Geometry & Entanglement

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AdS/CFT correspondence

AdS_{d+1} : $(d+1)$ dim. anti de Sitter space.
time.

→ described by embedding it in $(d+2)$ dimensional space.

$T_1, T_2, X_1, \ldots, X_d$ subject to

$$T_1^2 + T_2^2 - \sum_{i=1}^{d} X_i^2 = R^2$$ constant.

Metric in $(d+2)$-dim $ds^2 = -dT_1^2 - dT_2^2 + \sum_{i=1}^{d} dX_i^2$

→ Has $SO(d,2)$ isometry.

acts transitively → any point can be moved to any other point later.

— like the surface of a sphere S^2

$SO(3)$

Intrinsic coordinates on AdS_{d+1} : $t, r, \theta_1, \cdots \theta_{d-1}$

$T_1 = \sqrt{R^2+r^2}\cos t$ | $X_1 = r\cos\theta_1$, $X_2 = r\sin\theta_1\cos\theta_2$;

$T_2 = \sqrt{R^2+r^2}\sin t$. | .... $X_d = r\sin\theta_1 \cdots \sin\theta_{d-1}$

$$T_1^2 + T_2^2 - \underbrace{\sum_{i=1}^{d} X_i^2}_{r^2} = R^2$$

t, $t+2\pi$ describe same points in the original space-time.

In defining AdS_{d+1} we drop this identification.

$-\infty < t < \infty \Rightarrow$ universal cover.

Ex. $ds^2 = -(R^2+r^2)\, dt^2 + \dfrac{dr^2}{1+r^2/R^2} + r^2 d\Omega_{d-1}^2$

$d\Omega_{d-1}^2$: Metric on unit $(d-1)$ sphere S^{d-1}.

L. parameters $\theta_1, \dots, \theta_{d-1}$.

Slightly different coordinate system:

$r = R\tan\rho$ $\quad 0 \le \rho \le \frac{\pi}{2}$ as $0 \le r < \infty$

$ds^2 = R^2 \sec^2\rho \left(-dt^2 + d\rho^2 + \sin^2\rho \, d\Omega_{d-1}^2 \right)$

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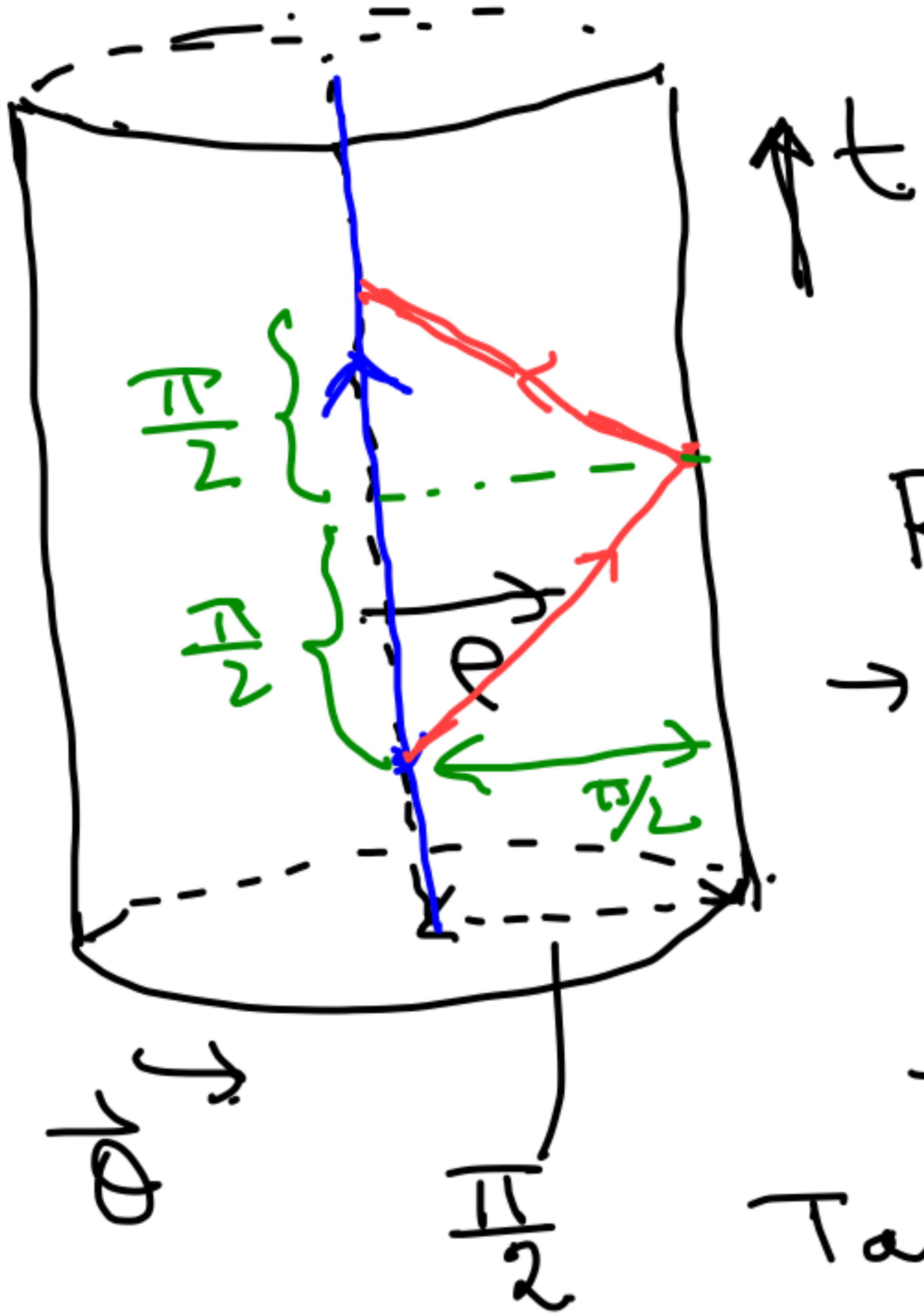

Light travels along null geodesic

$ds^2 = 0$

Radial null geodesic

$\rightarrow$ no motion in angular direction $\Rightarrow d\Omega_{d-1} = 0$

$$-dt^2 + d\rho^2 = 0 \Rightarrow \rho = \pm t + C$$

Take an observer at $\rho = 0$.

Light ray comes back in time π even though the boundary is ∞ distance away.

AdS_{d+1} metric satisfies.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \, T_{\mu\nu}.$$

$$\frac{d(d-1)}{8\pi G R^2} g_{\mu\nu}.$$

-ve cosmological constant.

Quantization of a scalar field in AdS_{d+1}:

$$S = -\frac{1}{2}\int d^{d+1}x \sqrt{-\det g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right).$$

$\Rightarrow$ Eq. of motion $(\Box - m^2)\phi = 0$

$$\Box\phi = D^\mu D_\mu\phi = \frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu\phi\right)$$

Basis of classical sol^n^s (analog of plane waves).

$$f_{n\ell\vec{m}}(t, r, \vec{\theta}) = \psi_{n\ell}(r)\, Y_{\ell\vec{m}}(\vec{\theta})\, e^{-i\omega_{n\ell}t}$$

$n, \ell \geq 0$ integers.

& $f^*_{n\ell\vec{m}}$:

$\psi_{n\ell}$: known.

$Y_{\ell\vec{m}}$: $(d-1)$ dim spherical harmonics

$\vec{m}$: collection of integers.

$$\omega_{n\ell} = \Delta + \ell + 2n \,, \quad \Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2+4m^2R^2}$$

Note: Discrete spectrum.

The metric on AdS_{d+1} provides a kind of attractive potential

→ responsible for quantization.

Large r behaviour of the wave fn:

$$\psi_{n\ell} \sim r^{-\left(\frac{d}{2}+\frac{1}{2}\sqrt{d^2+4m^2R^2}\right)}$$
$$\sim r^{-\Delta}$$

Other soln: discarded

$$r^{-\left(\frac{d}{2}-\frac{1}{2}\sqrt{d^2+4m^2R^2}\right)}$$

→ reason why ω is quantized.

Inner product of h, f.

$$(h,f) = i \int_\Sigma d^d x \sqrt{\det \gamma}\; n^\mu \left(h^* \partial_\mu f - \partial_\mu h^* f \right)$$

induced metric on Σ

normal to Σ

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

① (h,f) does not depend on how we fill the interior of Σ

② If h, f are normalizable, then (h,f) does not depend on Σ at all.

For $t = c$ slices, (h,f) is indep. of c.

$$(f_{n\ell\vec{m}}, f_{n'\ell'\vec{m}'}) = \delta_{nn'}\,\delta_{\ell\ell'}\,\delta_{\vec{m}\vec{m}'}$$

Any soln. of eom can be expanded as

$$\phi = \sum_{n,\ell,\vec{m}} \left(a_{n\ell\vec{m}}\, f_{n\ell\vec{m}}(t,r,\vec{\theta}) + a^*_{n\ell\vec{m}}\, f^*_{n\ell\vec{m}}(t,r,\vec{\theta}) \right)$$

constants

$\hookrightarrow a^\dagger_{n\ell\vec{m}}$

<u>Quantization</u> ① Regard $a_{n\ell\vec{m}}$ as operator

② Impose commutation relation. $\hbar = 1$

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\,\delta^{(d)}(\vec{x}-\vec{y})$$

$\hookrightarrow r, \vec{\theta}$

$\longrightarrow \delta L/\delta(\partial_0\phi) = \sqrt{-\det g}\; g^{00}\partial_0\phi$

$$[a_{n\ell\vec{m}}, a^{\dagger}_{n'\ell'\vec{m}'}] = \delta_{nn'}\,\delta_{\ell\ell'}\,\delta_{\vec{m}\vec{m}'}$$

$$[a_{n\ell\vec{m}}, a_{n'\ell'\vec{m}'}] = 0 \quad \text{etc.}$$

Spectrum Define $|0\rangle$ such that $a_{n\ell\vec{m}}|0\rangle = 0$

Excited states: $a^{\dagger}_{n\ell\vec{m}}|0\rangle$, $a^{\dagger}_{n\ell\vec{m}}\, a^{\dagger}_{n'\ell'\vec{m}'}|0\rangle$, - - - .

$E = \omega_{n\ell}$

$E = \omega_{n\ell} + \omega_{n'\ell'}$

Full $SO(d,2)$ symmetry is not manifest in this formalism.

Manifest symmetry: ① time translation
② $SO(d)$ rotation

→ Similar to Lorentz trs. in QFT in flat space-time.

$SO(d,2)$ relates all single particle states
→ like Lorentz trs. in Minkowski space.

Correlation functions.

$$\langle \prod_i \phi(x_i) \rangle = \langle 0 | \underbrace{T}_{\text{time ordering.}} \left(\prod_i \phi(x_i) \right) | 0 \rangle$$

$$\phi(t, r, \vec{\theta}) = \sum_{n l \vec{m}} \left(f_{n l \vec{m}} \, a_{n l \vec{m}} + h.c. \right)$$

$$a_{n l \vec{m}} | 0 \rangle = 0 \, , \quad \langle 0 | a^{\dagger}_{n l \vec{m}} = 0$$

$$\langle \prod_i \phi_i(x_i) \rangle \rightarrow r_i^{-\Delta} \quad \text{as } r_i \rightarrow \infty$$

Boundary correlation fn..

$$\left\langle \prod_i \phi(x_i) \right\rangle_B = \lim_{\substack{r_i \to \infty \\ \forall i}} \prod_i r_i^{+\Delta} \left\langle \prod_i \phi(x_i) \right\rangle$$

$t_i, \vec{\theta}_i$

$\left\langle \prod_i \phi(x_i) \right\rangle$ is invariant under $SO(d,2)$

→ not manifest in the canonical formulation

→ can be seen using a path integral description of the correlation fn..

$\langle \prod_i \phi(x_i) \rangle_B$ is not invariant under $so(d,2)$.

Reason: r transforms under $so(d,2)$

For large r, $r \to f(t, \vec{\theta}, \text{trs. par.})\, r$

$\langle \prod_i \phi(x_i) \rangle_B$ transform covariantly under $so(d,2)$.

Generalizes to multiple scalar fields of different masses, higher spin fields (vector, tensor, ...)

Interacting theories may be analyzed using perturbation theory assuming that the interactions are small.

In the full quantum gravity theory we do not expect $\langle \prod_i \phi(x_i) \rangle$ to be well defined but $\langle \prod_i \phi(X_i) \rangle_B$ are okay.

$\Leftrightarrow$ no local observables in quantum gravity since under metric fluctuation space-time points lose their identity.

AdS/CFT correspondence.

The boundary correlation fn.s in a quantum gravity theory on AdS_{d+1} correspond to correlation fn.s of a d-dimensional CFT (no gravity) at the boundary of AdS_{d+1}

for the coordinate system we have chosen

$$\mathbb{R} \times S^{d-1}$$

t $\qquad$ $\theta_1, \dots, \theta_{d-1}$

We need to understand the meaning of CFT_d:

Consider a QFT in d space-time dim.

has energy-momentum tensor $T_{\mu\nu}$.

① $\int d^{d-1}x\, T^{0\mu}$: μ-th component of the field momentum.

② If we vary the background metric $g_{\mu\nu}$, by $\delta g_{\mu\nu}$, then $\delta S \propto \int d^dx\, \delta g^{\mu\nu}(x)\, T_{\mu\nu}(x)$,

$\Rightarrow \langle \prod_i O_i(\vec{x}_i) \rangle_{g+\delta g} - \langle \prod_i O_i(x_i) \rangle_g \propto \int d^dx \langle \prod_i O_i(x_i)\, \delta g^{\mu\nu}(x)\, T_{\mu\nu}(x) \rangle$

CFT_d in flat space-time $g_{\mu\nu} = \eta_{\mu\nu}$.

Special class of QFT_d for which $\eta^{\mu\nu} T_{\mu\nu} = 0$.

In general background metric:

$$\boxed{g^{\mu\nu} T_{\mu\nu} = 0.}$$ (not quite true)

$$\Rightarrow \left\langle \prod_i O_i(x) \right\rangle_g = \left\langle \prod_i O_i(x) \right\rangle_{g+\delta g}$$

$g + \delta g \cong (1+2\Omega) g$

infinitesimal trs.

Under $\delta g_{\mu\nu} = 2\Omega g_{\mu\nu}$, $\delta g^{\mu\nu} = -2\Omega g^{\mu\nu}$

$\delta g^{\mu\nu} T_{\mu\nu} = -2\Omega g^{\mu\nu} T_{\mu\nu} = 0$

$$\left\langle \prod_i O_i(x_i)\right\rangle_g = \left\langle \prod_i O_i(x_i)\right\rangle_{(1+2\Omega(x))g}$$

infinitesimal

$$\Rightarrow \left\langle \prod_i O_i(x_i)\right\rangle_g = \left\langle \prod_i O_i(x_i)\right\rangle_{e^{2\Omega}g}$$

finite version

$\Omega(x)$ is any fn.

subtleties

$$\left\langle \prod_i O'_i(x_i)\right\rangle_{e^{2\Omega}g} = \left\langle \prod_i O_i(x_i)\right\rangle_g \times \exp\left(A[g_{\mu\nu},\Omega]\right)$$

for primary ops.

$$O'_i(x_i) = e^{\Delta_i \Omega(x_i)}\, O_i(x_i)$$

$\hookrightarrow$ Known.

- Drops out of normalized corrln. fn.