For $\lambda < \lambda_{eq}$, $\lambda' < \lambda_{eq}$ and we can ignore Ω_m & Ω_Λ terms.

$$t = H_0^{-1} \int^{\lambda} \frac{\lambda' d\lambda'}{\sqrt{\Omega_r}} = \frac{1}{2} H_0^{-1} \Omega_r^{-1/2} \lambda^2$$

$$\Rightarrow \quad \lambda(t) = \left(2 H_0 \Omega_r^{1/2} t\right)^{1/2} \equiv C_r t^{1/2}$$

For $\lambda_{eq} << \lambda << 1$, Ω_Λ can be ignored

$$t = H_0^{-1} \int_0^{\lambda} \frac{\lambda' d\lambda'}{\sqrt{\Omega_r + \Omega_m \lambda'}}$$

$\lambda_{eq} << \lambda_1 << \lambda$

some value.

$$H_0 t \simeq \int_0^{\lambda_1} \frac{\lambda' d\lambda'}{\sqrt{\Omega_r + \Omega_m \lambda'}} + \int_{\lambda_1}^{\lambda} \frac{\lambda' d\lambda'}{\sqrt{\Omega_m \lambda'}}$$

$$< \int_0^{\lambda_1} \frac{\lambda' d\lambda'}{\sqrt{\Omega_m \lambda'}}$$

$$= \frac{2}{3} \Omega_m^{-1/2} \lambda_1^{3/2}$$

negligible.

$$\frac{2}{3} \Omega_m^{-1/2} \left(\lambda^{3/2} - \lambda_1^{3/2}\right)$$

neglig

$$\Rightarrow H_0 t \simeq \frac{2}{3} \Omega_m^{-1/2} \lambda^{3/2}, \quad H_0 t < \frac{2}{3} \Omega_m^{-1/2}$$

well into Matter dominated era.

$$\lambda \simeq \left(\frac{3}{2} H_0 t\, \Omega_m^{1/2}\right)^{2/3} = C_m t^{2/3}$$

~~Check~~ : 9

$$\lambda > \left(\frac{3}{2} H_0 t\, \Omega_m^{1/2}\right)^{2/3}$$

Similar analysis for $\lambda >> 1$:

$$t = \int_{\lambda_i}^{\lambda} \frac{\lambda' \, d\lambda'}{\sqrt{\Omega_\Lambda}\,(\lambda')^2} = \frac{H_0^{-1}}{\sqrt{\Omega_\Lambda}} \ln \frac{\lambda}{\lambda_1}$$

$$\lambda \simeq \lambda_1 \exp\left(H_0 (t - t_0) \sqrt{\Omega_\Lambda}\right)$$

some constant ~ 1

Both matter & Λ give same result for $t \sim t_0$

Check: $\left(\frac{3}{2} H_0 t\, \Omega_m^{1/2}\right)^{2/3} \sim \left(2 H_0 t\, \Omega_r^{1/2}\right)^{1/2}$

for $t \sim t_{eq}$.

$\Rightarrow$ near $t = t_{eq}$ we can get the correct order of magnitude of $\lambda(t)$ using either radiation dominated or matter dominated formula.

Current age of the universe:

$$H_0 t_0 = \int_0^{\lambda_0 = 1} \frac{\lambda' \, d\lambda'}{\sqrt{\Omega_r + \Omega_m \lambda' + \Omega_\Lambda \lambda'^4}}$$

can be ignored.

(We are well into matter dominated era).

$$= \int_0^1 \frac{\lambda' \, d\lambda'}{\sqrt{\Omega_m \lambda' + \Omega_\Lambda \lambda'^4}} = .99$$

$$\Rightarrow \quad t \simeq H_0^{-1} = 1.38 \times 10^{10} \text{ years}$$

13.8 billion years.

Larger H_0 → lowers the age.

Age > age of known objects.

Earth: 4.5 billion years → safe

Oldest star ~13.2 billion years old.

(need theory of stellar evolution).

~~Cosmological constant dominated era:~~

$$\cancel{r_H(t) - r_H(t_0) \simeq \int_{\lambda_0}^{\lambda} d\lambda'}$$

$t = 0 \Rightarrow \lambda(t) = 0.$

→ Big bang singularity.

(Known physical laws break down).

- Horizon: $d_H(t)$: The maximum distance light travels at time t.

$$\frac{dt}{a(t)} = dr \quad \text{(for radial propagation)}$$

$$\int_0^{r_H} dr = \int_0^t \frac{dt'}{a(t')}$$

$$r_H(t) = \int_0^t \frac{dt'}{a(t')} = a_0^{-1} \int_0^t \frac{dt'}{\lambda(t')}$$

$$d_H(t) = a(t)\, r_H(t)$$

$$= \lambda(t) \int_0^t \frac{dt'}{\lambda(t')}$$

⇒ gives the maximum distance between two points which could communicate by time t.

Focus on $r_H(t)$.

Radiation dominated era:

$$r_H = a_0^{-1}\int_0^t \frac{dt'}{\lambda(t')} = a_0^{-1}\int_0^t \frac{dt'}{c_r (t')^{1/2}}$$

$$= a_0^{-1}\, c_r^{-1}\, 2t^{1/2}$$

Well into the matter dominated era:

$$r_H = a_0^{-1}\int_0^t \frac{dt'}{\lambda(t')} = a_0^{-1}\int_0^{t_1} \frac{dt'}{\lambda(t')} + a_0^{-1}\int_{t_1}^{t} \frac{dt'}{\lambda(t')}$$

$t_{eq} \ll t_1 \ll t$

$$a_0^{-1}\int_{t_1}^{t_0} \frac{dt'}{c_m t'^{2/3}}$$

$$= a_0^{-1} c_m^{-1}\, 3\left(t^{1/3} - t_1^{1/3}\right)$$

ignore.

$$< a_0^{-1}\int_0^{t_1} \frac{dt'}{c_m t'^{2/3}}$$

$$= a_0^{-1} c_m^{-1}\, 3 t_1^{1/3} \to \text{small.}$$

$$\Rightarrow r_H(t) \simeq 3 a_0^{-1} c_m^{-1} t^{1/3}$$

Ex. Both formula give same order of results for $t \sim t_{eq}$.

Today: $a_0 r_H(t_0) = 3 a_0^{-1} c_m^{-1} t_0^{1/3}$

$$= 3\left(\tfrac{3}{2} H_0 \Omega_m^{1/2}\right)^{-2/3} t_0^{1/3} = \left(\tfrac{3}{2}\Omega_m\right)^{-2/3} H_0^{-1}$$

cosmological constant dominated era:

$$r_H(t) - r_H(t_0) = a_0^{-1} \int_{t_0}^{t} \frac{dt'}{\lambda(t')}$$

$$= a_0^{-1} \int_{t_0}^{t} dt' \, \lambda_1^{-1} \exp(-H_0(t-t_0))$$

$$= a_0^{-1} \lambda_1^{-1} \frac{1}{H_0} \left(1 - e^{-H_0(t-t_0)}\right)$$

$r_H(t_0)$ will reach a finite value

There will be a maximal comoving separation between points which could ever communicate.

Thermal history.

current microwave spectrum:

black body.

$$P_0(\nu_0)\, d\nu_0 = \frac{8\pi h \nu_0^3 \, d\nu_0}{e^{\frac{h\nu_0}{kT_0}} - 1} \qquad T_0 = 2.7\,K$$

How did it look earlier.

~~$P_0(\nu_0)\,d\nu$~~ $\nu_0 \to \nu = \frac{a_0}{a} \nu_0$

$P_0 \to P = \left(\frac{a_0}{a}\right)^4 P_0$

(43)

$$P(\nu)\,d\nu = \left(\frac{a_0}{a}\right)^4 \times \frac{8\pi h \nu^3 d\nu}{e^{\frac{h\nu}{kT_0}\frac{a}{a_0}} - 1} \left(\frac{a}{a_0}\right)^3$$

$$\Rightarrow T = \cancel{k_B} T_0 \frac{a_0}{a} = T_0/\lambda$$

At ~~...~~ t_{eq} $\lambda = 1.6 \times 10^{-4}$

$$\Rightarrow T = T_0/\lambda = \frac{2.7\cancel{0}}{1.6} \times 10^4\ K.$$

$k_B = 8.62 \times 10^{-5}\ eV/K$

$k_B T \sim 8.60 \times 10^{-5} \times 10^4 \sim 1\ eV.$

At $\lambda \sim 10^{-5}$, $k_B T \sim 10\ eV.$

Ionization energy of $H \sim 13.6\ eV.$

Above ~~Around~~ this temperature the H in the universe was fully ~~...~~ ionized.

~~e~~ photons interacted strongly & were in thermal equilibrium.

As temperature fell $e^- p \Rightarrow H$

$\Rightarrow$ interaction with ~~...~~ photons decreased.

By $\lambda \sim 10^{-3}$ the universe was almost fully ~~... ionized~~. transparent to γ.

The CMB radiation today last interacted with matter at $\lambda \sim 10^{-3}$.

$z \sim 10^{3}$.

$\Rightarrow$ CMB is the image of the universe at $\lambda \sim 10^{-3}$.

Fact: CMB is highly isotropic

$\Delta T \sim 10^{-5}$.

$\Rightarrow$ The universe at $\lambda = 10^{-3}$ was highly isotropic

$\rightarrow$ homogeneous.

$\Rightarrow$ Best test that early universe was extremely homogeneous.

Even earlier T was higher.

For $kT >$ rest mass the particle - anti-particle pair could be produced.

Early universe: soup of particle-anti-particle pairs of all kinds

$\rightarrow$ still radiation dominated.

Review

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

co-moving coordinates (r, θ, ϕ).

$\lambda(t) \equiv a(t)/a_0$ ← a_0 present value.

radiation dominated — matter dominated — Cosmological const. dominated

$\lambda = 0$ — λ_{eq} — 1 → λ

big bang

$\lambda_{eq} \sim 10^{-4} \times 1.6$

neutrinos $\boxed{3 \times 10^{-4}}$

$k = 0$ for all practical purposes.

r small for observed universe

Horizon

comoving coordinates: $\displaystyle\int_0^t \frac{dt'}{a(t')} = \int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}}$

$= r_H$

→ Distance travelled by light since $t = 0$ in comoving coordinates

Physical Horizon: $a(t)\, r_H(t) = \lambda(t) \displaystyle\int_0^t \frac{dt'}{\lambda(t')}$

Results

Radiation dominated

$\lambda(t) = c_r t^{1/2}$, $c_r = \left(2 H_0 \Omega_r^{1/2}\right)^{1/2}$

$r_H(t) = 2 a_0^{-1} c_r^{-1} t^{1/2}$ $\quad d_H(t) = 2t$

Matter dominated

$$\lambda(t) = C_m t^{2/3}, \quad C_m = \left(\frac{3}{2} H_0 \Omega_m^{1/2}\right)^{2/3}$$

$$r_H(t) = 3 a_0^{-1} C_m^{-1} t^{1/3}, \quad d_H(t) = 3t$$

Cosmological constant dominated

$$\lambda = K e^{H_0 \sqrt{\Omega_\Lambda}(t - t_0)} \qquad K \sim 1$$

(t independent as $t \to \infty$)

$$r_H(t) \approx a_0^{-1} H_0^{-1} \times \text{~~order term~~}$$

$$d_H(t) \sim H_0^{-1} e^{H_0 \sqrt{\Omega_\Lambda}(t - t_0)}$$

Thermal history

ν_0 : frequency of CMB at present.

$$\underbrace{P(\nu_0)\, d\nu_0}_{} = \frac{8\pi h \nu_0^3 d\nu_0}{e^{\frac{h\nu_0}{k_B T_0}} - 1} \qquad c = 1 \text{ unit.}$$

Energy density between $(\nu_0, \nu_0 + d\nu_0)$

At time t the same photons would have frequency $\nu = \nu_0 \dfrac{a(t_0)}{a(t)} = \nu_0 / \lambda(t)$.

Total energy density $= \left(\dfrac{a(t_0)}{a(t)}\right)^4 \times$ present energy density

$$\Rightarrow P(\nu)\, d\nu = \frac{a(t_0)^4}{a(t)^4} \times 8\pi h \frac{\nu_0^3 d\nu_0}{e^{\frac{h\nu_0}{k_B T_0}} - 1}$$

$$= \frac{1}{\lambda(t)^4} 8\pi h. \lambda(t)^4 \frac{\nu^3 d\nu}{e^{\frac{h\nu\lambda(t)}{k_B T_0}} - 1}$$

$$= 8\pi h \frac{\nu^3 d\nu}{e^{\frac{h\nu}{k_B T(t)}} - 1}$$

$T(t) = T_0/\lambda(t)$

⇒ The spectrum remains black body at temperature $T(t)$. → explains why we see black body spectrum

At $\lambda = \lambda_{eq}$, $T(t_{eq}) = \frac{T_0}{\lambda_{eq}} \sim \frac{2.7 \times 10^4}{1.6 \to 3} K$

$k_B = 8.62 \times 10^{-5}$ eV/K

$k_B T(t_{eq}) \sim 8.62 \times 10^{-5} \times \frac{2.7}{\cancel{1.6}\,3} \times 10^4$ eV

~ 1 eV.

At $\lambda \sim 10^{-5}$ $k_B T \sim 10$ eV.

Recall: H ionization energy = 13.6 eV.

⇒ Much above this temperature all the Hydrogen would have been ionized.

⇒ Photons would have interacted with free electrons & protons & exchanged energy with them.

Could this affect separate conservation law for matter & radiation?

Note: Matter is still non-relativistic

⇒ The kinetic energy is negligible compared to rest mass energy, ~~the~~ → plays little role in energy balance.

→ can continue to use conservation law for radiation energy.

At even higher temperature (smaller λ) the matter would become relativistic, but by then radiation energy heavily dominates.

New effect! $k_B T >> m$ ⇒ a particle of mass m could be produced in pairs & could exchange energy with radiation

But these new relativistic particles can now be regarded as part of radiation

⇒ The universe continues to be radiation dominated & we do not need to change our equation.

→ hold at least ~~down~~ up to (SG) $T \sim 100$ GeV (standard model scale)

Come back to $\lambda \sim 10^5$, $k_B T \sim 10$ eV.

At this temperature the free electrons & protons begin to bond & start forming hydrogen atoms.

→ recombination.

Actually this happens at a lower temperature due to various effects to be discussed later.

By $\lambda \simeq 10^{-3}$ most free electrons & protons are gone.

⇒ scattering amplitude of photon from matter drops to almost zero.

The microwave spectrum we see today are the photons travelling without scattering from $\lambda \sim 10^{-3}$.

→ Gives us image of the sky at $\lambda \sim 10^{-3}$.

→ Last scattering surface.

$\lambda_L \sim 10^{-3}$. Ex. Calculate t_L

Fact: CMB is highly isotropic

$$\frac{\Delta T}{T} \sim 10^{-5}$$

$\Rightarrow$ The universe at $a = 10^{-3}$ was highly isotropic

$\Rightarrow$ homogeneous if we are not special.

$\rightarrow$ Best evidence for homogeneity and isotropy of the early universe.

Consider a microwave photon we receive today.

Where is it coming from?

$r = 0$ present position

$r = r_L$ location of last scattering surface.

$$\int_0^{r_L} dr = \int_{t_L}^{t_0} \frac{dt}{a(t)} = a_0^{-1} \int_{t_L}^{t_0} \frac{dt}{a(t)}$$

Use matter dominated formula:

$$a(t) = C_m t^{2/3}$$

$$r_L = a_0^{-1} C_m^{-1} \int_{t_L}^{t_0} dt\, t^{-2/3} = 3 a_0^{-1} C_m^{-1} \left(t_0^{1/3} - t_L^{1/3}\right)$$

Comoving distance between two ~~points~~ diametrically opposite points on the last scattering surface:

$$2r_L \cong 6 a_0^{-1} c_m^{-1} t_0^{1/3}.$$

Compare this with ~~comoving~~ horizon in comoving coordinates at t_L

$$r_H(t_L) = 3 a_0^{-1} c_m^{-1} t_L^{1/3}$$

$$\text{Ratio}: \frac{r_H(t_L)}{2r_L} = \frac{1}{2}\left(\frac{t_L}{t_0}\right)^{1/3} = \frac{1}{2}\left(\frac{a_L}{a_0}\right)^{1/2}$$

$$= \frac{1}{2}\lambda_L^{1/2} << 1.$$

$\Rightarrow$ The diametrically opposite points on the last scattering surface had no previous communication.

How could the temperatures be identical?

Consider two points on last scattering surface at angular separation $\Delta\theta$.

$\Rightarrow$ comoving distance at $t_L = r_L \Delta\theta$

$$\frac{r_L \Delta\theta}{r_H(t_L)} = \frac{\Delta\theta\, t_0^{1/3}}{t_L^{1/3}}$$

$\Rightarrow$ They could communicate for $\Delta\theta < \left(\frac{t_L}{t_0}\right)^{1/3} = \left(\frac{a_L}{a_0}\right)^{1/2} \simeq \frac{1}{30}$ radians

$= \frac{1}{30} \times \frac{180}{\pi}{}^\circ \simeq 2^\circ$

~~Possible~~ Possible solution:

$$\text{Recall: } a_0 r_H(t_L) = \int_0^{t_L} \frac{dt'}{\lambda(t')} \left(\frac{3}{2} H_0 \Omega_m^{1/2}\right)^{2/3} \simeq 6 C_m^{-1} (t_L)^{1/3}$$

$$\ll a_0 r_L = 6 C_m^{-1} t_0^{1/3} \sim H_0^{-1} \text{ (Ex)}$$

If the integral receives some unknown contribution from $t' < t_L$ that is $\geq 6 C_m^{-1} t_0^{1/3}$ $\underset{\text{Ex.}}{\sim H_0^{-1}}$ then we have a possible resolution.

Matter dominated + Radiation dominated for $t < t_L$ gives $a_0 r_H(t_L) \sim 3 C_m^{-1} (t_L)^{1/3}$

$\to$ ~~must~~ We must have some new physics.

upto $k_B T \sim 100$ GeV we understand the microscopic theory well.

$\Rightarrow$ No deviation from the standard big bang model till then.

Recall: $\lambda \sim 10^{-4}$ for $k_B T \sim 1$ eV.

$\Rightarrow$ For $k_B T \sim 100$ GeV, $\lambda \sim 10^{-4} \dfrac{1}{100 \times 10^9} \sim 10^{-15}$

whatever new physics is invoked it must have happened at $\lambda < 10^{-15}$ i.e. $k_B T > 100$ GeV.

λ_I : The value of λ at which new physics takes place

$\downarrow$

Inflation.

Postulate: For $\lambda < \lambda_I$, the energy density was stored in cosmological constant instead of radiation.

At $\lambda = \lambda_I$ it got converted to radiation and the standard big bang picture holds since then.

First study the consequence, then look for how it could happen.

~~At $t = t_I$ ($\lambda = \lambda_I$)~~

Λ_I : cosmological constant for $t < t_I$

$$\sim \rho_r(t_I) \sim \rho_{r0} \left(\frac{1}{\lambda_I}\right)^4$$

$$\left(\frac{\dot\lambda}{\lambda}\right)^2 = \frac{8\pi G}{3}\Lambda_I \Rightarrow \lambda = C_I \exp\left(\sqrt{\frac{8\pi G}{3}\Lambda_I}\, t\right)$$

$$\mathcal{H}_I = \sqrt{\frac{8\pi G}{3}\Lambda_I} \qquad = C_I \exp(\mathcal{H}_I t)$$

$$r_H(t_I) - r_H(t_*) = \int_{t_*}^{t_I} \frac{dt}{a(t)}$$

t_*: beginning of inflation

$$= \frac{1}{a_0} c_I^{-1} \frac{1}{H_I} \left\{ \exp(-H_I t_*) - \exp(-H_I t_I) \right\}$$

$$\simeq \frac{1}{a_0} c_I^{-1} \frac{1}{H_I} \exp(-H_I t_*).$$

Note: t_* could go negative since $\lambda(t)$ does not vanish in the past as long as Λ_I dominates.

Boundary condition

$$\lambda(t_I) = \lambda_I$$

$$\Rightarrow \quad c_I \exp(+H_I t_I) = \lambda_I$$

$$c_I = \lambda_I \exp(-H_I t_I)$$

$$\therefore \quad r_H(t_I) - r_H(t_*) = \frac{1}{a_0} \lambda_I^{-1} H_I^{-1} \exp(\underbrace{H_I(t_I - t_*)}_{N})$$

N: # of e-folding.

(e^N measures by what factor the $a(t)$ has increased during inflation)

$$r_H(t_I) = r_H(t_*) + \frac{1}{a_0} \lambda_I^{-1} H_I^{-1} \exp(N)$$

$$r_H(t_L) = r_H(t_*) + \frac{1}{a_0} \lambda_I^{-1} H_I^{-1} \exp(N)$$

$$+ \frac{1}{a_0} \underbrace{\int_{t_I}^{t_L} \frac{dt'}{\lambda(t')}}_{\text{small}}$$

Can this be made larger than

$$6 a_0^{-1} C_m^{-1} t_0^{1/3} \sim a_0^{-1} H_0^{-1}$$

For given λ_I, H_I, there is no bound on N.

$\Rightarrow$ By choosing N to be large enough we can make

$$a_0 r_H(t_L) > H_0^{-1}$$

$\Rightarrow$ solves the horizon problem.

How large N needs to be depends on λ_I.

$$e^N > H_0^{-1} \lambda_I H_I$$

$$\underset{\approx 1}{\lambda_I / \lambda_0} = T_0 / T_I, \quad \rho(\lambda_I) \approx (k_B T_I)^4 \text{ for } \hbar = c = 1.$$

$$H_I \sim \sqrt{\frac{8\pi G}{3} \rho_I} = \sqrt{\frac{8\pi G}{3}} (k_B T_I)^2$$

$$e^{N} > H_0^{-1} \frac{k_B T_0}{k_B T_I} \times \sqrt{\frac{8\pi G}{3}} \left(k_B T_I\right)^2$$

Also set $G = 1$ (Planck units).

Unit of length = Planck length $\sim 10^{-33}$ cm.

Unit of mass = Planck mass $\sim 10^{19}$ GeV.

Unit of time = Planck time $\sim 10^{-43}$ seconds

Suppose $k_B T_I \sim 10^{16}$ GeV. (Grand unification scale)

$\sim 10^{-3}$ in Planck Units.

$$\frac{k_B T_0}{k_B T_I} = \frac{2.7 \times 8.6 \times 10^{-5}\ eV}{10^{16} \times \underbrace{10^{9}}_{GeV \to eV}}$$

$$\sim 10^{-29}$$

$H_0^{-1} \sim 4 \times 10^{17}$ s. $\sim 10^{60}$ in Planck units.

$$e^{N} > 10^{60} \times 10^{-29} \times 10^{-6} \sim 10^{25}$$

$N > 25 \ln_e 10 \sim 60.$

Lower $k_B T_I \Rightarrow$ lower no. of e-folding.

Why cosmological constant?

Ex. Assume at for $t < t_I$ all the energy was in some ~~form~~ state satisfying

$p = \omega \rho$ (constant)

$\omega = \frac{1}{3}$ for radiation
$= 0$ for matter
$= -1$ for C.C.

At $t = t_I$, $\lambda(t)$ and $\rho(t)$ are continuous.

Ex. Find the condition on ω needed to solve the Horizon problem. $(\omega < -\frac{1}{3})$

Physical picture: A small region inflates & covers the whole of the visible universe today.

Scalar field driven inflation:

Consider gravity coupled to a scalar field ϕ:

$$S = \int d^4x \sqrt{\det g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

some function (potential)

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{\det g} \; \delta g^{\mu\nu} T^{\phi}_{\mu\nu}$$

$\to$ defines $T^{\phi}_{\mu\nu}$

Explicit analysis

$$\delta S = \int d^4x \sqrt{-\det g}\,\Big[-\tfrac{1}{2}\,\delta g^{\rho\sigma}\,\partial_\rho\phi\,\partial_\sigma\phi - \tfrac{1}{2}\,\delta g^{\rho\sigma}\, g_{\rho\sigma}\left(-\tfrac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right)\Big]$$

$$= -\tfrac{1}{2}\int d^4x \sqrt{-\det g}\;\delta g^{\rho\sigma}\,\underbrace{\Big[\partial_\rho\phi\,\partial_\sigma\phi + g_{\rho\sigma}\left(-\tfrac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right)\Big]}_{T_{\rho\sigma}}$$

Homogeneity & isotropy $\Rightarrow$ $\phi(\vec{x},t) = \phi(t)$

(later will relax this)

$$\Rightarrow\quad T_{00} = (\partial_t\phi)^2 - \tfrac{1}{2}(\partial_t\phi)^2 + V(\phi) = \underbrace{V(\phi) + \tfrac{1}{2}(\partial_t\phi)^2}_{\rho}$$

$$T_{ij} = g_{ij}\,\underbrace{\left(+\tfrac{1}{2}(\partial_t\phi)^2 - V(\phi)\right)}_{p}$$

$$\rho = V(\phi) + \tfrac{1}{2}\dot{\phi}^2$$
$$p = -V(\phi) + \tfrac{1}{2}\dot{\phi}^2$$

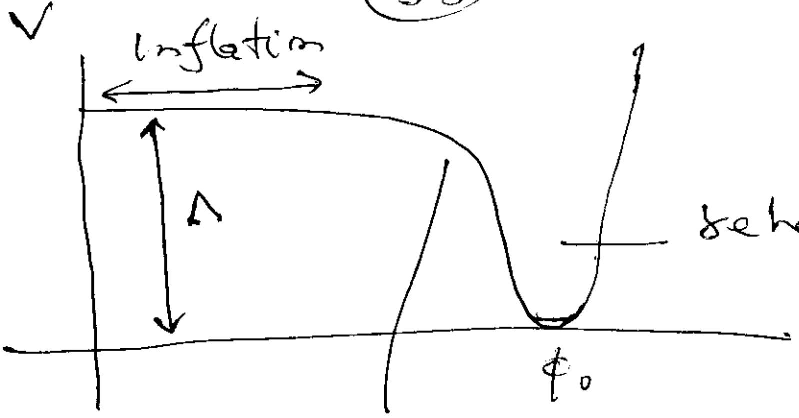
$p \simeq -\rho$ if $\dot{\phi}$ is small.

$\to$ behaves like cosmological constant

$$\Lambda = V(\phi).$$

ϕ eq: $\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0.$

$\dot{\phi}$ small $\Rightarrow$ $V'(\phi)$ small.



$V''(\phi_0)$: Mass^2 of ϕ at present

Need $V''(\phi_0) > (100\,\text{GeV})^2$ to avoid contradiction with expt.

Note: Since Einstein eq. is local, we could apply it locally.

If $\vec{\nabla}\phi$ & $\dot{\phi}$ were small in some region of space that region will inflate. $g_{ij} \gg \nabla_i\phi\nabla_j\phi$

It ① removes any inhomogeity to one part in e^N.

② Dilutes all particle densities.

$D_\mu J^\mu = 0 \Rightarrow \frac{d}{dt}(n a^3) = 0$
↳ conserved charge

$n_I = n_* e^{-3H_I(t_I - t_*)}$ → negligible
($\ll 1$ in the visible universe)

3. Explains why the curvature term k/a^2 is small.

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

Suppose at $t = t_*$, $\frac{k}{a_*^2} \sim \rho_*$

At $t = t_I$, $\rho_I \simeq \rho_*$

$$\frac{k}{a_I^2} = \frac{k}{a_*^2}\, \underbrace{e^{-2H_I(t_I - t_*)}}_{e^{-2N}}$$

Becomes extremely small.

After this universe enters radiation dominated era.

$$\cancel{\frac{k}{a^2} \Big/ \rho \sim \frac{k}{a^2}\,\rho_*}$$

$$\frac{k}{a^2} = \frac{k}{a_I^2} \times \frac{a_I^2}{a^2}, \quad \rho = \frac{\rho_* a_I^4}{a^4} \quad \left(\sim \rho_{eq}\frac{a_{eq}^3}{a^3} \text{ after } t_{eq}\right)$$

ρ decreases at a faster rate than k/a^2

Nevertheless with $N \simeq 60$, the initial smallness of $\frac{k}{a_I^2}$ is enough to explain the smallness of k/a^2 today.

$\lambda(t) = \frac{a(t)}{a_0}$

Inflation — Known

$\lambda \sim 10^{-15}$ microscopic

$T \sim 100\,\mathrm{GeV}$ physics

$\longleftrightarrow$

Unknown microscopic physics.

Can we predict something about the evolution of the universe without knowing the unknown microscopic physics between $\lambda \sim 10^{-15}$ and $\lambda = \lambda_I$?

λ_I: end of inflation.

We can due to near statistical equilibrium during most of the evolution

$\Rightarrow$ erases data from the past although not as efficiently as inflation.

Main tool: statistical mechanics will be reviewed now.

Microcanonical ensemble

A gas of total energy between $E, E+\delta E$, total no. of particles N, in a volume V.

\# of quantum states $\Omega_{N,V}(E)\,\delta E$

Entropy $S(E,N,V) = \ln \Omega(E, N, V)$

S is extensive.

$$\Rightarrow S(E,N,V) = V\, f\left(\frac{E}{V}, \frac{N}{V}\right)$$

$$S(\lambda E, \lambda N, \lambda V) = \lambda S(N,E,V)$$

$$\Rightarrow \quad E \frac{\partial S}{\partial E} + N \frac{\partial S}{\partial N} + V \frac{\partial S}{\partial V} = S.$$

Some definitions.

$$\beta = \frac{1}{k_B T} = \frac{\partial S}{\partial E}, \quad \mu = -T \frac{\partial S}{\partial N}, \quad p = T \frac{\partial S}{\partial V}$$

set $k_B = 1$ — temperature — chemical potential — pressure.

$n = \frac{N}{V} \rightarrow$ number density,

$s = \frac{S}{V} \rightarrow$ entropy density.

$\rho = \frac{E}{V} \rightarrow$ energy density.

T, μ, p, n, s, ρ do not scale with volume.

# of independent variables 2.

$\rightarrow$ can be taken to be ρ, n. or any other quantity.

Once we know $f(\rho, n)$ we can find relation between these variables.

$$V \frac{\partial S}{\partial V} + E \frac{\partial S}{\partial E} + N \frac{\partial S}{\partial N} = S$$

$$\Rightarrow V \frac{p}{T} + \frac{E}{T} - N \frac{\mu}{T} = S$$

$$\Rightarrow \boxed{\frac{p + \varepsilon - n\mu}{T} = s}$$

Under a small change

$$\delta S = \frac{\partial S}{\partial V} \delta V + \frac{\partial S}{\partial E} \delta E + \frac{\partial S}{\partial N} \delta N$$

$$= \frac{p}{T} \delta V + \frac{1}{T} \delta E - \frac{\mu}{T} \delta N$$

$$\Rightarrow \boxed{T \delta S = p \delta V + \delta E - \mu \delta N}$$

$\rightarrow$ 1st law of thermodynamics.

If we keep V fixed at 1 then

$$\boxed{T \delta s = \delta \varepsilon - \mu \delta n}$$

Computing $\Omega(E, N, V)$ & hence $S(E, N, V)$ is difficult.

Strategy: Use grand canonical ensemble

$$\mathcal{Q}(\mu, \beta, V) = \sum_{\alpha} e^{-\beta E_\alpha + \mu \beta N_\alpha}$$

E_α — energy of $|\alpha\rangle$; N_α — # of particles in $|\alpha\rangle$; sum over α — all quantum states $|\alpha\rangle$

$$\frac{\partial \mathcal{Q}}{\partial \beta} = -\sum_\alpha E_\alpha e^{-\beta E_\alpha + \mu\beta N_\alpha} = \mathcal{Q}\,\bar{E}$$

average energy

$$\frac{\partial \mathcal{Q}}{\partial \mu} = \beta \sum_\alpha N_\alpha e^{-\beta E_\alpha + \mu\beta N_\alpha} = \beta \mathcal{Q}\,\bar{N}$$

$$\Rightarrow \bar{E} = -\frac{\partial}{\partial \beta} \ln \mathcal{Q}$$

$$\bar{N} = \frac{1}{\beta}\frac{\partial}{\partial \mu} \ln \mathcal{Q}.$$

$$\mathcal{Q} = \sum_{N'} \int dE'\, \Omega(E', N', V)\, e^{-\beta E' + \mu\beta N'}$$

$$\Omega(E', N', V) = e^{S(E', N'; V)}$$

→ gets maximum contribution from

$E' = E,\ N' = N$ satisfying

$$\frac{\partial}{\partial E'}\left(S(E', N', V) - \beta E' + \mu\beta N'\right)\Big|_E$$

$$\frac{\partial}{\partial N'}\left(S(E', N', V') + \mu\beta N' - \beta E'\right)\Big|_N = 0$$

$$\beta = \frac{\partial S(E', N', V)}{\partial E'}, \quad \mu\beta = -\frac{\partial S(E', N', V)}{\partial N'}.$$

Call the solutions $E' = E,\ N' = N$ → $\bar{E} = E,\ \bar{N} = N$

$$\Rightarrow \frac{1}{T} = \frac{\partial S(E, N, V)}{\partial E}, \quad \frac{\mu}{T} = -\frac{\partial S(E, N, V)}{\partial N}$$

⇒ same as from microcanonical

$\bar{E} = E, \quad \bar{N} = N$

$$\mathcal{Q} \simeq \Delta E \, \Omega(E,N,V) \, e^{-\beta E + \mu\beta N}$$

$$\Rightarrow \ln \mathcal{Q} = \ln \Delta E + S(E,N,V) - \beta E + \mu\beta N.$$

$$\Rightarrow \boxed{s = \beta\rho - \mu\beta n + \frac{1}{V}\ln\mathcal{Q}.}$$

$$\boxed{\rho = -\frac{1}{V}\frac{\partial}{\partial\beta}\ln\mathcal{Q}}$$

$$\boxed{p = Ts - \rho + n\mu = \frac{T}{V}\ln\mathcal{Q}.}$$

$$\boxed{n = +\frac{T}{V}\frac{\partial}{\partial\mu}\ln\mathcal{Q}}$$

at fixed V & (μ or β).

We have assumed that the # of particles in the microcanonical ensemble does not change.

If this is not the case then in the defn of microcanonical ensemble $\Omega(E)\,\delta E$ = # of states between $E, E+\delta E$ with arbitrary no of particles.

In order to get the grand canonical ensemble to agree with microcanonical we need, to set $\mu = 0$.

Results for ideal gases:

Bose:

$$\ln Q = -\frac{V}{8\pi^3}\int_0^\infty 4\pi k^2 dk \ln\left(1 - e^{-\beta(e(k) - \mu)}\right)$$

Fermi: $\frac{V}{8\pi^3}\int_0^\infty 4\pi k^2 dk \ln\left(1 + e^{-\beta(e(k) - \mu)}\right)$

$$e(k) = \sqrt{k^2 + m^2}$$

This assume single state per particle.

If s states per particle then the result is multiplied by s.

e.g. photon: $s = 2$ (two helicities)
electron: $s = 2$ (two spin states)
etc.

For photon $\mu = 0$ since # of photons is not conserved.

$$\Rightarrow \ln Q = -\frac{V}{4\pi^3}\int_0^\infty 4\pi k^2 dk \ln\left(1 - e^{-\beta k}\right)$$

$$\rho = -\frac{1}{V}\frac{\partial}{\partial\beta}\ln\mathcal{Q} = \frac{1}{4\pi^3}\times 4\pi\int_0^\infty dk\, k^2 \frac{k}{1-e^{-\beta k}}$$

$$= \frac{1}{\pi^2}\int_0^\infty dk\; k^3 \frac{e^{\beta k}}{e^{\beta k}-1}$$ → energy density between photon momentum k, $k+dk$

→ black body spectrum

K

Now consider a mixture of gases.

$$\mathcal{H} = \sum_{i=1}^{K} H_i \quad \to \text{Hamiltonian}$$

$$+\ \mathcal{H}_{int}$$

└ small enough not to contribute to $\ln\mathcal{Q}$ but large enough to achieve equilibrium.

$$\Rightarrow \ln\mathcal{Q} = \sum_i \ln\mathcal{Q}_i$$

$$T_i = T.$$

What about μ_i's?

If the # of particles of each type is conserved then μ_i's are independent

$\Rightarrow$ system characterized by $\mu_1, \ldots \mu_K$

What if this is not true?

Example: A mixture of A, B, C with following possible reaction

$A + B \to C + C$

N_A, N_B, N_C are not conserved.

Should we set $\mu_A = \mu_B = \mu_C = 0$? X

General procedure: Look for conserved charges:

$\tilde{N}_1 = N_A - N_B$ is conserved.

$\tilde{N}_2 = N_A + N_B + N_C$ is conserved.

Def. of grand canonical ensemble

$$\ln \mathcal{Q} = \sum_\alpha e^{-\beta E_\alpha + \tilde{\mu}_1 \beta \tilde{N}_1 + \tilde{\mu}_2 \beta \tilde{N}_2}$$

$$= \sum_\alpha e^{-\beta E_\alpha + \beta(\tilde{\mu}_1 + \tilde{\mu}_2) N_A + \beta(\tilde{\mu}_2 - \tilde{\mu}_1) N_B + \beta \tilde{\mu}_2 N_C}$$

$\Rightarrow$ set

$\mu_1 = \tilde{\mu}_1 + \tilde{\mu}_2, \quad \mu_2 = \tilde{\mu}_2 - \tilde{\mu}_1, \quad \mu_3 = \tilde{\mu}_2$

$\tilde{\mu}_1$ & $\tilde{\mu}_2$ are independent variables.

$\ln \mathcal{Q} = (\ln \mathcal{Q}_A + \ln \mathcal{Q}_B + \ln \mathcal{Q}_C) \leftrightarrow$ fn. of $V, \beta, \tilde{\mu}_1, \tilde{\mu}_2$

General Case

Suppose we have K types of particles and L conserved charges.

$$\tilde{N}_\ell = \sum_{i=1}^{K} C_i^{(\ell)} N_i \qquad \ell = 1, \dots L$$

conserved charges. $\Downarrow$ fixed numbers

$$\ln Q = \sum_\alpha \exp\Big(-\beta E_\alpha + \beta \sum_{\ell=1}^{L} \tilde{\mu}_\ell \tilde{N}_\ell\Big)$$

$$\Downarrow$$

$$\beta \sum_{\ell=1}^{L} \sum_{i=1}^{K} C_i^{(\ell)} N_i \tilde{\mu}_\ell$$

$$= \beta \sum_{i=1}^{K} \mu_i N_i \qquad \mu_i = \sum_{\ell=1}^{L} C_i^{(\ell)} \tilde{\mu}_\ell$$

Note: $\tilde{\mu}_\ell$ are independent variables for $\ell = 1, \dots L$.

$$\ln Q = \sum_{i=1}^{K} \ln Q \Big| \quad \beta_i = \beta, \quad \mu_i = \sum_{\ell=1}^{L} C_i^{(\ell)} \tilde{\mu}_\ell$$

$\therefore \rho = \sum_{i=1}^{K} \rho_i$, $s = \sum_{i=1}^{K} s_i$, $p = \sum_{i=1}^{K} p_i$ with this restriction on μ_i's.

Now place such a system in the expanding universe & assume equilibrium at each t.

Independent variables:

1. a (or λ)
2. T
3. $\tilde{\mu}_\ell$ $\ell = 1, \ldots K$

$\Big\}$ $K+2$ variables.

Equations

$$\left.\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}\rho \\ \frac{d}{dt}(\rho a^3) &= -3 p a^2 \dot{a} \end{aligned}\right\} \text{2 eqs.}$$

We need K more equations.

$\rightarrow$ Conservation laws of $\tilde{N}_\ell$.

In general relativity we have ~~particle~~ conserved current $\tilde{J}_{(\ell)}$ satisfying:

$\tilde{J}^0_{(\ell)} = n_\ell \quad (\equiv \tilde{N}_\ell / V)$ $\quad$ ($\tilde{J}^i_{(\ell)} = 0$ isotropy)

$D_\mu \tilde{J}^\mu_{(\ell)} = 0 \underset{\text{Ex.}}{\Rightarrow} \dfrac{d}{dt}(n_{(\ell)} a^3) = 0 \rightarrow K$ eqs. $\quad \ell = 1, \ldots K$.

This gives $k+2$ equations.

Initial conditions.

Normally we are interested in studying physics around a specific temperature T_1 rather than a specific time t_1: defined to be the time at which $T = T_1$ (say)

$a(t_1)$: irrelevant if $k = 0$

$n_{(\ell)}(t_1)$:

→ ① Either compute from microphysics alone $T = T_1$

② Or determine by comparing predictions with current values of $n_{(\ell)}$.

A consequence

$$s = \frac{\sum_i \rho_i + \sum_i p_i - \sum_i \mu_i n_i}{T}$$

$$= \frac{\rho + p - \sum_\ell \tilde{\mu}_\ell \tilde{n}_{(\ell)}}{T}$$

$$T\,\delta s = \delta\rho - \sum_i \mu_i\,\delta n_i$$
$$= \delta\rho - \sum_i\sum_\ell \tilde{\mu}_\ell\, c_i^{(\ell)}\,\delta n_i$$
$$= \delta\rho - \sum_\ell \tilde{\mu}_\ell\,\delta\tilde{n}_{(\ell)}$$

$$\Rightarrow T\frac{ds}{dt} = \frac{d\rho}{dt} - \sum_\ell \tilde{\mu}_{(\ell)}\frac{d\tilde{n}_{(\ell)}}{dt}$$

$$T\frac{d}{dt}(s a^3) = T a^3\frac{ds}{dt} + 3a^2\dot{a}\,T\,s$$

$$= a^3\frac{d\rho}{dt} - a^3\sum_\ell \tilde{\mu}_{(\ell)}\frac{d\tilde{n}_{(\ell)}}{dt}$$
$$+ 3a^2\dot{a}\left(\rho + p - \sum_\ell \tilde{\mu}_{(\ell)}\, n_{(\ell)}\right)$$

$$= \underbrace{\frac{d}{dt}(\rho a^3) + 3a^2\dot{a}p}_{0} - \sum_\ell \tilde{\mu}_{(\ell)}\underbrace{\frac{d}{dt}(\tilde{n}_{(\ell)}a^3)}_{0}$$

$$= 0$$

$$\Rightarrow \frac{d}{dt}(s a^3) = 0$$

$\Rightarrow$ Total entropy / comoving volume remains constant

$\rightarrow$ Consequence of assumption of equilibrium

Condition for achieving equilibrium:

$H = \frac{\dot{a}}{a}$ = expansion rate.

$H = \sqrt{\frac{8\pi G}{3}\rho} \rightarrow$ calculable.

σ_A : average x section of scattering of particle A with other particles.

n : # density of particles.

Per unit time collision rate

$= n v \sigma \simeq n\sigma$ for $v \simeq 1$

Each scattering exchanges energy $\sim kT$ and particle # ~ 1. (assume)

$\Rightarrow$ We need

$n\sigma >> H$ for achieving equilibrium.

It could happen that particles exchange energy faster than H but # exchange is slower than $H \Rightarrow$ treat particle no. as conserved.

$\xleftarrow{T}$ Known $\longrightarrow$

100 GeV ——— 10 MeV

Unknown

$$\frac{1}{V}\ln Q_i = \mp \frac{g_i}{8\pi^3}\int 4\pi k^2\, dk\, \ln\left(1 \mp e^{-\beta\left(\sqrt{k^2+m_i^2}-\mu_i\right)}\right)$$

$$\rho_i = -\frac{\partial}{\partial\beta}\left(\frac{1}{V}\ln Q_i\right) = \frac{g_i}{8\pi^3}\int 4\pi k^2\, dk\, \frac{\sqrt{k^2+m_i^2}}{e^{\beta\left(\sqrt{k^2+m_i^2}-\mu_i\right)} \mp 1}$$

$$n_i = \frac{1}{\beta}\frac{\partial}{\partial\mu_i}\left(\frac{1}{V}\ln Q_i\right) = \frac{g_i}{8\pi^3}\int 4\pi k^2\, dk\, \frac{1}{e^{\beta\left(\sqrt{k^2+m_i^2}-\mu_i\right)} \mp 1}$$

In any conservation law the N_i's appear in combination $N_i - \bar{N}_i$

particles, antiparticles

$$\tilde{N}_\ell = \sum c_i^{(\ell)} N_i \to \sum c_i^{(\ell)} (N_i - \bar{N}_i)$$

$$\sum_\ell \tilde{\mu}_\ell N_\ell = \sum c_i^{(\ell)} \tilde{\mu}_\ell (N_i - \bar{N}_i)$$

$$\Rightarrow \mu_i = \sum_\ell c_i^{(\ell)} \tilde{\mu}_\ell, \quad \bar{\mu}_i = -\sum_\ell c_i^{(\ell)} \tilde{\mu}_\ell$$

$$\Rightarrow \mu_i = -\bar{\mu}_i$$

Now note that if $T \ll m_i$ then ρ_i, n_i get factors of $e^{-\beta(m_i-\mu_i)}$

$\to$ small unless $m_i - \mu_i \ll m_i$

In that case $m_i - \bar{\mu}_i = m_i + \mu_i \simeq 2m_i$

Either n_i or $\bar{n}_i$ must be small.

$\Rightarrow$ Both n_i and $\bar{n}_i$ are small

unless $n_i - \bar{n}_i \neq 0$ due to conservation laws.

At 10 MeV we can ignore heavy particles e.g. $\mu^{\pm}$, $\pi^{\pm}$ etc.

What about p, n?

At present $n_p - \bar{n}_p \neq 0$. $n_n - \bar{n}_n \neq 0$

$\Rightarrow$ They would have been non-zero at $T \sim$ MeV.

Recall: Matter radiation equality

at $\sim 10^{4}$ K $\sim$ eV.

~~Rat~~ ~~At~~ Since $T \sim \frac{1}{a}$ for radiation dominated, at $T \sim 10$ MeV, a was ~ 1 eV

10^7 times smaller.

$$\Rightarrow \left.\frac{\rho_r}{\rho_m}\right|_{T \sim 10\,\mathrm{MeV}} \sim 10^7$$

$$\frac{n_r}{n_m} = \frac{\rho_r / kT}{\rho_m / 1\,\mathrm{GeV}} \sim 10^{7} \times \frac{1\,\mathrm{GeV}}{10\,\mathrm{MeV}} \sim 10^9$$

proton mass