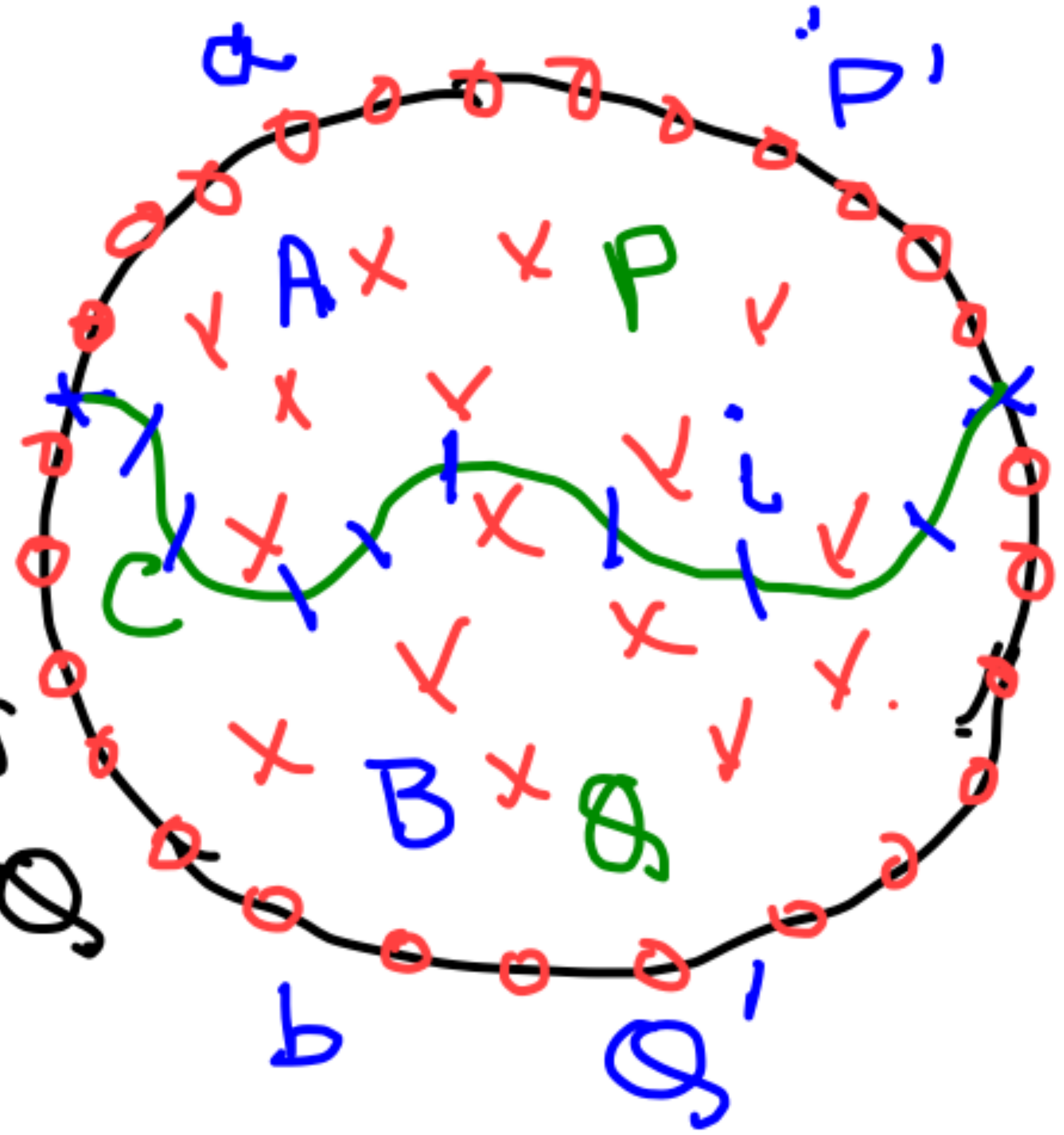
M_{iAa}: Product of perfect tensors in P



C cuts n_c links.

N_{iBb}: Product of perfect tensors in Q

Bulk state $P_A|A\rangle \otimes Q_B|B\rangle$ (summation convention).

$\xrightarrow{\text{maps to}}$ $|P_i\rangle \otimes |Q_i\rangle$, $|P_i\rangle = M_{iAa} P_A |a\rangle$

$|Q_i\rangle = N_{iBb} P_B |b\rangle$ | $\rho(P') = |P_i\rangle\langle P_j| \langle Q_j|Q_i\rangle$

$S(\rho(P')) \leq n_c \ln 2 \rightarrow$ saturated if M_{iAa} and N_{iBb} are isometries from $iA \to a$, $iB \to b$.

If M_{iAa} and N_{iBb} are isometries then C must be the minimal cut.

Converse is not automatic

Minimal cut $\nRightarrow$ M_{iAa} and N_{iBb} are isometries.

Can we prove this? $\Rightarrow$ RT formula.

We'll first study some other consequences of M, N being isometries before discussing the proof (or lack of proof).

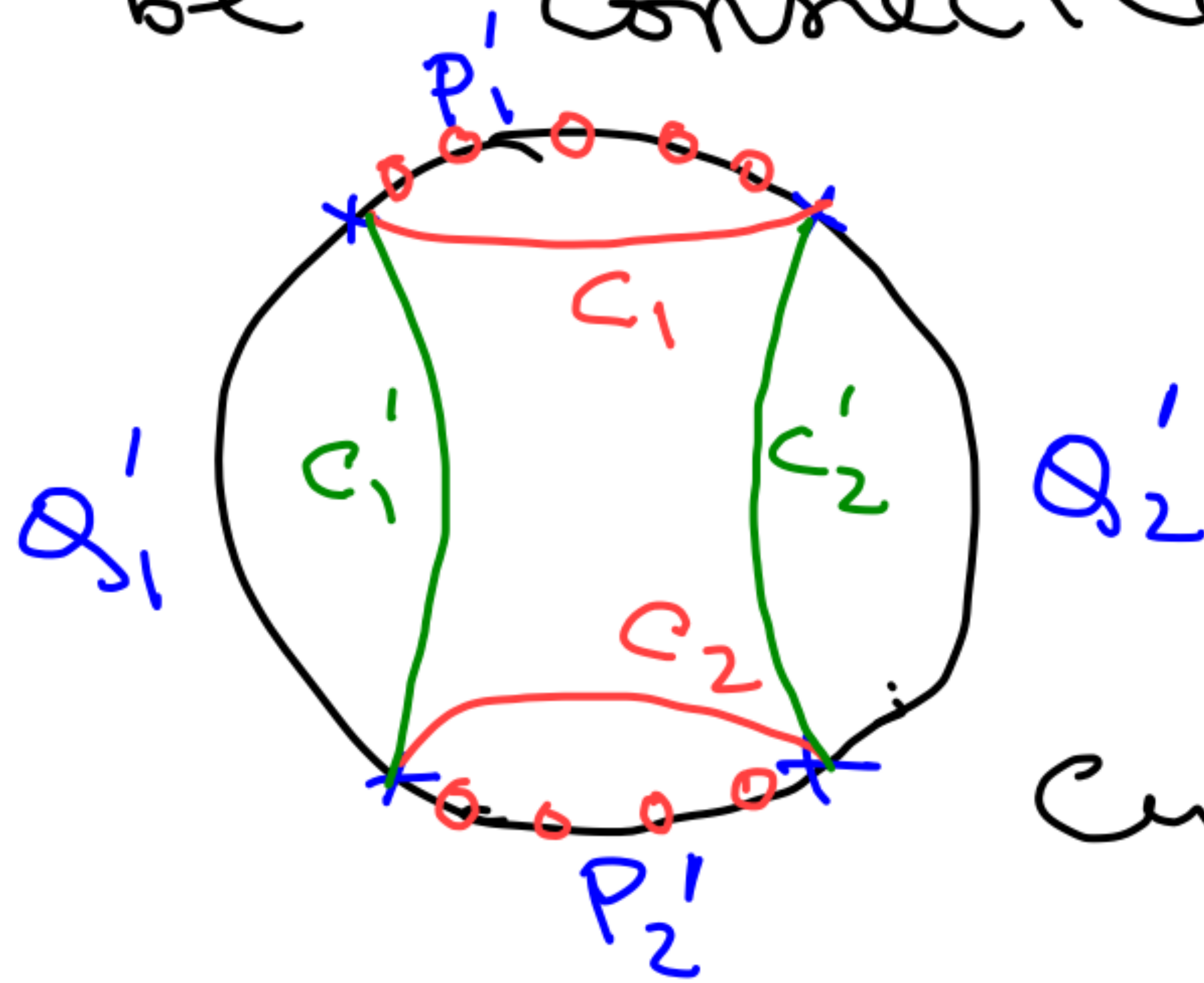
Note: The formula does not require P' and Q' to be connected.

$P' = P_1' \cup P_2'$

$Q' = Q_1' \cup Q_2'$

$C = C_1 \cup C_2$

$C' = C_1' \cup C_2'$



$S(Q) \leq n_c \ln 2$ will hold for both kinds of cuts.

Minimal cut will require searching among both kinds of cuts.

What happens if the bulk state is not a product state? (Assuming M, N are isometries)

$|\psi\rangle = f_{AB} |A\rangle \otimes |B\rangle$

Bulk → boundary isometry is given by

$$M_{iAa} N_{iBb} : (AB| \to (ab)$$

Image of $|\psi\rangle$ in the boundary:

$$M_{iAa} N_{iBb} f_{AB} |a\rangle \otimes |b\rangle$$

$\Rightarrow \delta_{ii'} \delta_{BB'}$

$$\rho(P') = M_{iAa} N_{iBb} f_{AB} M^*_{i'A'a'} N^*_{i'B'b'} f^*_{A'B'} |a\rangle\langle a'| \, |b\rangle\langle b'|$$

$\delta_{bb'}$

$$= M_{iAa} M^*_{iA'a'} f_{AB} f^*_{A'B} |a\rangle\langle a'|$$

bulk AA' ρ

$//$

$$\rho(p') = \underbrace{M_{iAa} M^*_{iA'a'} \rho^{bulk}_{AA'}}_{\rho_{aa'}} |a\rangle\langle a'|$$

Define

$$M_{a,iA} = M_{iAa} \Rightarrow \overset{\rho_{aa'}}{\overset{\shortparallel}{\rho}} = M \rho^{bulk} \otimes \underset{i,j \text{ space}}{I} M^{\dagger}$$

$$\rho_{aa'} \propto M_{a,iA} \rho^{bulk}_{AA'} \delta_{ij} M^*_{a',jA'} \quad M^{\dagger}_{jA',a'}$$

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = -\mathrm{Tr}(\rho' \ln \rho')$$

$$= -\mathrm{Tr}(\rho_{bulk} \ln \rho_{bulk}) + n_c \ln 2 \quad \rho' \propto \rho^{bulk} \otimes I$$

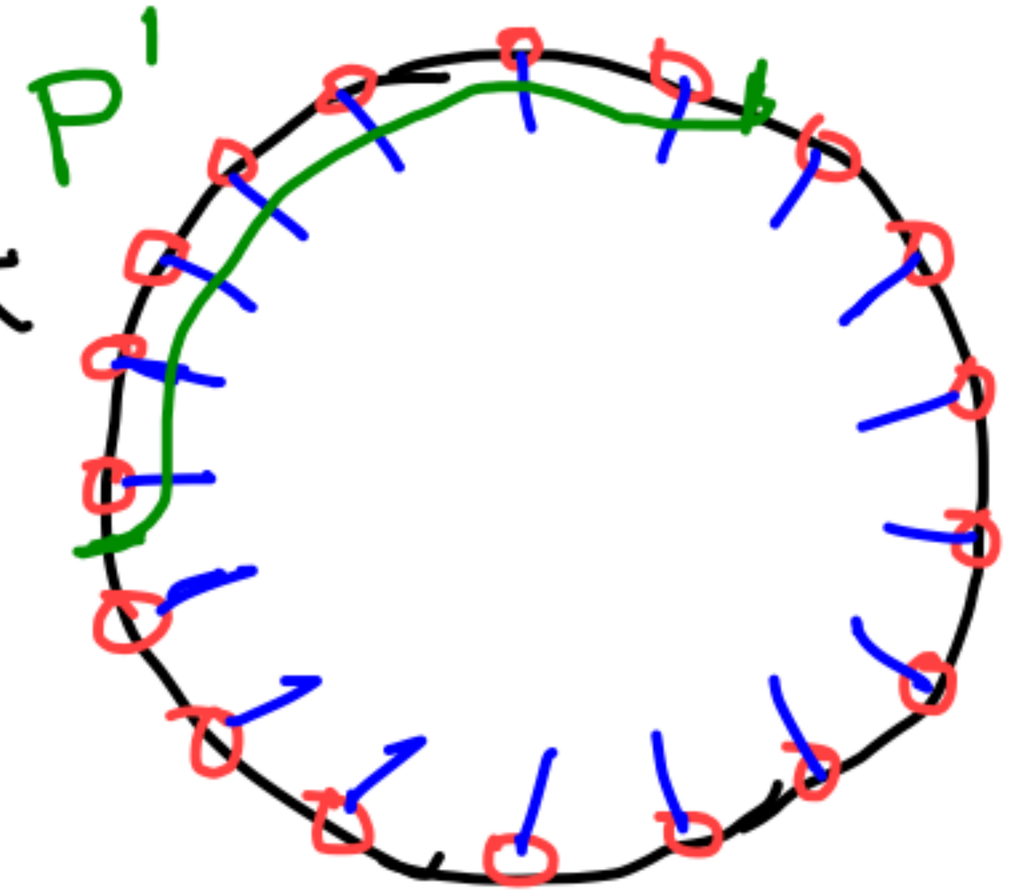
$$= S^{bulk} + n_c \ln 2$$

Is it possible to find a cut for which M_{iAa} and N_{iBb} are isometries?

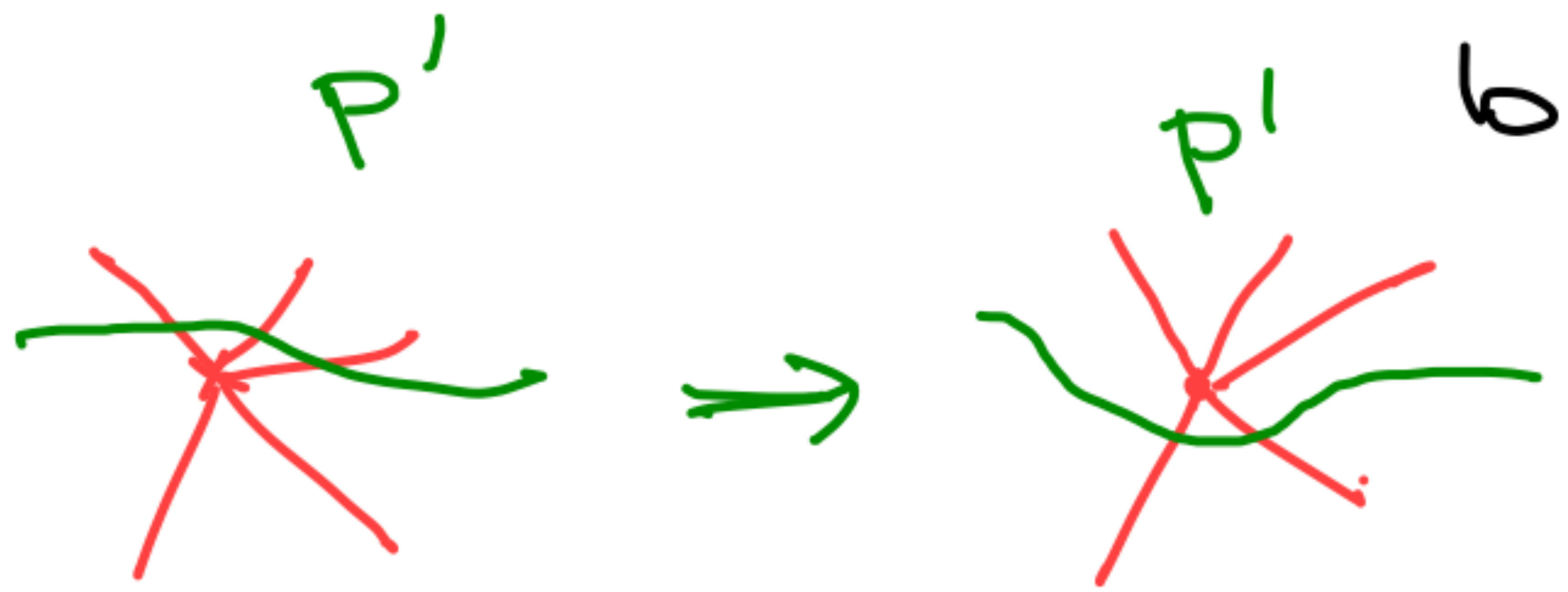
Greedy algorithm: attempts to find the minimal cut for which M or N is an isometry.

Do the analysis separately for P' and Q'.

① Begin with the trivial cut $M_{ia} = \delta_{ia}$
$\Rightarrow$ obviously an isometry.



② Move the cut through one bulk point at a time s.t. at least 3 links are directed from the bulk point towards p'.



ⓐ reduces the no. of links cut.

ⓑ The vertex is an isometry from the bulk towards p'.

$\Rightarrow$ The new cut has M_{ia} an isometry.

Repeat this process.
Stop when we cannot move the cut any more.
Call this cut C_0. M_{iAA} for C_0 is an isometry.

Now we begin with Q' and carry out the same procedure.
Suppose we have to stop at C_0'.
N_{iBB} for C_0' is an isometry.

If C_0 coincides with C_0' then we are done.

If C_0 and C_0' does not coincide, we still have some useful results.

Define: $C_0 \cap C_0'$ as the collection of links that are common to C_0 and C_0'.

The one can show that $S(\rho(P')) \geq n_{C_0 \cap C_0'} \ln 2$.

$S(\rho(P')) \leq n_{C_0} \ln 2, \; n_{C_0'} \ln 2$.

If $n_{\mathrm{conc}_0'} \simeq n_{c_0}$ or $n_{c_0'}$ then we have a narrow range of allowed values of $S(\rho(p'))$

However for this code the exact equality does not hold in all cases.
$[c_0 = c_0']$

We'll come back to this at the end.

Back to geometry.

A more "precise" version of RT formula:

We'll work on the $t=0$ slice of static geometry (otherwise we need HRT formula)

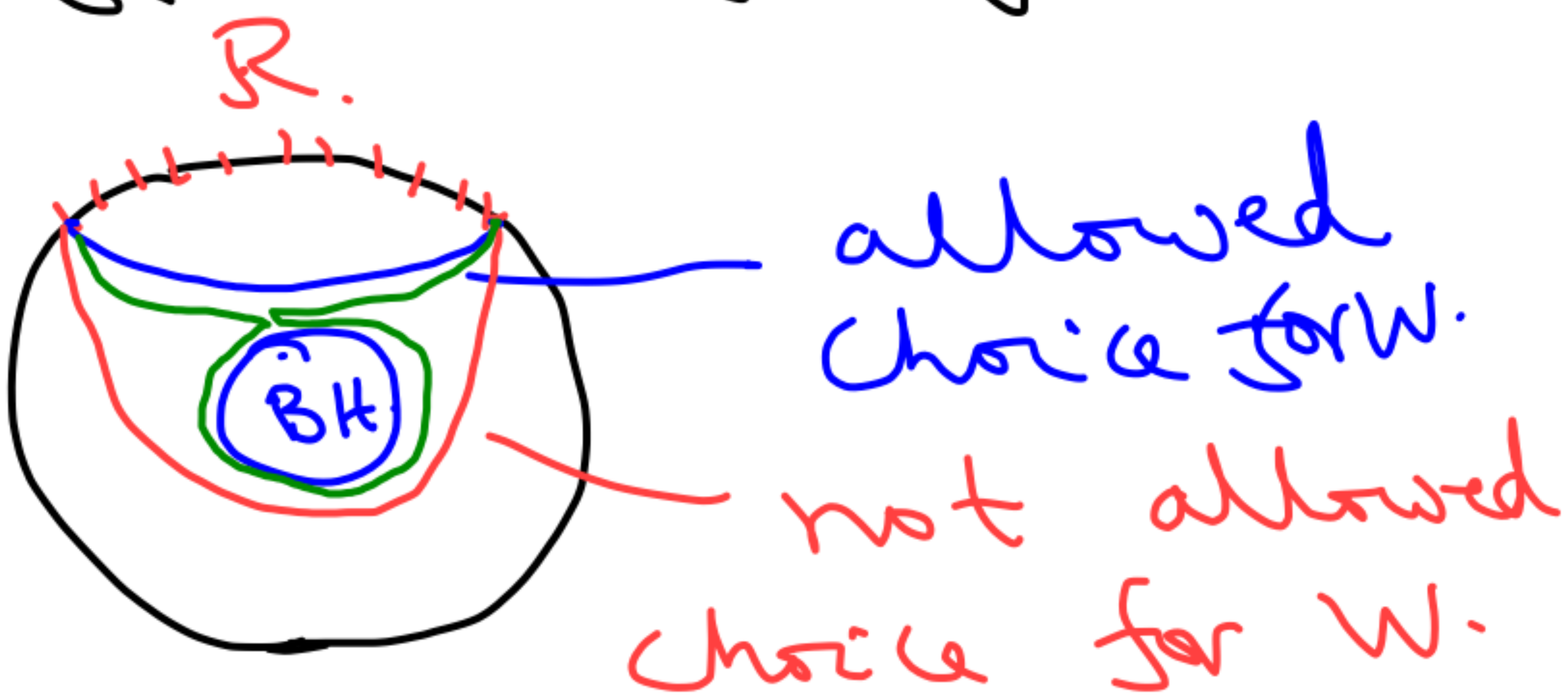
Consider some region R on the boundary, not necessarily connected.

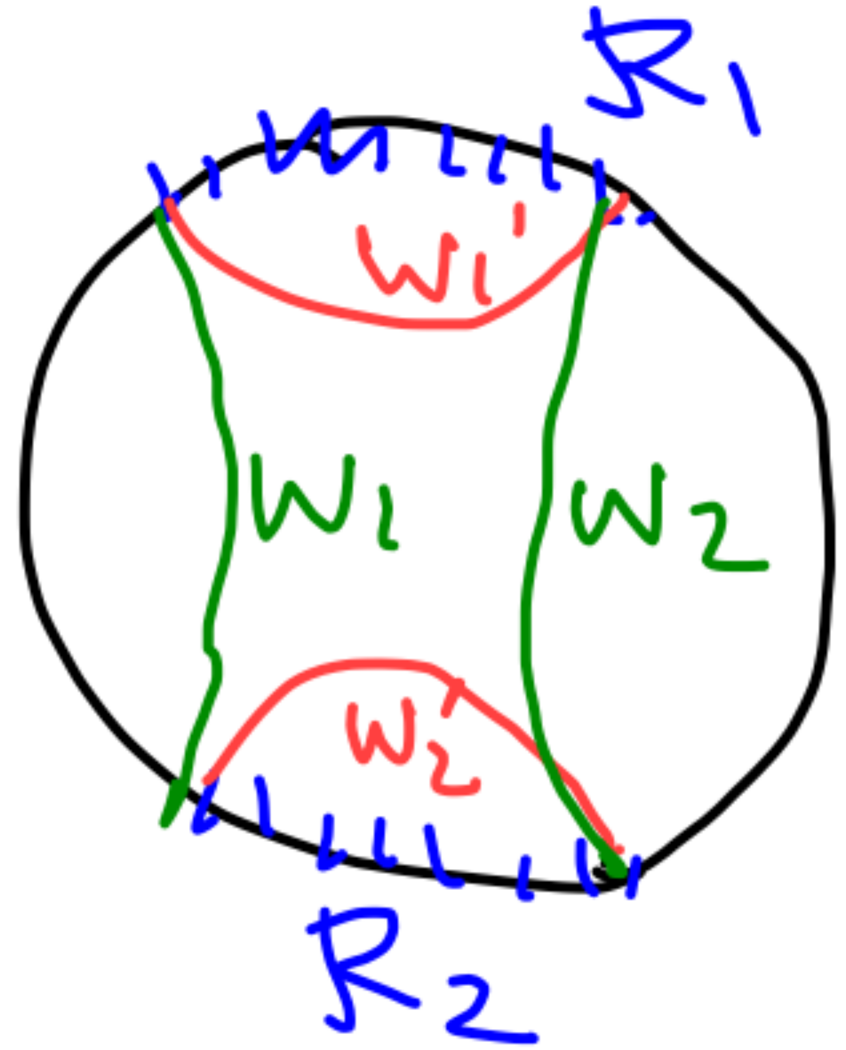
Consider a "surface" W in the bulk that is homologous to R.

$\Downarrow$

W should be continuously deformable to R → stronger than $\partial W = \partial R$

Ex-1





$R = R_1 \cup R_2$

$W_1' \cup W_2'$ and $W_1 \cup W_2$ are both allowed choice of W.

V: region enclosed by W and R.

Now consider

$$S(W) \equiv \frac{\text{Area}(W)}{4G_N} + S_{bulk}(Y).$$

Y: region bounded by W and R

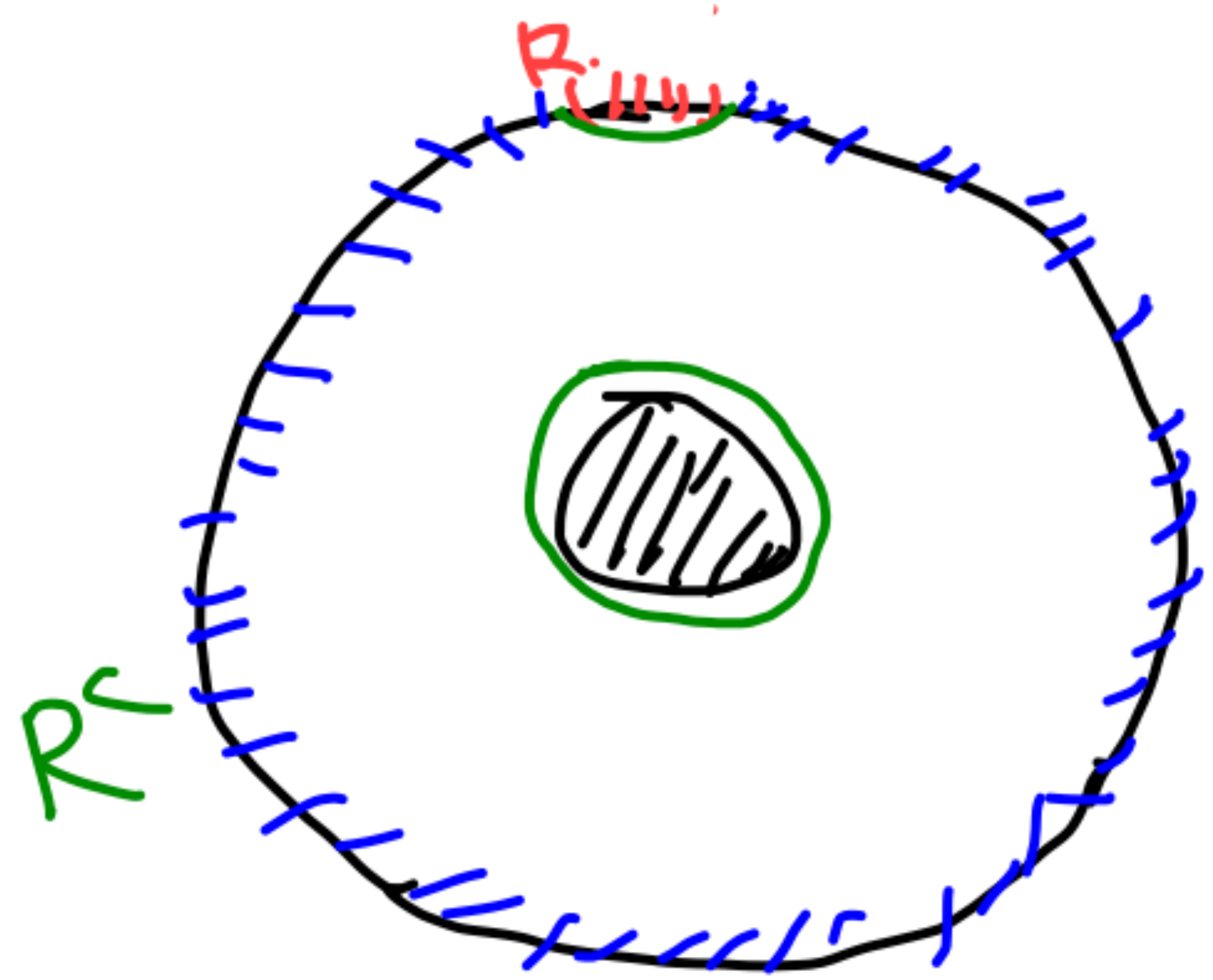
somewhat ill-defined due to UV divergence in entanglement entropy

Find W that minimizes S(W).

Min(S(W)) gives the entanglement entropy of R in the boundary.

arXiv: 1307.2892 Faulkner, Lewkowycz, Maldacena.

$$S(R^c) = \frac{\text{Area}(W)}{4G_N} + \frac{\text{Area}(\text{Horizon})}{4G_N} \simeq 0$$



Minimal surface W is small.

$$S(R) = \frac{\text{Area}(W)}{4G_N} \simeq 0.$$

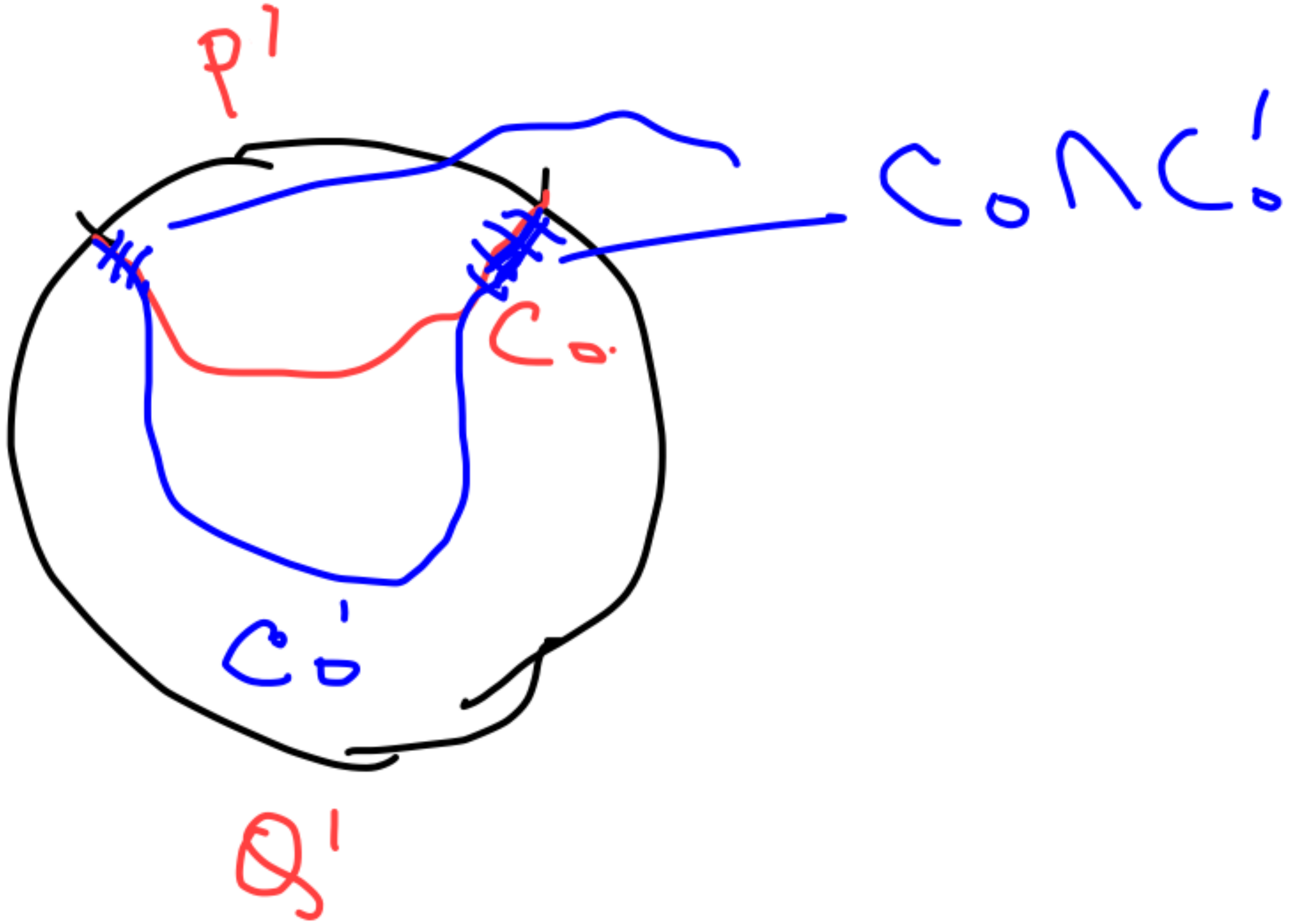
$$= \frac{\text{Area}(\text{Horizon})}{4G_N}$$

$\Rightarrow$ Boundary is in a mixed state (thermal state with $T = T_{BH}$)

So that $S(R^c)$ = Black hole entropy

The results that we saw are similar to the ones we found in the holographic code, except that for the holographic code most of the results are upper bounds instead of equality.

This problem was solved in (arXiv: 1601.01694) using random tensor network. ⇒ also shows that the information from the bulk region V is encoded in R for the minimal surface.



$$|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$$