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Suppose we have $R(x)$ $\tilde{R}(y)$

Is there $y \overset{?}{=} f(x)$ such that

$\tilde{R}(f(x)) = R(x)$

Many of choices of $f(x)$ possible → It doesn't fix $f(x)$ uniquely

If R is a const., we can't do this, as in the case of sphere.

so

Look at other quantities :-

$R_{ij} R^{ij}$, $\tilde{R}_{ij} \tilde{R}^{ij}(y)$

We must have $R^{ij}(x) R_{ij}(x) = \tilde{R}_{ij} \tilde{R}^{ij}(f(x))$

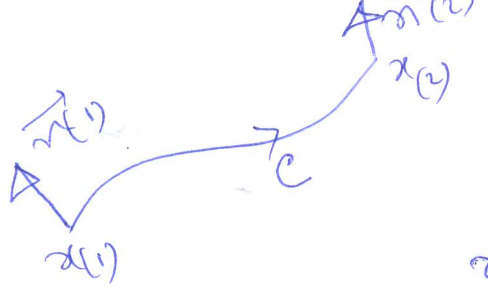
It gives another condn on $f(x)$

In 2-D it fixes the coord. trs.

Look at a 3rd scalar:-

If $R_{ijkl}(x) R^{ijkl}(x) = \tilde{R}_{ijkl} \tilde{R}^{ijkl}(f(x))$

Now it's a consistency check & we have no freedom in choosing $f(x)$



$$\frac{dn^i(u)}{du} + \Gamma^i_{lk}(x(u)) \frac{dx^l}{du} n^k(u) = 0$$

$n^i(u=0) = n^i_{(1)}$

$n^i(u=1) = n^i_{(2)}$

$\{x^i(u)\}$: Path connecting $x_{(1)}$ to $x_{(2)}$
$0 \le u \le 1$

$n_{(2)} = M(x_{(1)}, x_{(2)}, C)\, n_{(1)}$

2) Rules for $\parallel$ transport are reversible
(Clearly, if $\vec{n} \parallel \vec{m}$, $\vec{m} \parallel \vec{n}$ in the local coord. sys.)

changing u to $(1-u)$ $\quad \frac{dn^i(1-u)}{du} + \Gamma^i_{lk}\left(x(1-u)\right)\frac{dx^l}{du}\, n^k(1-u) = 0$

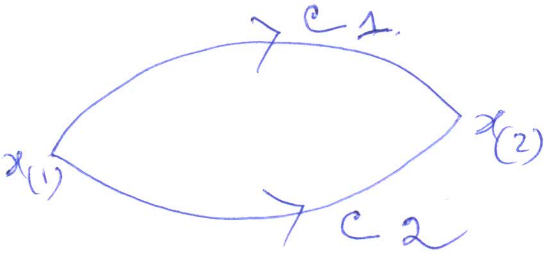
[follows from $\frac{dn^i(u)}{du} + \Gamma^i_{lk}(x(u))\frac{dx^l}{du}\, n^k(u) = 0$]

[The opp. happens in Newtonian mech. in presence of friction — $\frac{d^2x}{dt^2}$ & $\frac{dx}{dt}$ pick up opp. sign on reversal]

$$\left.\begin{aligned} n^i(1-u) &= n^i_{(2)} \quad \text{at } u=0 \\ &= n^i_{(1)} \quad \text{at } u=1 \end{aligned}\right\} \rightarrow \text{B.C.}$$

$$M(x_{(2)}, x_{(1)}; \overset{\text{path reversed}}{-C})\, M(x_{(1)}, x_{(2)}; C) = \mathbb{1}$$

$$M(x_{(2)}, x_{(1)}; -C) = \left(M(x_{(1)}, x_{(2)}, C)\right)^{-1}$$



$M(x_{(1)}, x_{(2)}, C_1) \overset{?}{=} M(x_{(1)}, x_{(2)}, C_2)$

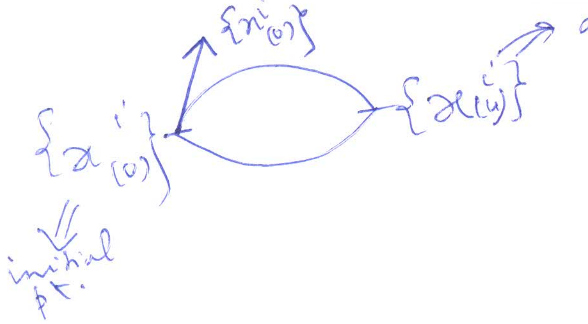
(Under what condns can these be equal?)

If they're equal, then $\quad M(x_{(1)}, x_{(2)}, C_2)^{-1}\left(M(x_{(1)}, x_{(2)}, C_1)\right) = \mathbb{1}$

$$\Rightarrow M(x_{(1)}, x_{(2)}, -C_2)\, M(x_{(2)}, x_{(1)}, +C_1) = \mathbb{1}$$

So equivalent question :- Under what condition $\parallel$ transport along the closed curve $C_1 - C_2$ brings a vector to the same vector?

We'll study this in the context of small loops.



length of the loop is

$$\int du \sqrt{g_{ij}(x(u)) \frac{dx^i}{du} \frac{dx^j}{du}}$$

$$\sim \epsilon$$

$\frac{dx^i}{du}$ should be small to give the length $\sim \epsilon$

[u is an irrelevant parameter]

so we'll take the range of u : $(0, \epsilon)$

(then $dx^i/du \sim 1$)

$\therefore \quad x^i(0) = x^i_{(0)}$

$$x^i(\epsilon) = x^i_{(0)} \qquad \frac{dx^i(u)}{du} \sim 1$$

(It's just a matter of convention to keep track of factors of ϵ)

[We will try to do the calculation of $n^i(u)$ upto $O(\epsilon^2)$]

$$\frac{dn^i}{du} = -\Gamma^i_{jk}(x(u))\, n^j(u) \frac{dx^k}{du}$$

$n^i = n^i_{(0)}$ at $u = 0$

$x^i(u) = x^i_{(0)}$ at $u = 0, \epsilon$

Goal: Find $n^i(u)$ to $O(\epsilon^2)$ & then evaluate $n^i(\epsilon)$ (by solving the above Diff. Eqn.).

(We will solve it iteratively) $\rightarrow$

<u>1st iteration</u> :-

$$\frac{dn^i}{du} = -\Gamma^i_{jk}(x^{(0)})\, n^j(0) \frac{dx^k(u)}{du} + O(\epsilon)$$

[Int. over u, with range $\sim \epsilon$, will give an error of $O(\epsilon^2)$]

$$\therefore\ \eta^i(u) = \eta^i_{(0)} - \Gamma^i_{jk}(x_{(0)})\,\eta^j_{(0)}\left(x^k(u) - x^k_{(0)}\right) + O(\epsilon^2)$$

<u>Second iteration</u> :-

$$\frac{d\eta^i}{du} = -\Gamma^i_{jk}(x(u))\,\eta^j(u)\,\frac{dx^k(u)}{du}$$

$$= -\left\{\Gamma^i_{jk}(x_{(0)}) + \partial_l \Gamma^i_{jk}(x_{(0)})\left(x^l(u) - x^l_{(0)}\right) + O(\epsilon^2)\right\}$$

$$\left\{\eta^j_{(0)} - \Gamma^j_{st}\,\eta^s_{(0)}\left(x^t(u) - x^t_{(0)}\right) + O(\epsilon^2)\right\}$$

$$\times\, dx^k/du$$

$$= -\Gamma^i_{jk}(x_{(0)})\,\eta^j_{(0)}\,\frac{dx^k}{du} - \partial_l \Gamma^i_{jk}(x_{(0)}) \times \left(x^l(u) - x^l_{(0)}\right)\frac{dx^k}{du}\,\eta^j_{(0)}$$

$$+ \Gamma^i_{jk}(x_{(0)})\,\Gamma^j_{st}(x_{(0)})\,\eta^s_{(0)}\left(x^t(u)\ x^t_{(0)}\right)\frac{dx^k}{du}$$

$$+ O(\epsilon^2)$$

$$= -\Gamma^i_{jk}(x_{(0)})\,\eta_0^j\,\frac{dx^k}{du} - \left\{\partial_t \Gamma^i_{sk}(x_{(0)}) - \Gamma^i_{jk}(x_{(0)})\,\Gamma^j_{st}(x_{(0)})\right\}$$

$$\times\, \eta^s_{(0)} \times \left(x^t(u) - x^t_{(0)}\right)\frac{dx^k}{du}$$

$$+ O(\epsilon^2)$$

$$\frac{d}{du}\left(x^k(u) - x^k_{(0)}\right)$$

Integrating, we get,

$$\eta^i(u) = \eta^i_{(0)} - \Gamma^i_{jk}(x_{(0)})\,\eta^j_{(0)}\left(x^k(u) - x^k_{(0)}\right)$$

$$- \left\{\partial_t \Gamma^i_{sk}(x_{(0)}) - \Gamma^i_{jk}(x_{(0)})\,\Gamma^j_{st}(x_{(0)})\right\}\eta^s_{(0)} \times \int_0^u \left(x^t(u') - x^t_{(0)}\right)\frac{d\left(x^k(u') - x^k_{(0)}\right)}{du'}$$

$$+ O(\epsilon^3)$$

Set $u = \epsilon$ $\Rightarrow \left[\left(x^{k}(\epsilon) - x^{k}_{(0)}\right) = x^{k}_{(0)} - x^{k}_{(0)} = 0 \right]$

Then we get,

$$x^{i}(\epsilon) - x^{i}_{(0)} = -\left\{ \partial_t \Gamma^{i}_{sk}(x_{(0)}) - \Gamma^{i}_{jk}(x_{(0)}) \Gamma^{j}_{st}(x_{(0)}) \right\}$$
$$x^{s}_{(0)} \underbrace{\oint \left(x^{t}(u') - x^{t}_{(0)}\right) \frac{d}{du'} \left(x^{k}(u') - x^{k}_{(0)}\right)}$$

antisymm. in t & k (bcos we can int. by parts & boundary terms vanish for this closed loop)

$$= -\frac{1}{2} \left\{ \partial_t \Gamma^{i}_{sk}(x_{(0)}) - \Gamma^{i}_{jk}(x_{(0)}) \Gamma^{j}_{st}(x_{(0)}) - \partial_k \Gamma^{i}_{st}(x_{(0)}) + \Gamma^{i}_{jt}(x_{(0)}) \Gamma^{j}_{sk}(x_{(0)}) \right\}$$

$\Rightarrow$ $\boxed{\text{using the antisym. of the integral}}$

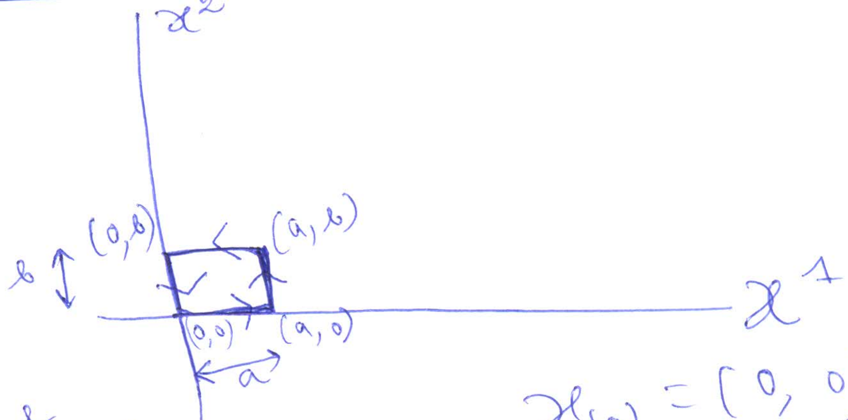
$$\times x^{s}_{(0)} \oint \left(x^{t}(u') - x^{t}_{(0)}\right) \frac{d}{du'} \left(x^{k}(u') - x^{k}_{(0)}\right)$$

$$= -\frac{1}{2} R^{i}_{\ skt}(x_{(0)}) \, x^{s}_{(0)} \underbrace{\oint \left(x^{t}(u') - x^{t}_{(0)}\right) \times \frac{d}{du'} \left(x^{k}(u') - x^{k}_{(0)}\right)}$$

in general non-zero

[geom. interpretation $\Rightarrow$ area of the projection the 2-dim. plane planned by x^t & x^k]

e.g. :-



only 1 indep. comp. in the 2-d plane bcos of antisym.

$x_{(0)} = (0, 0)$

Take $t = 1$, $k = 2$

$$\oint x^1(u') \frac{d}{du'} x^2(u')$$ ⇒ Divide into 4 parts

$$= 0 + ab + 0 + 0$$

$$= ab = \text{area of the rectangle}$$

Start with a coord. sys. which lies along the plane of the loop, ~~~~ then it is the area of the loop — now go to a general coord. system & we will see it'll be the projection of the area in the x^i-x^k plane

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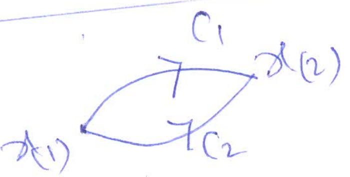
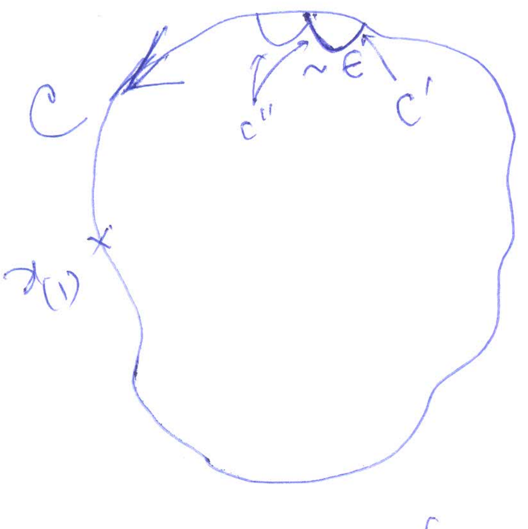


Let the Riemann tensor be zero.

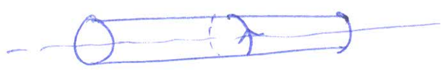
$\int E \cdot dl$
$= \int \nabla \times E \cdot dS$

Then, $M(x_{(1)}, x_{(2)}; C_1)$ ~~~~ & $M(x_{(1)}, x_{(2)}; C_2)$ differ by ~~~~ at most $O(\epsilon^3)$ terms.



loop of area of order 1

We assume this loop is covered with an area — not all loops are like this ⇒ e.g.



$$M(x_{(1)}, x_{(1)}, C) = M(x_{(1)}, x_{(1)}, C') + O(\epsilon^3)$$

[we have reduced the area of the loop by $\sim\epsilon^2$]

$$M(x_{(1)}, x_{(1)}, C') = M(x_{(1)}, x_{(1)}, C'') + O(\epsilon^3)$$

(In $\frac{1}{\epsilon^2}$ steps, we will get zero area)

In $\sim \frac{1}{\epsilon^2}$ steps we can make the final loop to a point.

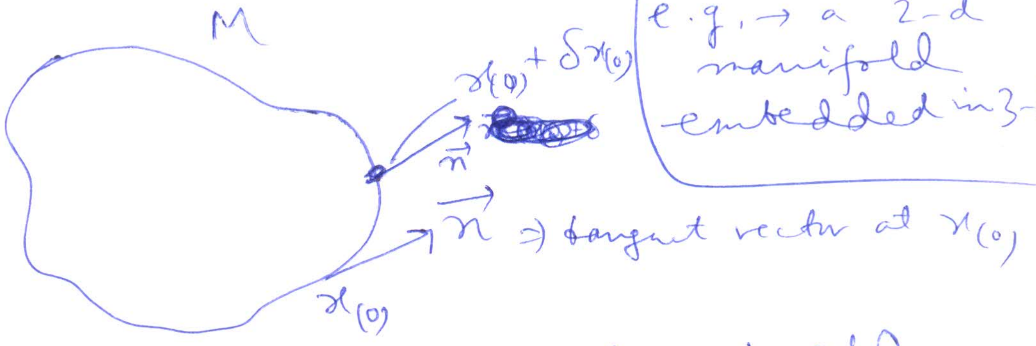
$M = I$ for the final loop

(total error $\Rightarrow O(\epsilon^3/\epsilon^2) = O(\epsilon)$)

$$M(x_{(1)}, x_{(2)}, C) \sim \frac{1}{\epsilon^2} \times \epsilon^3 + I$$
$$= I \text{ in } \epsilon \to 0 \text{ limit}$$

[So || trs. produces the same result along a closed loop if the Riemann tensor is zero]

Notion of || trs. can be defined as an intrinsic prop. of the manifold, but if we can embed the manifold in a higher dim. space, we can give || trs. a diff. interpretation.



(e.g., $\to$ a 2-d manifold embedded in 3-d)

Think of $\vec{n}$ as a vector in the embedding space (3-dim. say)

(At $x_{(0)} + \delta x_{(0)}$, $\vec{n}$ may no longer be the tangent)

Project $\vec{n}$ at the tangent plane at $x_{(0)} + \delta x_{(0)}$

$\Rightarrow$ vector in the tangent space at $x_{(0)} + \delta x_{(0)}$

(2-dim. if M is 2-dim.)

Claim :- This notion of ‖ trs. as of trs. the vector in the embedding space coincides with our previous (intrinsic) defn. of ‖ trs.

Steps :-

Take $\{x^i\}$: coordinates on M

$\{y^\mu\}$: coordinates of the embedding space

M is described by a set of functions:

$y^\mu = f^\mu(x)$

Example :- 2-d sphere in 3-dim

$y^1 = a \sin\theta \cos\phi$
$y^2 = a \sin\theta \sin\phi$
$y^3 = a \cos\theta$

If M has a wrong signature we have to embed it in the Minkowski space

Take a pt. $x_{(0)}$ on M & a tangent vector $\{n^i\}$ at $x_{(0)}$

(The following is assuming Euclidean geometry!)

① The vector n^i at $x_{(0)}$ corresponds to the vector

$$N^\mu = n^i(x_{(0)}) \left.\frac{\partial f^\mu}{\partial x^i}\right|_{x_{(0)}}$$

in the embedding space.

② Take the same vector & regard this as a vector at

$$\left\{ y^\mu_{(0)} + \delta y^\mu_{(0)} = f^\mu\left(x_{(0)} + \delta x_{(0)}\right)\right\}$$

Use the notion of ‖ trs. in the Euclidean embedding space

(The embedding space may be 1 or more dim. higher)

$\boxed{\text{Aside}}$

Show that $g_{ij}(x) = \frac{\partial f^{\mu}}{\partial x^{i}} \frac{\partial f^{\nu}}{\partial x^{j}} \delta_{\mu\nu}$

metric induced on the lower dim. surface of the manifold

$\Downarrow$ Euclidean metric

Show that

$$\tilde{N}^{\mu} = n^{i}(x_{(0)} + \delta x_{(0)}) \left.\frac{\partial f^{\mu}}{\partial x^{i}}\right|_{x_{(0)}+\delta x_{(0)}}$$

where $n^{i}(x_{(0)} + \delta x_{(0)})$ is the $\parallel$ transport of $n^{i}(x_{(0)})$ with the metric g_{ij}.

We are to show that $N^{\mu} = \tilde{N}^{\mu}$

Q.E.D.

H/W

$$\Gamma^{i}_{lk} = \frac{1}{2} g^{is}\left[\frac{\partial g_{ls}}{\partial x^{k}} + \frac{\partial g_{ks}}{\partial x^{l}} - \frac{\partial g_{lk}}{\partial x^{s}}\right]$$

$$= \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\frac{\partial x^{s}}{\partial f_{\mu}}\left[\frac{\partial^2 f^{\nu}}{\partial x^{k}\partial x^{l}}\frac{\partial f_{\nu}}{\partial x^{s}} + \frac{\partial f^{\nu}}{\partial x^{l}}\frac{\partial^2 f_{\nu}}{\partial x^{k}\partial x^{s}} + \frac{\partial^2 f^{\nu}}{\partial x^{l}\partial x^{k}}\frac{\partial f_{\nu}}{\partial x^{s}} + \frac{\partial f^{\nu}}{\partial x^{k}}\frac{\partial^2 f_{\nu}}{\partial x^{l}\partial x^{s}} - \frac{\partial^2 f^{\nu}}{\partial x^{s}\partial x^{l}}\frac{\partial f_{\nu}}{\partial x^{k}} - \frac{\partial f^{\nu}}{\partial x^{l}}\frac{\partial^2 f_{\nu}}{\partial x^{s}\partial x^{k}}\right]$$

$$= \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\frac{\partial^2 f^{\mu}}{\partial x^{k}\partial x^{l}} + \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\cancel{\frac{\partial^2 f_{\nu}}{\partial x_{\mu}\partial x^{k}}}^{\to 0}\frac{\partial f^{\nu}}{\partial x^{l}}$$

$$+ \frac{1}{2}\cancel{\frac{\partial x^{i}}{\partial f^{\mu}}}\cancel{\frac{\partial^2 f_{\mu}}{\partial x^{l}\partial x^{k}}} + \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\cancel{\frac{\partial^2 f_{\nu}}{\partial f_{\mu}\partial x^{l}}}^{\to 0}\frac{\partial f^{\nu}}{\partial x^{\mu}}$$

$$- \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\cancel{\frac{\partial^2 f^{\nu}}{\partial f^{\mu}\partial x^{l}}}^{\to 0}\frac{\partial f_{\nu}}{\partial x^{k}} - \frac{1}{2}\frac{\partial x^{i}}{\partial f^{\mu}}\cancel{\frac{\partial^2 f_{\nu}}{\partial f_{\mu}\partial x^{k}}}^{\to 0}$$

$$= \frac{\partial x^{i}}{\partial f^{\mu}}\frac{\partial^2 f^{\mu}}{\partial x^{l}\partial x^{k}}$$

Now, $\delta n^i_{(0)} = -\Gamma^i_{lk}(x_{(0)}) n^k_{(0)} \, \delta x^l_{(0)}$

$$\Rightarrow n^i(x_{(0)} + \delta x_{(0)}) = n^i(x_{(0)}) - \frac{\partial x^i}{\partial f^\mu} \frac{\partial^2 f^\mu}{\partial x^l \partial x^k} n^k \delta x^l_{(0)}$$

$$\therefore N'^\mu \equiv n^i(x_{(0)} + \delta x_{(0)}) \left[\frac{\partial f^\nu}{\partial x^i_{(0)}} + \frac{\partial^2 f^\nu}{\partial x^i \partial x^s}\bigg|_{x_{(0)}} \delta x^s_{(0)} \right]$$

$$= n^i(x_{(0)}) \frac{\partial f^\nu}{\partial x^i_{(0)}} + n^i(x_{(0)}) \frac{\partial^2 f^\nu}{\partial x^i \partial x^s}\bigg|_{x_{(0)}} \delta x^s_{(0)}$$

$$- \underline{\frac{\partial x^i}{\partial f^\mu}} \frac{\partial^2 f^\mu}{\partial x^l \partial x^k} n^k \delta x^l_{(0)} \underline{\frac{\partial f^\nu}{\partial x^i_{(0)}}} + O(\delta x^2_{(0)})$$

δ^ν_μ

$$= n^i(x_{(0)}) \frac{\partial f^\nu}{\partial x^i_{(0)}} + \cancel{n^i(x_{(0)}) \frac{\partial^2 f^\nu}{\partial x^i \partial x^s}\bigg|_{x_{(0)}} \delta x^s_{(0)}}$$

$$- \cancel{n^i(x_{(0)}) \frac{\partial^2 f^\nu}{\partial x^l \partial x^i}\bigg|_{x_{(0)}} \delta x^l_{(0)}} + O(\delta x^2)$$

$$= N^\mu + O(\delta x^2)$$

Note

$$ds^2 = \delta_{\mu\nu} \, dy^\mu \, dy^\nu$$

$$= \underbrace{\left[\delta_{\mu\nu} \frac{\partial f^\mu}{\partial x^i} \frac{\partial f^\nu}{\partial x^j} \right]}_{\substack{\shortparallel \\ g_{ij}}} dx^i \, dx^j$$

$\boxed{N^\mu = \frac{dy^\mu}{du}}$

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$$D_i\left(A^{i_1\ldots i_p}{}_{j_1\ldots j_q}\, B^{k_1\ldots k_{p'}}{}_{l_1\ldots l_{q'}}\right)$$

$$= D_i\left(A^{i_1\ldots i_p}{}_{j_1\ldots j_q}\right) B^{k_1\ldots k_{p'}}{}_{l_1\ldots l_{q'}}$$

$$+ A^{i_1\ldots i_p}{}_{j_1\ldots j_q}\, D_i\left(B^{k_1\ldots k_{p'}}{}_{l_1\ldots l_{q'}}\right)$$

[Identities involving tensors can be proved by going to a local coord. sys. where $\Gamma^i{}_{jk} = 0$, but not its derivative; then cov. or contravar. deriv. reduces to ordinary deriv.]

Other identities

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

Bianchi identity

$$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0$$

↓ multiply by g^{ik} to get

$$\Rightarrow D_s\left(g^{ik} R_{ijkl}\right) + g^{ik} D_k R_{ijls} + D_l\left(g^{ik} R_{ijsk}\right) = 0$$

$$\Rightarrow D_s\left(R_{jl}\right) + g^{ik} D_k R_{ijls} - D_l R_{js} = 0$$

↓ multiply by g^{js}

$$\Rightarrow D_s\left(g^{js} R_{jl}\right) + g^{ik} D_k \Big(\underbrace{g^{js} R_{ijls}}_{\rightsquigarrow R_{il}}\Big) - D_l R = 0$$

$$\Rightarrow D_k\left(g^{jk} R_{jl}\right) + D_k\left(g^{ik} R_{il}\right) - D_k\left(\delta^k_l R\right) = 0$$

$$\Rightarrow 2 D_k\left(g^{jk}\left(R_{jl} - \tfrac{1}{2} g_{jl} R\right)\right) = 0$$ → like a cov. divergence

Einstein tensor $\Rightarrow R_{jl} - \frac{1}{2} g_{jl} R$

[The same formalism can be carried out for a metric of any signature]

e.g. $\rightarrow$

$$\begin{pmatrix} \underbrace{-1 \;\; -1 \;\; \cdots \;\; -1}_{k_1} & \\ & \underbrace{1 \;\; 1 \;\; \cdots \;\; 1}_{k_2} \end{pmatrix} = L$$

metric of k_1 -ve & k_2 +ve signature $\xleftarrow{\text{implies}}$

$$\boxed{M L M^{+} = L}$$

$$ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^3)^2 \qquad \boxed{x^0 = ct}$$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

standard Minkowski space-time $\Rightarrow$ where $\eta = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Proper time $d\tau^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu$

Generalize to a Riemannian space:-

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad \text{where } \mu, \nu = 0, 1, 2, 3$$

[Does this have a physical significance? Here we have replaced Minkowski $\eta_{\mu\nu}$ to $g_{\mu\nu}(x)$]

<u>Claim</u> :- This describes the theory of gravity.

(basic postulate of GTR)

(This will be a generalization of Newtonian theory of gravity.)

Question ① How does a particle move for a given $g_{\mu\nu}(x)$?

Question ② What rule tells us what kind of $g_{\mu\nu}$ is produced in the presence of matter?

In special relativity, free particle trajectories are straight lines.

Massive particle must have $ds^2 < 0$.

Massless particle " " $ds^2 = 0$.

[st. lines with $ds^2 > 0$ don't describe free particle trajectories.]

In general metric, a particle moves along a geodesic.

Geodesics are generalisations of st. lines in curved spacetime

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} = 0$$

this is an assumption

this is a postulate of GTR

We'll see it gives us the principle of equivalence.

Massive :-

$$g_{\mu\nu}(x) \frac{dx^\mu}{du} \frac{dx^\nu}{du} < 0$$

Massless :-

$$g_{\mu\nu}(x) \frac{dx^\mu}{du} \frac{dx^\nu}{du} = 0$$

These are the restrictions

We must show that in the appropriate limit these reproduce Newtonian eqns.

Newtonian gravity

① Non-relativistic limit: velocities small

$$\left|\frac{dx^i}{du}\right| << \left|\frac{dx^0}{du}\right|$$

If we parametrise with u

② Weak gravitational field:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}(x)}_{\text{small}}$$

③ Time dependence of $h_{\mu\nu}(x)$ is slow. To first approximation we can ignore $\partial_0 h_{\mu\nu}(x)$.

Newton didn't say that it should be time-indep. – but we will see New. law holds in this case

[This is just one aspect – how a particle feels gravity. We must also check the other aspect → how a particle produces a grav. field → Then only we'll have proven the Newtonian limit]

Geodesic eqn:

$$\frac{d^2x^\mu}{du^2} + \underbrace{\Gamma^\mu_{00}\left(\frac{dx^0}{du}\right)^2}_{\text{leading contrib. comes from this}} \simeq 0 \quad \text{(acc. to ①)}$$

Now, $\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \underbrace{\partial_\nu g_{00}}_{\text{dominant term acc. to ③}})$

$$= -\frac{1}{2} g^{\mu\nu}\partial_\nu g_{00}$$

$$g = \eta + h = \eta(1 + \eta^{-1}h)$$

$$\Rightarrow g^{-1} = (1+\eta^{-1}h)^{-1}\eta^{-1}$$

$$= (1-\eta^{-1}h)\eta^{-1}$$

$$\therefore \; g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\rho} h_{\rho\sigma} \eta^{\sigma\nu}$$

Hence, $\Gamma^{\mu}_{00} = -\frac{1}{2}\left(\eta^{\mu\nu} - \eta^{\mu\rho} h_{\rho\nu}\eta^{\sigma\nu}\right)\partial_\nu h_{00}$

$$\simeq -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} \quad [\text{using } ②]$$

$$\simeq -\frac{1}{2}\eta^{\mu i}\partial_i h_{00} \quad \text{using } ③]$$

$$\Rightarrow \Gamma^{0}_{00} \simeq 0 \;\; \& \;\; \Gamma^{j}_{00} = -\frac{1}{2}\underbrace{\eta^{ji}}_{\delta_{ji}}\partial_i h_{00} = -\frac{1}{2}\partial_j h_{00}$$

($\therefore$ we get the eqns $\Rightarrow$

$$\frac{d^2x^0}{du^2} = 0$$

&

$$\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{du}\right)^2 = 0$$

---

integrating

$$x^0 = \underbrace{a}_{\text{integration constant}} u + b \longrightarrow \text{set to } 0 \text{ by shifting } x^0$$

$\Rightarrow$ (origin of time)

Putting this in the 2^{nd} eqn, we get,

$$\frac{d^2x^i}{du^2} - \frac{1}{2}a^2\partial_i h_{00} = 0$$

Putting $u = \frac{ct}{a}$, we get,

$$\frac{a^2}{c^2}\frac{d^2x^i}{dt^2} - \frac{1}{2}a^2\partial_i h_{00} = 0$$

$$\Rightarrow \frac{d^2x^i}{dt^2} - \frac{1}{2}c^2\partial_i h_{00} = 0$$

(compare this with the Newton's eqn)

Newton's eqn. for the motion of a particle in the presence of ~~a~~ gravitational potential $\phi(\vec{x})$:-

$$m \frac{d^2x^i}{dt^2} = F = - m\, \partial_i \phi$$

$$\Rightarrow \frac{d^2x^i}{dt^2} = - \partial_i \phi$$

Comparing, we get, $\frac{1}{2} c^2 h_{00} = -\phi$

$$\Rightarrow h_{00} = -2\phi/c^2$$

[Newton's law didn't say ϕ should be small - but here in this language we see that even for ϕ of order 1, h_{00} is small, being suppressed by c^2]

[What about the other comp. of $h_{\mu\nu}$?
→ Even if they are non-zero, don't enter the eqns. of motion in Newtonian limit - they become relevant only if you go away from this limit - here there is no way to relate them - here, they have no meaning

e.g. → In electrostatics, vector pot. has no meaning]

$g_{\mu\nu}(x)$

$\times$ $x_{(0)}$

$$\frac{d^2x^\mu}{du^2} + \Gamma^{\mu}_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} = 0$$

Observer at $x_{(0)}$
→ I throw in diff. particles at diff. angles with diff. vel. - what kind of motion do they execute?

Suppose we have chosen a coordinate system such that $\Gamma^{\mu}_{\nu\rho}(x_{(0)}) = 0$ (Locally inertial frame)

we see → All of them obey $\dfrac{d^2x^\mu}{d\tau^2} = 0$

Suppose I don't know about general coordinate invariance

[Of course, if the exp. are carried out at diff. space pts. & diff. time, we won't get this result — Diff. from Minkowski spacetime can be seen only in this manner]

Principle of equivalence

[Won't be able to distinguish betn. presence or absence of grav. The eqn. $\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\dfrac{dx^\nu}{d\tau}\dfrac{dx^\rho}{d\tau} = 0$ can be attributed to a wrong choice of coord. sys. even in Minkowski spacetime — can't be distinguished from Minkowski spacetime
Gravity locally is equiv. to accln.
— We can make ourselves belive we are in flat spacetime by going to a diff. coord. sys.]

Principle of equivalence is not a consequence of invariance under general coordinate transformation

E.g. :-
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + R^\mu{}_\nu \frac{dx^\nu}{du}\, d\tau/du = 0$$

this eqn. is general coord. invariant

also inv. under reparametrisation of u → scaling of u

where $d\tau^2 = -g_{\mu\nu}\,dx^\mu dx^\nu$

We can't choose a coord. sys. where $R^\mu{}_\nu$ is zero — of course $\Gamma^{\mu}_{\nu\rho}$ can be ~~chosen~~ made to be "

(It violates principle of equivalence)
General coord. inv. is a symmetry principle

We try to see does geodesic eqn. follow from an underlying sym. principle?
→ We see that it doesn't follow from general coord. inv.

(Principle of equivalence says that we get the geodesic eqn. in a general coord. sys.)

But as happens in quantum theory, as it is not protected by any sym. principle, quantum corrections will make terms like $R^\mu{}_\nu \frac{dx^\nu}{du} \frac{dx}{du}$ appear, as they are allowed by general coord. inv.
Principle of equiv. is, most likely, violated in quantum th. — isn't a sacred principle

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There is an argument which shows that these correction terms will be small
→ Coeff. of these kind of terms will be very small as they'll involve powers of length, by dimensional argument bcos involve higher derivatives of $R^\mu{}_\nu$ metric — & we know that fundamental length scales of quantum mechanics are very small.

current exp. show correc. are very small

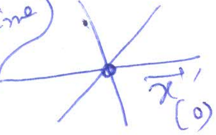
26/7/06 Suppose in the coordinate system $\{x'\}$:

$$g_{\mu\nu}(x'_{(0)}) = \eta_{\mu\nu}$$

$$\Gamma^{\mu}_{\nu\rho}(x'_{(0)}) = 0$$

Suppose

All expts. done at a single space time pt.



$$\frac{d^2 x'^{\mu}}{du^2} = 0$$ is the equation of motion in this coordinate system (when the particles through that space-time pt.)

Q) How does this equation look in a general coordinate system $\{x\}$?

$$x'^{\mu} = f^{\mu}(x)$$

$$x^{\mu} = \phi^{\mu}(x')$$

$\frac{d^2 x'^{\mu}}{du^2} = 0$ gives us

$$0 = \frac{d}{du}\left(\frac{dx'^{\mu}}{du}\right) = \frac{d}{du}\left(\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{dx^{\sigma}}{du}\right)$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{d^2 x^{\sigma}}{du^2} + \frac{dx^{\sigma}}{du}\left\{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\sigma}}\frac{dx^{\rho}}{du}\right\}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\left\{\frac{d^2 x^{\alpha}}{du^2} + \frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\nu}}{\partial x^{\rho}\partial x^{\sigma}}\frac{dx^{\rho}}{du}\frac{dx^{\sigma}}{du}\right\}$$

Now, $$\Gamma^{\alpha}_{\beta\gamma} = \Gamma'^{\mu}_{\nu\sigma}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x'^{\sigma}}{\partial x^{\gamma}} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\mu}}{\partial x^{\beta}\partial x^{\gamma}}$$

$\therefore$ At $x'_{(0)}$, $\Gamma'^{\mu}_{\nu\gamma} = 0$

$$\Gamma^{\alpha}_{\beta\gamma}(x_{(0)}) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\beta} \partial x^{\gamma}} \text{ at } x_{(0)}$$

Hence we get

$$0 = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left[\frac{d^2 x^{\alpha}}{du^2} + \Gamma^{\alpha}_{\rho\sigma} \frac{dx^{\sigma}}{du} \frac{dx^{\rho}}{du} \right]$$

[At every pt., in our new coord. sys., this eqn. holds — this can be shown by applying Principle of Eqv, at each pt., choosing a special coord. frame where $\Gamma^{\mu}_{\nu\rho} = 0$ & now convert this to our general coord. sys.]

---

later we'll try to generalise to eqns like

$$\frac{d^2 x^{\mu}}{du^2} = F^{\mu}$$ $\hookrightarrow$ say, in presence of EM field

---

We'll use the equality

$$\frac{d^2 x'^{\mu}}{du^2} \equiv \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left[\frac{d^2 x^{\alpha}}{du^2} + \Gamma^{\alpha}_{\rho\sigma} \frac{dx^{\sigma}}{du} \frac{dx^{\rho}}{du} \right]$$

Gravitational red shift :-

Massive particle :-

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} < 0$$

$\xrightarrow[\text{proper time}]{\text{define}}$ $$d\tau^2 = - g_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = -1$$

Suppose we have a time-indep.

gravitational field (which means)

$$g_{\mu\nu}(x) = g_{\mu\nu}(x^1, x^2, x^3)$$

Statement itself requires that we can choose a coord. sys. where $g_{\mu\nu}$ is a fn. of 3 spacelike variables

Consider a particle sitting at $\vec{x}_{(0)}$ at all time.

[particle isn't free — we are somehow fixing it at that pt. by some ext. force, whatever it is]

Imagine that the particle has a clock that moves 1 unit / second in free space without gravity.

internal clock

(By principle of equiv., in a local inertial frame it must hold)

Must also be true in locally inertial frame $g_{\mu\nu} = \eta_{\mu\nu}$

$\Rightarrow d\tau = dx^0$ in this frame

the clock moves 1 unit/second

$\Rightarrow$ per unit $\Delta\tau$

(Instantaneously, it may have an accln., but not velocity → accln. bcs some ext. force must be there to hold it at that pt.)

[moving of hand is an event — so any 2 such consecutive events must be separated by $\Delta\tau$ in any coord. sys.]

must be true in every coordinate system.

Now go back to the general coord. sys.

$$g_{\mu\nu} = g_{\mu\nu}(\vec{x})$$

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$$

$$\Rightarrow g_{00}(\vec{x}_{(0)}) \left(\frac{dx^0}{d\tau}\right)^2 = -1$$

$$\Rightarrow \frac{dx^0}{d\tau} = \left\{-g_{00}(\vec{x}_{(0)})\right\}^{-1/2}$$

$$\Delta x^0 = -\left\{g_{00}(\vec{x}_{(0)})\right\}^{-1/2} \Delta\tau$$

diff. betw. the 2 events — i.e., 2 ~~positions~~ movements of hand

(What is imp. is the ratio of periods of time on 2 diff. ~~clocks~~ pts. — Here we simply define time interval at a pt. w.r.t. some standard clock — whether we use $2t$ or t as units of time doesn't matter)

$\times$ $\vec{x}_{(0)}$

$\times$ $\vec{x}_{(1)}$

copy of the same clock (as at $\vec{x}_{(0)}$) here

$$\Delta x^0\Big|_{\text{clock at } \vec{x}_{(1)}} = \left\{-g_{00}(\vec{x}_{(1)})\right\}^{-1/2} \Delta\tau$$

(unit which tells how often the clock ticks)

(Now we can compare this with that at $\vec{x}_{(0)}$)

$$\frac{\Delta t_{(0)}}{\Delta t_{(1)}} \equiv \frac{\Delta x^0\Big|_{\text{clock at } \vec{x}_{(0)}}}{\Delta x^0\Big|_{\text{clock at } \vec{x}_{(1)}}} = \frac{\left\{-g_{00}(\vec{x}_{(0)})\right\}^{-1/2}}{\left\{-g_{00}(\vec{x}_{(1)})\right\}^{-1/2}}$$

Newtonian limit

$$h_{00} = -\frac{2V}{c^2}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

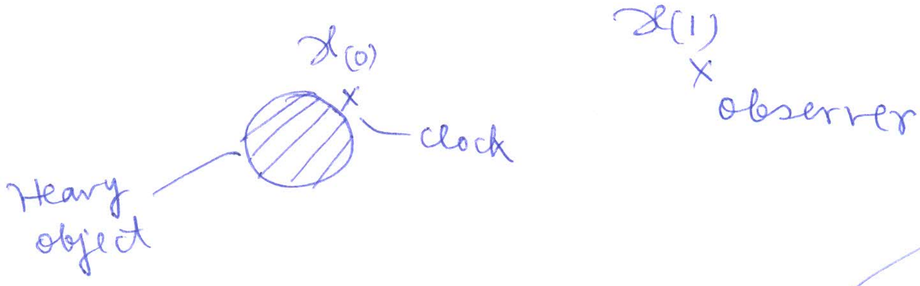
$$\eta_{00} = -1$$

$$g_{00} = -1 - \frac{2V}{c^2}$$

$$\frac{\Delta t_{(0)}}{\Delta t_{(1)}} = \sqrt{\frac{1 + \frac{2}{c^2}V(\vec{x}_{(1)})}{1 + \frac{2V(\vec{x}_{(0)})}{c^2}}}$$

$$\simeq 1 + \frac{V(\vec{x}_{(1)}) - V(\vec{x}_{(0)})}{c^2}$$

for $\frac{V}{c^2} << 1$



$V(\vec{x}_{(1)}) > V(\vec{x}_{(0)})$

bcos grav. pot. is -ve

$$\therefore \frac{\Delta t_{(0)}}{\Delta t_{(1)}} > 1$$

Clocks near the heavy object tick slower than clocks away

So far we have discussed how particles move in a grav. field - we'll then ask how grav. field behave in presence of particles

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Generalize the eqn of motion of a charged particle in electromagnetic field in the presence of gravity:-

Steps :- ① Write down the eqns in Minkowski space-time.

② Interprete them as eqns in locally inertial frame in the presence of gravity.

③ Make coordinate trs. to find the eqns. in a general coordinate system.

Manifestly Lorentz invariant form:-

$$m \frac{d^2 x^\mu}{d\tau^2} = e \, \eta^{\mu\nu} F_{\nu\rho} \frac{dx^\rho}{d\tau}$$

where $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$

$\{A_\mu\}$:- vector potential

e : charge of the particle.

$\boxed{\text{We have set } c = 1}$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$F_{0i} = \partial_0 A_i - \partial_i A_0 = -E_i$ (where $i = 1, 2, 3$)

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

$$\phi = A^0 = -A_0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k$$

[$\epsilon_{123} = 1$ & ϵ_{ijk} is totally antisymmetric]

$F_{12} = \epsilon_{123} B_3 = B_3$; $B_3 = \dfrac{\partial A_2}{\partial x^1} - \dfrac{\partial A_1}{\partial x^2}$

We get, $m \dfrac{d^2 x^i}{d\tau^2} = e \eta^{i\nu} F_{\nu\rho} \dfrac{dx^\rho}{d\tau}$

$$= e F_{i\rho} \frac{dx^\rho}{d\tau} = e F_{i0} \, dx^0/d\tau + e F_{ij} \frac{dx^j}{d\tau}$$

$$= e E_i \frac{dx^0}{d\tau} + e \epsilon_{ijk} B_k \frac{dx^j}{d\tau}$$

$\left(\frac{dx^0}{d\tau} \, dx^j/dt \right)$

Now, $m\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\left(m\frac{dx^i}{d\tau}\right) = \frac{dp^i}{d\tau}$

(We have thought τ as the indep. variable & , x^i & x^0 as fns of τ)

(τ can be eleminated in term of x^0 & ~~can~~ x^0 can now be taken as indep. variable)

$= \frac{dx^0}{d\tau}\frac{dp^i}{dt}$

$\therefore$ we get $\frac{dp^i}{dt} = eE_i + e\,\epsilon_{ijk}B_k\frac{dx^j}{dt}$

$\Rightarrow \frac{d\vec{p}}{dt} = e\vec{E} + e(\vec{v}\times\vec{B})$

$\boxed{v^j = \frac{dx^j}{dt}}$

[How x^0 depends on τ is obtd. from

$m\frac{d^2x^\mu}{d\tau^2} = e\eta^{\mu\nu}F_{\nu\rho}\,dx^\rho/d\tau$

But this info. is needed in our above calculations]

$\to \frac{dx^0}{d\tau}$ just cancelled on both sides

Study the eqn. of motion of a charged particle in the presence of gravity & electromagnetic field.

Go to the locally inertial frame x' at the point $x'_{(0)}$.

In that frame, we have $m\frac{d^2x'^\mu}{d\tau^2} = e\eta^{\mu\nu}F'_{\nu\rho}\frac{dx'^\rho}{d\tau}$

in a general coord. sys.

$\frac{\partial x'^\mu}{\partial x^\kappa}\left\{\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right\}$

[Suppose we know a gauge field in a local inertial frame, but how will we define trs. law for a general coord. sys.]

That largely depends on our defn. – they are completely indep. fields & depends on us what trs. laws we give for them
the other tensors we saw earlier were constructed out of $g_{\mu\nu}$

$$m\frac{\partial x'^\mu}{\partial x^\kappa}\left\{\frac{d^2x^\kappa}{d\tau^2}+\Gamma^\kappa_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right\}$$

$$= e\frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x^\kappa}{\partial x'^\alpha}\eta^{\alpha\nu}F'_{\nu\rho}\underbrace{\frac{\partial x'^\rho}{\partial x^\beta}\frac{dx^\beta}{d\tau}}_{\text{By chain rule}}$$

[Now, $g^{\alpha\beta}=\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta^{\mu\nu}$

$\eta^{\alpha\nu}=\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}g^{\gamma\delta}$]

$$= e\frac{\partial x'^\mu}{\partial x^\kappa}\underbrace{\frac{\partial x^\kappa}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\gamma}}_{\delta^\kappa{}_\gamma}\frac{\partial x'^\nu}{\partial x^\delta}g^{\gamma\delta}$$

$$\times F'_{\nu\rho}\frac{\partial x'^\rho}{\partial x^\beta}dx^\beta/d\tau$$

We want to write everything in terms of unprimed frame

$$= e\frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x'^\nu}{\partial x^\delta}g^{\kappa\delta}F'_{\nu\rho}\frac{\partial x'^\rho}{\partial x^\beta}\frac{dx^\beta}{d\tau}$$

[The natural choice of how $F_{\mu\nu}$ should transformation (in order to be able to write the eqn. completely in the new frame) is ~~?~~

Define:- $F_{\delta\beta}=\frac{\partial x'^\nu}{\partial x^\delta}\frac{\partial x'^\rho}{\partial x^\beta}F'_{\nu\rho}$

In other words, F transforms as a ~~contrava~~ covariant vector of rank 2.

$$\Rightarrow m\left\{\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right\} = e\, g^{\kappa\delta} F_{\delta\beta}\frac{dx^\beta}{d\tau}$$

(It's not completely arbitrary — we'll do some consistency checks)

(This defn. has some nice features)

$$F'_{\mu\nu} = \partial_\mu' A_\nu' - \partial_\nu' A_\mu'$$
$$= D_\mu' A_\nu' - D_\nu' A_\mu'$$

$\Downarrow$ implies

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where $A_\mu = \dfrac{\partial x'^\alpha}{\partial x^\mu} A'_\alpha$

i.e., A_μ is rank 1 covariant tensor.

(Diff. betw. L.H.S. is zero in unprimed frame; so its must be zero in any frame as each side, by construction, is a tensor)

[The above eqn. doesn't change its form under a general coord. trs. — it's in a perfectly covariant form — it's also is unchanged under Lorentz trs.]

(Take the special case) $x'^\alpha = \Lambda^\alpha{}_\mu x^\mu$ where Λ: Lorentz trs. motion

(We get the usual Lor. trs. of A_μ)

[But the reverse isn't true — we could add some non-linear terms in the trs. of A_μ & Lor. trs. being linear, that will give the same transformed A_μ as in STR — but then A_μ won't be a tensor in GTR & the eqn. above won't be covariant]