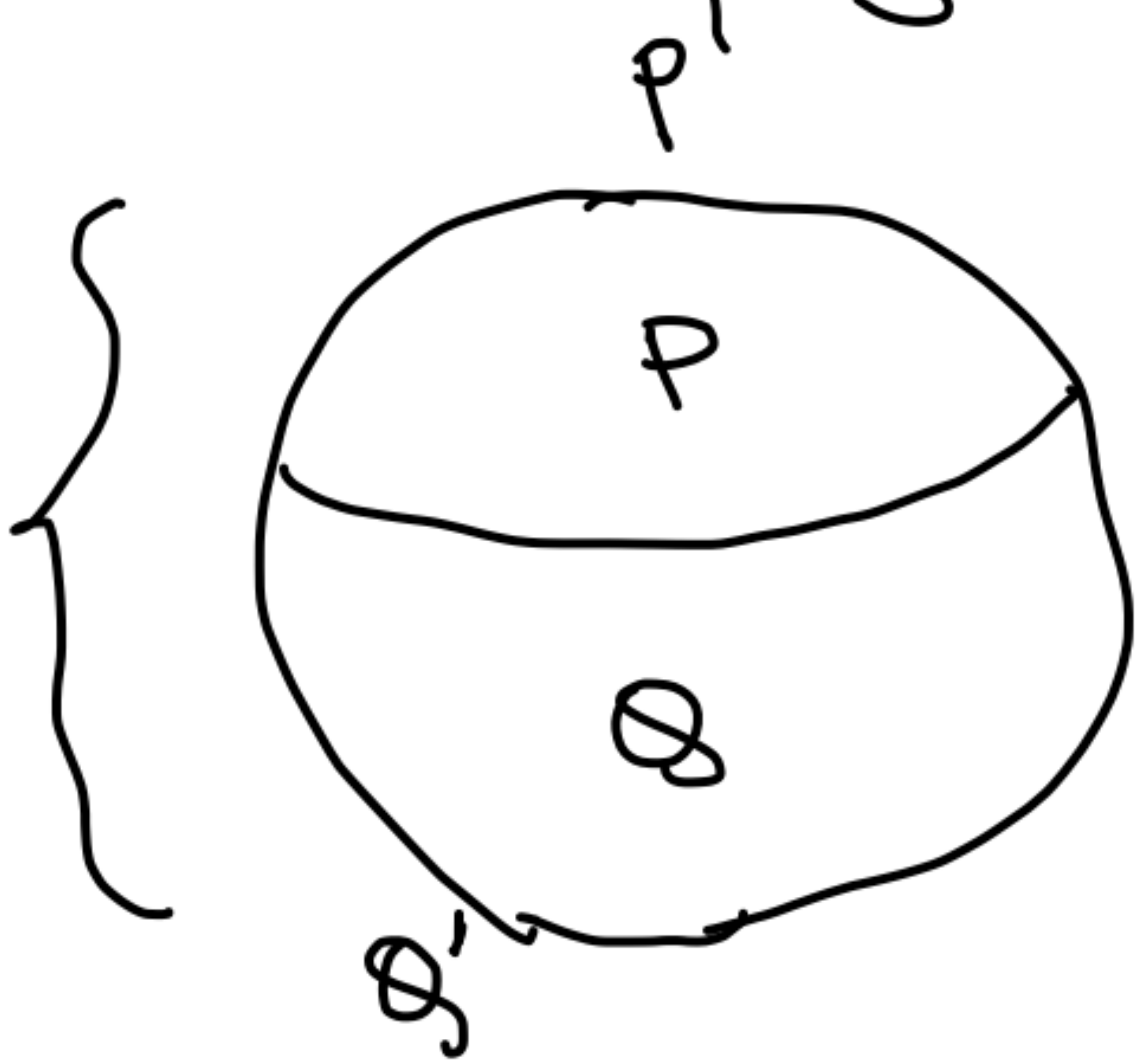
$|i\rangle \otimes |\tilde{P}\rangle$ and $|j\rangle \otimes |\tilde{P}\rangle$ are orthonormal.

$$|i\rangle \otimes |A\rangle \to M_{iAa}|a\rangle \quad \ni \quad |i\rangle \otimes \sum_A P_A |A\rangle$$

$$|i\rangle \otimes |\tilde{P}\rangle \to |P_i\rangle \ni \langle P_i|P_j\rangle \propto \delta_{ij} \quad \Big| \to \sum_A M_{iAa} |a\rangle P_A = |P_i\rangle$$

$\langle \mathcal{Q}_i | \mathcal{Q}_j \rangle \propto \delta_{ij}$ if N_{iBb} is an isometry from $iB \to b$.

Suppose that M and N are both isometries for the minimal length cut.



Bulk operators in P can be pushed to P'. Bulk operators in Q can be pushed to Q'.