Quantum Field Theory:

→ It is a technique to deal with a system of many particles.
  • In particular, this technique is essential when the number of particles is not conserved.
  • For example in a photon gas, the number of photons is not conserved.
  • For example, the number of electrons is not conserved in high-energy, $e^-e^+$ pairs are created. Then, one needs quantum field theory.
  • It is also a convenient tool, when the number of particles is large, but conserved.

→ We begin with a system, where the number of particles is conserved.
  • Let us first consider the simplest possible system:−
    A single $(1+1)$ particle moving in some potential $V(x)$.

    The Schrödinger equation is:

$$\frac{\partial}{\partial t} \psi(x,t) = \hat{H} \psi(x,t);$$

where $\hat{H}$ is the Hamiltonian operator and

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(x).$$

Suppose, we want to find a general solution.
We want to find the eigenstates of $\hat{H}$ and then put in the time-evolution.
Eigenvalue of $\hat{H}$:

The eigenvalue equation is:

$$\hat{H}U_n(\vec{r}) = \epsilon_n U_n(\vec{r})$$

where $U_n(\vec{r})$ are the eigenfunctions and $\epsilon_n$ are the eigenvalues.

$\{U_n(\vec{r})\}$ forms a complete basis of states.

It comes out that we can write a general solution of the Schrödinger equation as:

$$\psi(\vec{r},t) = \sum_{n} a_n(t) U_n(\vec{r})$$

where $a_n(t) = a_n(0)e^{-i\epsilon_n t/\hbar}$

and $\psi(\vec{r},t)$ is written as a linear combination of the basis states, only with the coefficients having a certain time dependence.

Now, let us take, instead of one particle, $N$ identical bosonic (non-interacting) particles moving under the same potential.

Let us denote the $i^{th}$ particle's coordinates by:

$$\vec{r}_i \equiv (x_i, y_i, z_i)$$

and $\nabla_i \equiv (\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i})$ being the gradient operator for the $i^{th}$ particle's coordinates.

Then, a generic wave function will be described by $N$ of these coordinates, e.g.,

$$\psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t)$$

and this follows the Schrödinger equation.
\[ \hat{\psi}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) = \hat{\psi}_{\text{in}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) \]

where \( \hat{\psi}_{\text{in}} = \sum_{i=1}^{N} \hat{\psi}_i \) and \( \hat{\psi}_i = -\frac{\hbar^2}{2m} \nabla_i^2 + V(\vec{r}_i) \)

Now, if the particles are identical, the wave function should be symmetric for bosons and antisymmetric for fermions under the exchange of two particles.

So, here, \( \psi(\vec{r}_1, \cdots, \vec{r}_N, t) \) is symmetric under \( \vec{r}_i \leftrightarrow \vec{r}_j \) (for any pair of \( "i", \"j" \)).

We are now to introduce a basis of \( N \)-particle states which will be suitable for the evolution of a wave-function like this.

We write the basis of states as:

\[ \langle \vec{r}_1, \cdots, \vec{r}_N \rangle = U_{\mu_1}^{(1)} U_{\mu_2}^{(2)} \cdots U_{\mu_N}^{(N)} \]

But this is not a proper choice of a basis, as we want a function which should be symmetric under the exchange of any two particles clearly the above function does not satisfy this.

The basis should, therefore, be also symmetric under the exchange of particles.

Thus we add to this all terms with permutation of \( \vec{r}_1, \cdots, \vec{r}_N \) and we write:

\[ \langle \vec{r}_1, \cdots, \vec{r}_N \rangle = \frac{1}{\sqrt{N!}} \left( \sum_{\mu} U_{\mu_1}^{(1)} U_{\mu_2}^{(2)} \cdots U_{\mu_N}^{(N)} \right) \]

\[ + \cdots \left( N! \right) \text{-factorial term} \]
We have also made it automatically symmetric under the
exchange of $n_i$'s instead of just the $\pi_i$'s.

For example, $\phi_{3,1,2} = \phi_{1,2,3}$. We should not count them
separately.

- We want to check the normalization of this function:

Now, $\int d^3\pi_1 \cdots d^3\pi_N \, \phi^* \phi = \sum_{n_1, \ldots, n_N} (\pi_1, \ldots, \pi_N) \phi^* \phi_{n_1, \ldots, n_N}$. 

We will assume that the single particle wave functions are
normalized to 1, i.e.,

$$\int d^3\pi \, \phi^* \phi = 1$$

[orthonormal basis state].

- Let us first consider that $\{n_1, \ldots, n_N\}$ are all different.

Then the value of the integral will be:

$$1 \cdot \frac{1}{N!} \prod_{i=1}^N \phi^*(\pi_i) \phi(\pi_i)$$

We get something like,

$$\phi^* \phi = \sum_{n_1, \ldots, n_N} (\pi_1, \ldots, \pi_N) \phi^* \phi_{n_1, \ldots, n_N}$$

will be zero.

- Now, suppose, some of the indices $(n_i)$ are the same.

We use something called an *occupation number
representation.*

To see how many $n_i$'s are 1, i.e., how many
$\phi^* \phi$'s, then we see how many $n_i$'s are 2, then how
many $n_i$'s are 3 and so on.
Let $m_i$ of the $n_i$'s be 1.

Let $m_i$ of the $n_i$'s be 2.

and so on...

We have, $\sum m_i = N$.

Example: $m_1 = 1$, $m_2 = 0$, $m_3 = 2$, all others are zeros.

Then, $U(\vec{r_1}, \vec{r_2}) = \frac{1}{\sqrt{2}} \left[ U_1(\vec{r_1}) U_2(\vec{r_2}) + U_1(\vec{r_2}) U_2(\vec{r_1}) \right]$.

Here $m_1 = 1$

$m_2 = 2$

The function is normalized.

Example: $m_1 = 2$, $m_2 = m_3 = \cdots = 0$.

Then, $U(\vec{r_1}, \vec{r_2}) = \frac{1}{\sqrt{2}} \left[ U_1(\vec{r_1}) U_2(\vec{r_2}) + U_2(\vec{r_2}) U_1(\vec{r_1}) \right]

= \frac{1}{\sqrt{2}} U_1(\vec{r_1}) U_2(\vec{r_2})

This state is clearly not normalized.

We will try to find the norm of a state like this by using the occupation number representation.

If some of the $m_i$'s are > 1, then some of the terms are identical in $\vec{r}_2$.

There will be $(m_1! m_2! \cdots)$ identical terms.

If we permute these $(m_i)$ times, we get the same term.

So, altogether, there are $N!$ different terms.

$\sum m_i !=$
\[ u_{\frac{p}{m_1}, \ldots, \frac{p}{m_N}} (\bar{m}_1, \ldots, \bar{m}_N) = \frac{1}{\sqrt{N!}} \sum_{\pi} [T_1 + T_2 + \cdots + T_p], \]

where \( p = \frac{N!}{m_1! \cdots m_N!} \).

\[ T_1 = \sum_{\pi} u_{\pi}(\bar{m}_1) u_{\pi}(\bar{m}_2) \cdots u_{\pi}(\bar{m}_N) \]
\[ \times u_{\pi}(\bar{m}_{m_1+1}) \cdots u_{\pi}(\bar{m}_{m_1+m_2+1}) \cdots \]

\[ T_2, T_3, \ldots \text{ are obtained from } T_1 \text{ by} \]

inequivalent permutations \( \pi (\bar{m}_1, \ldots, \bar{m}_N) \).

If we permute any \( \bar{m}_i \) to \( \bar{m}_{m_1+i} \) terms, we get the same term. There are thus \( m_1! \) of such terms.

If we permute \( \bar{m}_1 \) with \( \bar{m}_{m_1+i} \), we will get a separate term.

In our example, \( m_1 = 2, m_2 = m_3 = \cdots = 0 \), then \( p = \frac{2!}{2!} = 1 \),

\[ m_1! = 2! \]

and we get our previous result.

It is now easy to calculate the norm,

\[ T_1^* T_1 = 1; \quad T_2^* T_2 = 1; \quad T_3^* T_3 = 0 \quad [\text{for one of the } \bar{m}_i \text{'s, at least, the } \]
\[ u(\bar{m}_i) \text{ will be different}].\]

\[ \sum_{\pi} \left\{ d^2 \bar{m}_1, \ldots, d^2 \bar{m}_N u_{\pi}(\bar{m}_1, \ldots, \bar{m}_N) \right\} \]
\[ = \frac{1}{N!} \sum \left\{ \frac{N!}{m_1! \cdots m_N!} \left[ 1 + 1 + \cdots + \left( \frac{N!}{m_1! \cdots m_N!} \text{ terms} \right) \right] \right\} \]
\[
\hat{h}(N) = \sum_{i=1}^{N} \hat{h}_i ; \quad \text{where} \quad \hat{h}_i = -\hbar^2 \nabla_i^2 + V(\vec{r}_i)
\]

Now, \( \hat{h}_N = \sum_{i=1}^{N} \hat{h}_i \) \((\vec{r}_1, \ldots, \vec{r}_N) = \left( \sum_{i=1}^{N} e_{\vec{r}_i} \right) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} \)

\[
\begin{align*}
\hat{h}(N) & \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N) \\
& = (\epsilon_{\vec{r}_1} + \epsilon_{\vec{r}_2} + \cdots + \epsilon_{\vec{r}_N}) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N)
\end{align*}
\]

So, we take out \( \left( \sum_{i=1}^{N} \epsilon_{\vec{r}_i} \right) \) common.

Now, if we use the occupation number representation,

\[
\begin{align*}
\hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N) & = (m_1 \epsilon_1 + m_2 \epsilon_2 + \cdots) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N) \\
& = (\sum_{i=1}^{N} m_i \epsilon_i) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N)
\end{align*}
\]

\[
\begin{align*}
\hat{h}(N) & \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N) \\
& = [\epsilon_{\vec{r}_1} (m_1 \epsilon_1 + m_2 \epsilon_2 + \cdots) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N)] + [\epsilon_{\vec{r}_2} (m_2 \epsilon_2 + m_3 \epsilon_3 + \cdots) \hat{u}_{\vec{r}_1, \ldots, \vec{r}_N} (\vec{p}_1, \ldots, \vec{p}_N)] + \cdots
\end{align*}
\]
Let us now consider a different system involving two particles.

For every single energy eigenstate, we consider a harmonic oscillator.

Let, for every $\omega_k$, we introduce a pair of operators $(a_k, a_k^+)$, such that:

$$[a_k, a_k^+] = \delta_{kk'}$$
$$[a_k, a_k] = [a_k^+, a_k^+] = 0$$

The index $k'$ labels the single-particle eigenstates.

We also write the Hamiltonian as:

$$H = \sum_k \varepsilon_k a_k^+ a_k$$

[Actually, $\sum_k \varepsilon_k [a_k^+ a_k + \frac{1}{2}]$, but we subtract this constant ground state energy]

$a_k, a_k^+$ are like the annihilation and creation operators.

It is as if the $k$th harmonic oscillator has energy $\varepsilon_k$ with ground state energy, $\frac{1}{2}$ term removed (subtracted).

Let us consider an electron in a centrifugal potential and we introduce a harmonic oscillator with a frequency such that $\varepsilon_k = \hbar \omega_k$.

Good: Show that this system is equivalent to the earlier system, if we identify:
\( \Phi_{m_1, \ldots, m_N} \rightarrow a_{m_1}^+ \ldots a_{m_N}^+ |0> \)

where \(|0>\) is the direct product of the ground states of the \(N\) number of harmonic oscillators.

\(|0>\) satisfies,
\[
\begin{align*}
\Phi_{m_1, \ldots, m_N} &= 0 \\
\end{align*}
\]

for each \(k\).

We have the matrix elements in both the quantum systems should be the same.

Here one particle with many frequencies is being mapped to many particles in the previous system.

We can also use the occupation number representation.

For this case, it will be:

\[
\begin{align*}
(a_{m_1}^+)^{m_1} & (a_{m_2}^+)^{m_2} \ldots \ldots |0> \\
\end{align*}
\]

It does not matter in which orders they appear as they commute.

We need to check that the energy eigenvalues must match.

To find:
\[
\begin{align*}
\hat{\mathcal{A}} & a_{m_1}^+ a_{m_2}^+ \ldots \ldots a_{m_N}^+ |0> \\
\end{align*}
\]

Now,
\[
\begin{align*}
[\hat{\mathcal{A}}, a_{m}^+] &= [\sum_{k} e_k a_{m}^+ a_{k}, a_{m}^+] \\
&= [e_5 a_{m}^+, a_{m}^+] \\
&= e_5 a_{m}^+ \\
&= e_5 a_{m}^+ + a_{m}^+ \hat{\mathcal{A}} \\
\end{align*}
\]

\[
\begin{align*}
\hat{\mathcal{A}} & a_{m_1}^+ a_{m_2}^+ \ldots \ldots a_{m_N}^+ |0> \\
&= (e_{m_1} a_{m_1}^+ + a_{m_1}^+ \hat{\mathcal{A}}) a_{m_2}^+ \ldots \ldots a_{m_N}^+ |0> \\
&= e_{m_1} a_{m_1}^+ a_{m_2}^+ \ldots \ldots a_{m_N}^+ |0> + a_{m_1}^+ (e_{m_1} a_{m_1}^+ a_{m_2}^+ \ldots \ldots a_{m_N}^+) |0> \\
&= (e_{m_1} + e_{m_2} + \ldots + e_{m_N}) a_{m_1}^+ a_{m_2}^+ \ldots \ldots a_{m_N}^+ |0>.
\end{align*}
\]
If there is \((a_i, a_i^+)^3\), it will count the states 3 times etc. So the energy eigenvalues match with our previous case.

Now we need to check the normalization, i.e. it has the exact same norm as \(\nu\).

Now, we calculate the norm of this state (in either of the two representations).

Let us take the occupation number representation.

Norm of \((a_i)^{m_1_1} (a_i^+)^{m_1_2} \ldots \ldots 10\) is:

\[
\langle 0 | \ldots \ldots (a_i)^{m_1_2} (a_i^+)^{m_1_1} (a_i^+)^{m_2_1} (a_i)^{m_2_2} \ldots \ldots 10 \rangle
\]

Now, \((a_i)^{m_1_1} \equiv a_i \ldots \ldots a_i^{m_1_1} \equiv (a_i)^{m_1_1} = a_i^1 \ldots \ldots a_i^{m_1_1}

\]

\[
\vdots \quad (a_i)^{m_1_1} (a_i^+)^{m_1_2} \ldots \ldots 10
\]

Again, \([a_i, a_i^+] = 1\]

\[
\vdots \quad a_i a_i^+ = 1 + a_i^+ a_i
\]

Now, \(\langle 0 | a_i^2 (a_i^+)^2 | 10 \rangle \]

\[
= \langle 0 | a_i, a_i, a_i^+, a_i^+ | 10 \rangle
\]

\[
= \langle 0 | a_i, (1 + a_i^+ a_i) a_i^+ | 10 \rangle
\]

\[
= \langle 0 | a_i, a_i^+ | 10 \rangle + \langle 0 | a_i, a_i^+ a_i, a_i^+ | 10 \rangle
\]

\[
= \langle 0 | a_i, a_i^+ | 10 \rangle + \langle 0 | a_i, a_i^+ (1 + a_i^+ a_i) | 10 \rangle
\]

\[
= \langle 0 | a_i, a_i^+ | 10 \rangle + \langle 0 | a_i, a_i^+ a_i^+ a_i a_i^+ | 10 \rangle
\]

\[
[\text{Again } a_i^+ | 10 \rangle = 0]
\]

\[
= 2 \langle 0 | a_i, a_i^+ | 10 \rangle = 2 \langle 1 | 10 \rangle = 2
\]

In the same way, \((a_i^+)^{m_2_1} (a_i)^{m_2_2} \ldots \ldots 1 \equiv \nu_2\)

\[
m_2 (m_2 - 1) \ldots \ldots 1 \equiv \nu_2
\]
Similarly, \( (a_2)^m (a_2^+)^m \)

\[ \begin{align*}
\text{Thus, we will get, } m_1! m_1! & \quad \text{which is the final result.}
\end{align*} \]

\[ \begin{align*}
\text{Check:} & \quad \langle 0 | (a_2)^2 (a_2^+)^2 | 10 \rangle \\
& \quad = \langle 0 | (a_2)(a_2^+) [1 + a_2 a_2^-] (a_2^+)^2 | 10 \rangle \\
& \quad = \langle 0 | a_2 a_2^- + a_2^+ a_2^- | 10 \rangle \\
& \quad + \langle 0 | a_2^- a_2^- + a_2^+ a_2^+ | 10 \rangle \\
& \quad = 2 + [a_2^- a_2^+] = 0 \\
& \quad \text{etc.}
\end{align*} \]

\[ \begin{align*}
& \quad = \langle 0 | a_2 (1 + a_2 a_2^-) a_2^+ | 10 \rangle \\
& \quad = \langle 0 | a_2 a_2^+ | 10 \rangle + \langle 0 | a_2 a_2^- a_2 a_2^- | 10 \rangle \\
& \quad = \langle 0 | a_2 a_2^+ | 10 \rangle + \langle 0 | a_2 a_2^- (1 + a_2 a_2^-) | 10 \rangle \\
& \quad - 2\langle 0 | a_2 a_2^+ | 10 \rangle + \langle 0 | a_2 a_2^- a_2 a_2^- | 10 \rangle \\
& \quad = 2.
\end{align*} \]

\text{• Thus, here we do not have to worry about symmetrizing the wave functions, as there are no \( \hat{S}_z \) 's.}

\text{The above formalism only works for \( \hat{S}_z \) 's uniquely distinguishable.}

\text{Here, in this system, the eigenvalues and the norm shows that the particles are indistinguishable.}

\text{In this system, the total number of particles were not necessarily specified.}

\text{The total number of particles is the sum of \( m_i \) 's.}

\text{The ground state will map to a zero-particle state, total \( m = 0 \).}

\text{For the first excited state, total \( m = 1 \).}
\((a_i^+) \gamma_i \approx U(\gamma_i) U(\gamma_i^+)\) (with the normalisation constant \(\bar{U}\)).

In the original system, we have to fix \(N\); here we do not have to.

If our quantum system is such that the particle number is not conserved, it goes through.

For example:

Photon gas: The total energy is conserved but the number of particles may change.

In the original system, we can not have a transition of a system of \(N\) particles to a system of \((N-1)\) particles. It will then move us from one Hilbert space to another.

But, in the present system, the Hilbert space allows all such systems.

In the original system, the position operator is an observable. It needs to be seen how we calculate it in the new system.

\(a_i^+ |N\text{-particle state}\rangle \approx |(N+1)\text{-particle state}\rangle\).

The number of harmonic oscillators have to do with the number of energy eigenstates (in general \(\infty\)).

Let \(\hat{H} = \sum_k \epsilon_k a_k^+ a_k\).

If we add a small perturbation which takes a 2-particle state to a 3-particle state, for example:
\[\hat{H}' = \sum_k \epsilon_k a_k^+ a_k + \left(\lambda a_1^+ + \lambda^* a_1\right)\] Hamiltonian.
It can take a 2-particle state + a 3-particle state or vice versa.

We may as well take the perturbation to be:
\[ \left( a_1^+ a_2^+ a_3^+ \text{H.c.} \right) \]
where H.c. = Hermitian conjugate.

What perturbation we add, depends on the experiment we are performing.

12.1.2011

• Recap:
  • A particle in a potential is described by:

\[ i \hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \hat{H} \psi(\vec{r},t) \]
where \[ \hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \]

We denote the eigenstates of this operator by \[ \psi_n(\vec{r}) \]
and that:
\[ \hat{H} \psi_n(\vec{r}) = E_n \psi_n(\vec{r}) \]

The set \{ \psi_n(\vec{r}) \} forms a basis.

• We then considered \(N\) mutually non-interacting identical bosons, each moving in the same potential \(V(\vec{r})\).

For this system, the Schrödinger equation takes the form:

\[ i \hbar \frac{\partial \psi(\vec{r}_1, \ldots, \vec{r}_N, t)}{\partial t} = \hat{H}_N \psi(\vec{r}_1, \ldots, \vec{r}_N, t) \]
where
\[ \hat{H}_N = \sum_i \frac{\hbar^2}{2m} \nabla_i^2 + V(\vec{r}_i) \]

The state of the system can be obtained by taking the direct product of the individual states and then symmetrizing.
We have:

\[ u(n_1, \ldots, n_N) = \frac{1}{N!} \left( u_{n_1}(f_1) \ldots u_{n_N}(f_N) \right) + \text{all permutations of } f_1, \ldots, f_N. \]

We claimed that this system is equivalent to another system (of infinitely many S.H.O.'s) each corresponding to an energy eigenstate, where the frequency \( \omega_n \) was related to the energy eigenvalue of the previous system by:

\[ \omega_n = \hbar \omega_n. \]

Claim: This quantum system is equivalent to another quantum system of infinite number of harmonic oscillators with \( \omega_n = \omega_n \), and the Hamiltonian of this system is:

\[ \hat{H} = \sum_{n=1}^{\infty} \varepsilon_n \hat{a}^+_n \hat{a}_n; \] where \( \hat{a}^+_n \) is the creation operator and \( \hat{a}_n \) is the annihilation operator of the \( n \)-th harmonic oscillator.

Then follow the commutator relations as follows:

\[ [\hat{a}_m, \hat{a}^+_n] = \delta_{mn}, \]
\[ [\hat{a}_m, \hat{a}_n] = 0, \]
\[ [\hat{a}_m^+, \hat{a}_n^+] = 0. \]

Even for degenerate eigenstates, we introduce these many harmonic oscillators.

If \( \nu (k) = \frac{1}{k^2} \), then all the \( \varepsilon_n \)'s will be different.

This system does not have a knowledge about \( \nu \) beforehand.
en's are obtained by solving single particle quantum mechanics
- they have no knowledge about the en's, they only depend
  on what state we are.

- If two quantum systems have all their matrix elements same
  if all the observables have a one-to-one map in the two
  systems, then the two systems are equivalent.

Simple Harmonic Oscillators are associated with the
en's to one simple harmonic oscillator to every energy
state.

The \( \hat{x} \) and \( \hat{p} \) of this new system created by the
combination of \((a, a^\dagger)\) has nothing to do with the \( \hat{x} \) and
\( \hat{p} \) operators of our original system.

[N.B.: Later, we will think of the wave function as a
dynamical variable and obtain \((a, a^\dagger)\) from it.]

We made the claim that:

\[
(\frac{\hbar}{2m_1}, \ldots, \frac{\hbar}{2m_n}) \leftrightarrow a^\dagger \ldots a^\dagger \quad |0\rangle \equiv |n_1, n_2, \ldots, n_n\rangle;
\]

where \( a_k(0) = 0 \) for every \( k \).

The fact that we have \( N \)-selection operators correspond to the
fact that we have an \( N \)-particle system.

Let \( m_1 \) of the \( m \)'s are 1,
\( m_2 \) of the \( m \)'s are 2, and so on.

These \( m \)'s have nothing (will not really) to do with the new
formulation.
The Hamiltonian as such knows nothing about the $n_i$'s as:

$$\hat{H} = \sum_{m=1}^{\infty} E_m a_m^\dagger a_m.$$ 

We checked that the eigenvalue of $\hat{H}$ are the same and that is:

$$\sum_{m=1}^{\infty} E_m.$$ 

We also checked that:

$$\int d^3n_1 \cdots d^3n_N \langle \eta_1, \cdots, \eta_N | \hat{H} | \eta_1, \cdots, \eta_N \rangle = \langle n_1, \cdots, n_N | \eta_1, \cdots, \eta_N \rangle.$$ 

If two quantum systems are the same, they must have the same inner product.

How many particles in the new formulation, we have, depends on the number of $a^\dagger$'s we use, while in our original problem, it is there in the Hamiltonian.

- **The Number Operator**:

  It is defined as:

  $$\hat{N} = \sum_{k=1}^{\infty} a_k^\dagger a_k.$$ 

  Now,  

  $$\hat{N} \langle n_1, n_2, \cdots, n_N \rangle = \sum_{k} a_k^\dagger a_k a_1^\dagger a_2^\dagger \cdots a_N^\dagger \langle 10 \rangle.$$
\[
\begin{align*}
&= \sum_{n} a_n^+ (\sum_{\kappa} a_{\kappa} + a_{\kappa}^+ a_{\kappa}) a_n^+ \cdots a_{n+N}^+ 10) \\
&= \sum_{n} a_n^+ (\sum_{\kappa} a_{\kappa}, a_{\kappa}^+) \cdots a_{n+N}^+ 10) \\
&\quad + \sum_{\nu} a_{\nu}^+ a_{\nu}^+ (\sum_{m} a_{m}, a_{m}^+) a_{\nu}^+ \cdots a_{n+N}^+ 10) \\
&= a_n^+ \cdots a_{n+N}^+ 10) \\
&\quad + a_{n+1}^+ a_{n+1}^+ a_{n+2}^+ \cdots a_{n+N}^+ 10) \\
&\quad + \cdots \text{(N times)} \\
&= \prod_{i=1}^{N} (a_{i}^+ a_{i}) \cdots (a_{n+N}^+ a_{n+N}) \\
&\quad \text{[\textit{We had similarly,}} \\
&\quad \sum_{\kappa=1}^{\infty} (\sum_{\kappa} a_{\kappa} + a_{\kappa}^+ a_{\kappa}) (a_{1}^+ a_{1}) \cdots (a_{n}^+ a_{n}) \\
&\quad = (\sum_{\kappa} \psi_{\kappa}) (a_{1}^+ a_{1}) \cdots (a_{n}^+ a_{n}) \text{[refer page 9].}} \\
\end{align*}
\]

- **Other observables:**
  - **Hermitean operators:**
    - N-particle quantum mechanics.
    - We have \( (\hat{x}_1, \hat{y}_1, \hat{z}_1) \), \( (\hat{x}_2, \hat{y}_2, \hat{z}_2) \), \( \vdots \), \( (\hat{x}_N, \hat{y}_N, \hat{z}_N) \) as the position operators.
  - Let us consider the position operator \( \hat{x}_1 \).
  - Q: Does this make a good quantum observable?
  - Ans: No. The first particle does not make any sense as the particles are indistinguishable.
but on the other hand, $\sum \hat{\alpha}$ is a good quantum observable.

This (\textit{hat}) is a special case of what is known as a one-body operator, and a general one-body operator (though for an $N$-particle system), has the following form:

$$\hat{b}(\mathbf{r}) = \sum_{\alpha=1}^{N} \hat{b}_\alpha \mathbf{r}$$

where $\hat{b}_\alpha = \hat{b}_\alpha (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$.

Here are not particularly general, but they are of a certain class (each element in the sum depends on the coordinates and momenta of one particle).

Thus, $\sum \hat{b}_\alpha$ is not a one-body operator.

We want to calculate:

$$\langle \mathbf{r}_1', \ldots, \mathbf{r}_N' | \hat{b}(\mathbf{r}) | \mathbf{r}_1, \ldots, \mathbf{r}_N \rangle$$

We say that there is an exact map with the new formulation, the matrix elements must match.

**Problem:** Find an operator $\hat{B}$ such that the matrix element of this is equal to:

$$\langle \mathbf{r}_1', \ldots, \mathbf{r}_N' | \hat{b}(\mathbf{r}) | \mathbf{r}_1, \ldots, \mathbf{r}_N \rangle$$

We define:

$$\hat{B} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \hat{a}_\alpha^\dagger \hat{a}_\beta$$

where $b_{\alpha \beta} = \int d^3r \ U(\mathbf{r}) \ U(\mathbf{r})^* \ \delta_{\alpha \beta}$

We just have to calculate the $\delta$-integrals.

we can do them by manipulating the $U$'s and the $U^*$'s by combinations of $a$, $a^\dagger$'s.
We will try to establish that the:

\[ L.H.S. = R.H.S. \]

Now, \( R.H.S. = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl} <0|a_{m_n} \ldots a_{m_1} a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} <0|a_{m_n} \ldots [S_{mn} + a_{m_1}^+ a_{m_1}]|a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} S_{k'm_1} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ + \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} S_{k'm_1} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ + \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} S_{k'm_1} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ + \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} S_{k'm_1} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ + \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

\[ = \sum_{k,l=1}^{\infty} b_{kl} S_{km_n} S_{k'm_1} <0|a_{m_n} \ldots a_{m_1}^+ a_{m_1}^+ \ldots a_{m_N}^+|10> \]

Where \( m_n \) means \( a_{m_n} \) is missing and \( m_1^+ \) means \( a_{m_1}^+ \) is missing.

Here we have used the fact that \([a_{m_n}, a_{m_1}^+] = \delta_{m_n m_1}\).
Thus, $R.H.S. = \sum_{i=1}^{N} \sum_{j=1}^{N} b_{m,n}^{ij} \langle \alpha_{n_1}^{i} \cdots \alpha_{n_{N-1}}^{i} a_{n_{N}}^{j} \cdots a_{n_{N+1}}^{j} | \rangle_{m,n}^{+10}$.

We define in general, the Kronecker delta as:

$$\delta_{\mu, \nu} \cdots \delta_{\mu_{i}, \nu_{i}} \cdots$$

For example, $\delta_{\mu_{1}, \nu_{1}; \mu_{2}, \nu_{2}} = \delta_{\mu_{1}, \mu_{2}} \delta_{\nu_{1}, \nu_{2}} + \delta_{\mu_{1}, \nu_{2}} \delta_{\mu_{2}, \nu_{1}}$.

We can, very well, generalize this to higher terms.

So, in terms of this, the matrix element (the point in the box) can be given by a simple expression:

$$\delta_{\mu_{1}, \nu_{1}; \mu_{2}, \nu_{2}} \cdots \delta_{\mu_{i}, \nu_{i}; \mu_{i+1}, \nu_{i+1}} \cdots \delta_{\mu_{N}, \nu_{i}; \mu_{i+1}, \nu_{i+1}}$$

Let us wonder:

$$\langle \alpha_{n_1}^{i} \cdots a_{n_{N}}^{j} \cdots \alpha_{n_{N+1}}^{j+1} | \rangle_{m,n}^{+10}$$

we start pushing the a's towards the right.

First, we push $a_{n_{N}}^{+}$, this will pick up a commutator with $a_{n_{N}}^+$, and then we push $a_{n_{N-1}}^{+}$, which picks up a commutator with $a_{n_{N-1}}^-$. But, $a_{n_{N}}^+$ could have picked up a commutator with any other $a_{n_{i}}^+$.

For example:

$$\langle \alpha_{n_1}^{i} a_{n_{2}}^{j} \cdots a_{n_{N}}^{j+1} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \langle \alpha_{n_1}^{i} [\delta_{n_1, n_{N}} + a_{n_{N}}^{+} a_{n_{N}}^{-}] a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \langle \alpha_{n_1}^{i} \delta_{n_1,n_{N}} a_{n_{N}}^{+} a_{n_{N}}^{-} + \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} | a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \langle \alpha_{n_1}^{i} \delta_{n_1,n_{N}} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$+ \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \delta_{n_1,n_{N}} \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$+ \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \delta_{n_1,n_{N}} \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$+ \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$= \delta_{n_1,n_{N}} \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$

$$+ \langle \alpha_{n_1}^{i} a_{n_{N}}^{+} a_{n_{N}}^{-} a_{n_{N+1}}^{j+10} | \rangle_{m,n}^{+10}$$
\[
\phi(s_1, \ldots, s_N) = \delta(s_1, \ldots, s_N)
\]

Our claim is that:

\[
\langle \chi_{1}^{*} \cdots \chi_{k}^{*} \chi_{k+1} \cdots \chi_{N} \rangle = \delta(s_{1}, \ldots, s_{N})
\]

We write, the R.H.S. matrix element =

\[
\sum_{s_{1}=1}^{N} \sum_{s_{2}=1}^{N} \cdots \sum_{s_{k}=1}^{N} \cdots \sum_{s_{N}=1}^{N} \delta(s_{1}, \ldots, s_{N})
\]

Now, L.H.S. = \[\int d^{2}n_{1} \cdots d^{2}n_{N} \sum_{s_{1}=1}^{N} \sum_{s_{2}=1}^{N} \cdots \sum_{s_{k}=1}^{N} \cdots \sum_{s_{N}=1}^{N} \delta(s_{1}, \ldots, s_{N})
\]

Now, \(\delta(n_{1}, \ldots, n_{N})\) is a function which by construction

is completely symmetric under \(n_{i} \leftrightarrow n_{j}\) (for any pair of \(i, j\)).

Again, \(\delta(n)\) is also symmetric under the exchange of any two

particles.

\(\delta(n_{1}, \ldots, n_{N})\) is completely symmetric

under \(n_{i} \leftrightarrow n_{j}\) (for any pair of \(i, j\)).

Again, \(\delta(n_{1}, \ldots, n_{N}) = \frac{1}{N!} \sum_{\text{all permutations of } n_{i}\text{'s}} \times \)

This term is also symmetric under \(n_{i} \leftrightarrow n_{j}\) (for any

pair of \(i, j\)).

So, instead of calculating the \((N!)\) integrals separately,

we perform only the first and multiply the whole by

\((N!)\) as each of these integrals gives us the same value

because of symmetry under \(n_{i} \leftrightarrow n_{j}\).

So, we write the L.H.S. as:

\[
\int d^{2}n_{1} \cdots d^{2}n_{N} \times N! - \frac{1}{N!} \sum_{\text{all permutations of } n_{i}\text{'s}} \times \]

\[
\sum_{n_{1}, \ldots, n_{N}} \phi(n_{1}, \ldots, n_{N}) \delta(n_{1}, \ldots, n_{N})
\]

where \(N!\) is the sum over all permutations of \(i, \ldots, n_{N}\).
\( P(i) \) is the \( i \)th entry in the \( i \)th position of the permutation.

For example: 
\[ P(1) = 1, \quad P(2) = 2, \quad P(3) = 3 \]

For the second term, 
\[ P(1) = 1, \quad P(2) = 3, \quad P(3) = 1 \]

and so on.

So we get:

\[ \sum_{n=1}^{N} b_{m_1, m'_1} \sum_{k=1}^{N} S_{m_k, m'_k} \]

Now, we first pull the sums out side the integrals and write the

\[ \sum_{k=1}^{N} \sum_{n=1}^{N} b_{m_1, m'_1} S_{m_k, m'_k} \]

Now, the R.H.S. is:

\[ \sum_{k=1}^{N} \sum_{n=1}^{N} b_{m_1, m'_1} S_{m_k, m'_k} \]

Now, \( \sum S_{m_1, m'_1} \) has \( N! \) terms but not all are non-zero.

They are non-zero only when \( j = P(i) \).

Now, when \( P(i) = i \) the remaining \((N-1)\) terms can permute

in \((N-1)!\) ways, so there are \((N-1)!\) terms.

\[ \sum_{k=1}^{N} \sum_{n=1}^{N} b_{m_1, m'_1} S_{m_k, m'_k} = \sum_{k=1}^{N} \sum_{n=1}^{N} b_{m_1, m'_1} S_{m_k, m'_k} \]

In the product, there is no \( m_k \) (or \( k \neq i \)), again

\[ m_k = m'_k \quad \text{and} \quad k = i \],

Thus, \( m'_k = m'_k \), never appears.

\[ \sum_{k=1}^{N} \sum_{n=1}^{N} b_{m_1, m'_1} S_{m_k, m'_k} = \delta_{m_1, m'_1} \delta_{m'_k, m'_k} \]

This also has \((N-1)!\) terms.
This is how, we make a one-body operator in our original system, to a one-body operator in the harmonic oscillator problem.

- **Two-body operator.**

\[ \hat{\sum}_{N}^{(2)} = \sum_{i<j} \hat{\pi}_{i,j} \hat{\pi}_{i,j} \]

- **Problem:** To find an appropriate operator \( \hat{\mathcal{V}} \) in the harmonic oscillator problem which corresponds to this.

It will involve two creation and two annihilation operators.

\[ \mathcal{U}(\pi_1, \ldots, \pi_N) \leftarrow a_{\pi_1} a_{\pi_2} \cdots a_{\pi_N} 10 \]

Now, let \( \pi_1 = 1, \pi_2 = 1, \pi_3 = 2, \pi_4 = 3 \), then we write it as:

\[ a_{\pi_1} a_{\pi_2} a_{\pi_3} a_{\pi_4} 10 \]

Coming back to the two-body operator, we define \( \hat{\mathcal{V}} \) in the second description as:

\[ \hat{\mathcal{V}} = \sum_{\pi, \nu} \sum_{N}^{(2)} \nu_{\pi, \nu} a_{\pi} a_{\nu} \]

where

\[ \nu_{\pi, \nu} = \int d^3r d^3r' U_{\pi, \nu}(r) U_{\pi, \nu}(r') \]

such that,

\[ \langle \pi_1, \ldots, \pi_N | \hat{\mathcal{V}} | \pi_1', \ldots, \pi_N' \rangle = \int d^3r \ldots d^3r' U_{\pi_1', \pi_1}(r) \ldots U_{\pi_N', \pi_N}(r) \]

\[ \langle \pi_1, \ldots, \pi_N | \hat{\mathcal{V}}^{(2)} | \pi_1', \ldots, \pi_N' \rangle = \int d^3r \ldots d^3r' U_{\pi_1', \pi_1}(r) \ldots U_{\pi_N', \pi_N}(r) \]

\[ \langle \pi_1, \ldots, \pi_N | \hat{\mathcal{V}}^{(3)} | \pi_1', \ldots, \pi_N' \rangle = \int d^3r \ldots d^3r' U_{\pi_1', \pi_1}(r) \ldots U_{\pi_N', \pi_N}(r) \]

\[ \langle \pi_1, \ldots, \pi_N | \hat{\mathcal{V}}^{(4)} | \pi_1', \ldots, \pi_N' \rangle = \int d^3r \ldots d^3r' U_{\pi_1', \pi_1}(r) \ldots U_{\pi_N', \pi_N}(r) \]
\[
L.H.S. = \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
= \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
= \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
+ \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
+ \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
+ \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
\text{L.H.S.} = \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]

\[
= \sum_{\nu, \nu'} \left< \omega_{\nu, \nu'} \right| \left| \sum_{n \nu} \left< \omega_{n \nu} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| a_{n \nu} + a_{n \nu}^{+} \right| \cdots \right| a_{n \nu} + a_{n \nu}^{+} \right| \right>
\]
\[ R \cdot H \cdot S = \left\{ d^3 \pi_1 \cdots d^3 \pi_N U^*_{\pi_1} \cdots U^*_{\pi_N} \right\} N_N^{(2,4)} U_{\pi_1} \cdots U_{\pi_N} \]

\[ = \left\{ d^3 \pi_1 \cdots d^3 \pi_N x N! \frac{1}{N!} U_{\pi_1} \cdots U_{\pi_N} \sum_{N=1}^{N!} \frac{\frac{\pi_1}{N} \cdots \frac{\pi_N}{N}}{\pi_1 \cdots \pi_N} \right\} \]

\[ \frac{1}{N!} \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \cdots U_{\pi_N} \left( \pi_N \right) \]

[By the same argument as in page 21.]

where \( \sum_{p} \) is the sum over all permutations of \( 1, \ldots, N \).

\[ \therefore R \cdot H \cdot S = \sum_{N} \sum_{\pi_1 \cdots \pi_N} \left\{ d^3 \pi_1 \cdots d^3 \pi_N U^*_{\pi_1} \cdots U^*_{\pi_N} \right\} \frac{N \cdot N! \cdot N!}{N!} U_{\pi_1} \cdots U_{\pi_N} \left( \pi_i \right) \cdot \frac{1}{N!} \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \cdots U_{\pi_N} \left( \pi_N \right) \]

\[ = \sum_{N} \sum_{\pi_1 \cdots \pi_N} \frac{N \cdot N! \cdot N!}{N!} \frac{1}{N!} \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \cdots U_{\pi_N} \left( \pi_N \right) \]

\[ = \sum_{N} \sum_{\pi_1 \cdots \pi_N} \frac{N \cdot N! \cdot N!}{N!} \frac{1}{N!} \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \cdots U_{\pi_N} \left( \pi_N \right) \]

\[ \sum_{i=1}^{N} \sum_{\pi_1 \cdots \pi_N} \frac{N \cdot N! \cdot N!}{N!} \frac{1}{N!} \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \cdots U_{\pi_N} \left( \pi_N \right) \]

Now, \( \sum_{p} \prod_{i=1}^{N} U_{\pi_i} \left( \pi_i \right) \) has \( (N-2)! \) terms because if we

fix \( a, k, \) then we have \( (N-2) \) places which can be filled in

\( (N-2)! \) ways.

\[ \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) \text{ and } \pi_k \neq \pi_i \] for \( k \neq i \).

\[ \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) = \delta \left( \pi_1, \ldots, \pi_i \right) \cdots \delta \left( \pi_i, \ldots, \pi_N \right) \]

\[ \therefore \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) = \delta \left( \pi_i \right) \cdots \delta \left( \pi_i \right) = \delta \left( \pi_i \right) \]

\[ \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) = \delta \left( \pi_i \right) \cdots \delta \left( \pi_i \right) = \delta \left( \pi_i \right) \]

\[ \therefore \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) = \delta \left( \pi_i \right) \cdots \delta \left( \pi_i \right) = \delta \left( \pi_i \right) \]

\[ \prod_{k \neq i}^{N} U_{\pi_k} \left( \pi_k \right) = \delta \left( \pi_i \right) \cdots \delta \left( \pi_i \right) = \delta \left( \pi_i \right) \]
Hence, L.H.S. = R.H.S. (proved).
• We distinguish particles by their position wave function. Electrons inside a certain volume, for example. We are not making any distinction between the "first" electron and the "second" electron.

Just like we can ask, how many electrons within a certain energy are there, we can very well ask, how many electrons with a specific position wave function there are.

• The information of the potential in the first description has gone into the $c_n$'s in the harmonic oscillator formulation. We cannot see that here. We do not mean an infinite number of harmonic oscillators moving in a 3-D space. Construction of such a model system depends on the $c_n$'s which clearly depend on $V(r)$. We have $c_n = e^{i\omega_n t}$, where $\omega_n$ is the frequency of the harmonic oscillator with energy $E_n$.

• Two-body operator:

$$H^{(2)} = \sum_{\{\pm\}} \sum_{j=1}^{N} \sum_{k=1}^{N} \Delta \hat{H}_{ij}^{(2)}$$

To simplify it, we have to sum over all $\{\pm\}$. If $j = k$, then it becomes like a one-body operator, which we have already dealt with.

Q. What is the operator $\hat{\nabla}$ in the second description such that the following holds:

The matrix element $\langle m_1, \ldots, m_N | \hat{V} | n_1^*, \ldots, n_N^* \rangle$

$$= \int d\vec{r}_1 \cdots d\vec{r}_N \hat{V}(\vec{r}_1^*, \ldots, \vec{r}_N^*) \langle m_1, \ldots, m_N | \hat{H} | n_1^*, \ldots, n_N^* \rangle$$

If this is the case, we say that the two-body operator $\hat{\nabla}$ gets mapped $\hat{\nabla}$.
Answer: \[ \hat{V} = \sum_{\nu_1, \nu_2} n_{\nu_1, \nu_2} a_{\nu_1}^\dagger a_{\nu_2} \]

where \[ n_{\nu_1, \nu_2} = \int d^3 \pi \ d^3 \tau \ U_{\nu_1}^*(\vec{\pi}) U_{\nu_2}(\vec{\pi}) \]
\[ = \sum_{\nu}(\hat{n}_{\nu_1}, \hat{n}_{\nu_2}, \hat{\pi}_{\nu_1}, \hat{\pi}_{\nu_2}) U_{\nu_1}(\vec{\pi}) U_{\nu_2}(\vec{\pi}) \]

where \[ n(\pi_1, \pi_2, \vec{\pi}_1, \vec{\pi}_2) \] is a differential operator.

---

**Note:** If we did not choose the \( \hat{V} \)'s to be eigenstates of \( \hat{H} \), then we would have gotten cross terms for \( \hat{H} \) as well.

---

**Note:** If we simultaneously interchange \( \nu \to \lambda \) and \( \pi \to \phi \), then \( n_{\nu_1, \nu_2} \) and \( \hat{V} \) are symmetric.

---

**Note:** Here we did not assume that \( i < j \), but only assumed that \( \hat{\pi}_j \) is in the sum.

---

**Possible application of the two-body operator:** Suppose we have \( N \) particles, each moving under a potential \( V(\vec{r}) \), and now suppose further that each of the particles interacts with one another via a potential, \( \sum_{\nu, \nu'} n(\hat{n}_{\nu_1}, \hat{n}_{\nu_2}, \hat{\pi}_{\nu_1}, \hat{\pi}_{\nu_2}) \). These are non-mom particles.

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**Q:** What is an equivalent description with \( (a_\nu, a_\nu^\dagger) \)? Is it possible?

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**Ans:** Now, to our previous harmonic oscillator Hamiltonian, we add \( \hat{V} \), such that now the new effective Hamiltonian is:

\[ \hat{H} = \sum_{\nu} E_{\nu} a_{\nu}^\dagger a_{\nu} + \sum_{\nu_1, \nu_2} n_{\nu_1, \nu_2} a_{\nu_1}^\dagger a_{\nu_2} \]

We now find the eigenvalues of the original formulation.

We can find the eigenvalues of this new problem.
Claim: The eigenvalues in both the representations should be the same.

Here, in the new representation, the number operator

\[ \hat{N} = \sum_{k} a_{k}^{+} a_{k} \]

This operator commutes with the Hamiltonian \( \hat{H} \), i.e., \([\hat{N}, \hat{H}] = 0\).

Proof:

\[ [\hat{N}, \hat{A}] = \sum_{m, k} [a_{m}^{+} a_{m}, a_{k}^{+} a_{k}] + \sum_{m, k, l, k'} [a_{m}^{+} a_{m}, a_{l}^{+} a_{k'}] \]

Now, \([\hat{A}, \hat{B}] = [\hat{A}, \hat{C}] + \hat{B} [\hat{A}, \hat{C}]\)

\[ [a_{m}^{+} a_{m}, a_{k}^{+} a_{k}] = [a_{m}^{+} a_{m}, a_{k}] a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}] \]

Again, \([\hat{A}, \hat{B}, \hat{C}] = [\hat{A}, \hat{B}, \hat{C}] + [\hat{A}, \hat{C}] \hat{B}\)

\[ [a_{m}^{+} a_{m}, a_{k}^{+} a_{k}] = [a_{m}^{+} [a_{m}, a_{k}^{+}] + [a_{m}^{+} a_{k}], a_{m}^{+} a_{k}] \]

\[ = a_{m}^{+} [a_{m}^{+} a_{m}, a_{k}] + [a_{m}^{+} a_{m}, a_{k}] a_{m}^{+} \]

\[ = a_{m}^{+} a_{m}^{+} a_{m} a_{k} + a_{k}^{+} (a_{m} a_{m}^{+}) a_{m} \]

\[ = a_{m}^{+} a_{m}^{+} a_{m} a_{k} + a_{k}^{+} (-a_{m} a_{m}^{+}) a_{m} \]

\[ = a_{m}^{+} a_{m}^{+} a_{m} a_{k} + a_{k}^{+} a_{m} a_{m}^{+} a_{m} \]

\[ \therefore [a_{m}^{+} a_{m}, a_{k}^{+} a_{k}] = [a_{m}^{+} a_{m}, a_{k}^{+}] a_{k}^{+} a_{m}^{+} \]

\[ + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ = a_{m}^{+} a_{m} a_{k}^{+} a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ = a_{m}^{+} a_{m} a_{k}^{+} a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ = a_{m}^{+} a_{m} a_{k}^{+} a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ = a_{m}^{+} a_{m} a_{k}^{+} a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ = a_{m}^{+} a_{m} a_{k}^{+} a_{k}^{+} + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]

\[ + a_{k}^{+} [a_{m}^{+} a_{m}, a_{k}^{+}] \]
\[ [\hat{N}, \hat{a}^+_m \hat{a}_n + \hat{a}^+_n \hat{a}_m] = 2 \hat{a}^+_m \hat{a}^+_n \hat{a}_n \hat{a}_m + \hat{a}^+_m \hat{a}^+_n (-\delta_{mn}) \hat{a}_m \hat{a}_n + \hat{a}^+_n \hat{a}^+_m (-\delta_{mn}) \hat{a}_m \hat{a}_n \]

\[ = 2 \hat{a}^+_m \hat{a}^+_n \hat{a}_n \hat{a}_m - 2 \hat{a}^+_m \hat{a}^+_n \hat{a}_n \hat{a}_m \]

\[ = 0 \]

But, we must note that \( \hat{N} \) does not commute with any single term in \( \hat{a} \). Thus, the above statement is in no sense general. If the number of creation and annihilation operators in both the terms are not equal, then \( [\hat{N}, \hat{a}] \neq 0 \). In this case, in the first term, \( \hat{a} \) (inside the sum) there is one creation and one annihilation operator (each) while in the second term, there are two each.

Thus, we can always find a Hamiltonian which does not commute with \( \hat{N} \).

But, here, we can simultaneously diagonalise \( \hat{N} \) and \( \hat{a} \).

The eigenvalues of \( \hat{N} \) will be \( \hat{N} \) (the number of particles in the original system). Here in this new formulation, while writing down the Hamiltonian, we did not incorporate \( \hat{N} \) explicitly.

* For example: If we want solutions with \( \hat{N} = 19 \), we only write the Hamiltonian with the states whose eigenvalues with \( \hat{N} \) are equal to 19.

* We note that this trick still nowhere deals with fields.

* We now describe a third quantum system which, we will find out is equivalent to these two systems.

In this third approach, the meaning of Quantum Field Theory...
will become clear.

Let us (again) consider the single particle Schrödinger equation, e.g.,

\[ \frac{\hat{H}}{2m} \psi (\vec{r}, t) = \frac{\partial^2 \psi}{\partial t^2} + V(\vec{r}), \]

We now forget about the fact that this \( \psi (\vec{r}, t) \) describes a quantum state, let us think that this \( \psi (\vec{r}, t) \) describes a classical field and that the Schrödinger equation satisfies this. A field is supposed to be a function of \( \vec{r}, t \).

Here we treat \( \psi (\vec{r}, t) \) as a classical field and quantize it. This is known as second quantization.

**Quantum Field Theory / Second Quantization:**

Q: How do we quantize a classical system?

A: We find the canonically conjugate coordinates and momenta of the system. We find the Poisson brackets and then make their quantum (commutator) brackets.

**Step 1:** Find a Lagrangian / Action which upon varying, gives the Schrödinger equation as its classical equation of motion.

Answer:

\[ L = \left( \frac{d^2 \psi}{dt^2} - \hat{V}(\vec{r}, t) \right) \psi \]

Here \( \hat{V} \) is thought of as an ordinary differential operator.
Q.: What does a Lagrangian do?

A.: It takes a field configuration and generates a number at a

The action, \[ S = \int \! dt \! d^3 \! \mathbf{r} \, \psi^* (\mathbf{r}, t) \left( \partial_t \psi (\mathbf{r}, t) - \mathbf{\nabla} \psi (\mathbf{r}, t) \right) \]

We extremize this action. The first variation of the action should be zero. We change \( \psi \rightarrow \psi + \delta \psi \).

\[ \delta S = \int \! dt \! d^3 \! \mathbf{r} \, \psi^* (\mathbf{r}, t) \left( \partial_t \delta \psi (\mathbf{r}, t) - \mathbf{\nabla} \delta \psi (\mathbf{r}, t) \right) \]

Here \( \psi (\mathbf{r}, t) \) is a field which we vary. We vary it with respect to the field itself. Here \( \psi (\mathbf{r}, t) \) at every \( \mathbf{r} \) is a dynamical variable, \( \mathbf{r} \) is a continuous variable.

\[ \delta S = \int \! dt \! d^3 \! \mathbf{r} \, \psi^* (\mathbf{r}, t) \left( \partial_t \delta \psi (\mathbf{r}, t) - \mathbf{\nabla} \delta \psi (\mathbf{r}, t) \right) \]

\[ = \int \! dt \! d^3 \! \mathbf{r} \, \left\{ - \frac{\partial^2 \psi (\mathbf{r}, t)}{\partial t^2} - \mathbf{\nabla} \psi (\mathbf{r}, t) \right\} \delta \psi (\mathbf{r}, t) \]

\[ \left[ \frac{\partial^2 \psi (\mathbf{r}, t)}{\partial t^2} \right] \]

\[ \left[ \frac{\partial^2 \psi (\mathbf{r}, t)}{\partial t^2} \right] \]

The first term vanishing at the boundary.

again \( \frac{\partial^2 \psi (\mathbf{r}, t)}{\partial t^2} = -\frac{\hbar^2 \Delta}{2m} + V(\mathbf{r}) \).
\[ \int dt \int d^3r (\nabla^2 \Psi) \Psi = - \int d^3r \int d^3r \Psi \nabla \cdot (\nabla \Psi) \Psi + \int d^3r \int d^3r \Psi \nabla \times (\nabla \times \Psi) \Psi. \]

The first term vanishes at the boundary. We have

\[ \int dt \int d^3r (\nabla^2 \Psi) \Psi = - \int d^3r \Psi \nabla \cdot (\nabla \Psi) \Psi + \int d^3r \int d^3r \Psi \nabla \times (\nabla \times \Psi) \Psi. \]

Again, the second term vanishes for any arbitrary variation of \( \Psi \).

A useful trick: We can think of the variations of \( (S_4) \) and \( (S_4^*) \) as independent (though they are not so) variables. Why can we treat them so?

We imagine that we turn a complex variable \( \zeta \).

Support we have an equation like,

\[ F \bar{z} + \bar{F} z = 0. \]

What is the consequence?

Let \( F = F_1 + i F_2 \), \( \bar{z} = z_1 + i z_2 \), where \( F_1, F_2, z_1, z_2 \) are real functions.

\[ (F_1 + i F_2) (\bar{z}_1 + i \bar{z}_2) + (F_1 - i F_2) (\bar{z}_1 - i \bar{z}_2) = 0 \]

or

\[ 2 F_1 z_2 - 2 F_2 z_1 = 0 \]

Now, \( z_1 \) and \( z_2 \) are independent. So, if we vary \( z_1 \), we get \( F_1 = 0, F_2 = 0 \) or \( F = F_1 + i F_2 = 0 \).

Thus in this problem, we could have treated \( S_4 \) and \( S_4^* \) as independent variables and obtained the same result, i.e., \( F = 0 \).

Coming back to our problem, if \( \delta S = 0 \), then changing \( (S_4) \) and \( (S_4^*) \) separately gives us the equations:

\[ \delta S_4 = 0, \delta S_4^* = 0. \]
\[\frac{\partial \psi}{\partial t} - \hat{H} \psi = 0\]

and \(-i\hbar \frac{\partial \psi^*}{\partial t} = \hat{H}^* \psi^* = 0\)

which are nothing but the Schrödinger equation and its conjugate equation.

Alternatively: We write \(\psi = \psi_1 + i \psi_2\); where \((\psi_1), (\psi_2)\) are real.

The above formulation holds as long as the Hamiltonian is Hermitian.

\[\psi = \int d^3x \left[ (\psi_1 - i \psi_2) \left( \frac{\partial}{\partial t} + \hat{H} \right) \right] + \left( \psi_1 + i \psi_2 \right) \left( -i \hbar \frac{\partial}{\partial t} + \hat{H}^* \right)\]

Here we assume \(\int \psi \psi^* = \int (\psi_1 \psi_1^* + \psi_2 \psi_2^*)\). This and the work as long as \(\hat{H}\) is Hermitian.

\[\psi = \int d^3x \left[ (\psi_1 - i \psi_2) \left( \frac{\partial}{\partial t} + \hat{H} \right) \right] + \left( \psi_1 + i \psi_2 \right) \left( -i \hbar \frac{\partial}{\partial t} + \hat{H}^* \right)\]

We will treat in many cases as if \(\psi\) and \(\psi^*\) complex conjugate are independent variables.

Thus, here taking \(\psi = 0\) and \(\psi^* = 0\) and \((\psi_1), (\psi_2)\) independently, we get:

\[\frac{\partial \psi}{\partial t} - \hat{H} \psi = 0\]

and \(\frac{\partial \psi^*}{\partial t} + \hat{H}^* \psi^* = 0\)

Now, multiplying the second equation by \((\psi^*\psi)\) and adding (subtracting) it to (from) the first equation gives, thus:
Schrödinger equation (and its complex conjugate).

Now, we replace the continuous index \( k \) by a discrete index. We replace it, as we do not always have a good intuition of what these continuous variables actually do.

Suppose \( \psi_n(t) \) is a complete basis of states. It is not necessary to choose them as the eigenstates of \( \hat{H} \).

But for simplicity, we do take them to be the eigenstates of \( \hat{H} \).

Then, we write:

\[
\psi(\mathbf{r}, t) = \sum_n a_n(t) \psi_n(\mathbf{r}).
\]

We note that \( \psi(\mathbf{r}, t) \) has exactly the same information as \( a_n(t) \) (though incidentally has a discrete index).

**Note:** Free particles have continuous index, which we can discretize by forcing them in a box.

We impose here, consider the \( a_1(t), a_2(t), \ldots \) as the various coordinate variables.

i.e., \( a_n(t) = \) coordinate variables.

- A map between classical mechanics (whose basic objects are the trajectories, \( q(t) \)), the action must be extremized under the variation of these trajectories) and quantum field theory (specifying \( \psi(\mathbf{r}, t) \) is equivalent to specifying the \( a_n(t) \)).

The analog of these trajectories \( q(t) \), are these \( a_1(t), a_2(t), \ldots \).

Thus classical field theory is classical mechanics with infinitely many of variables.
• $\psi(t, r)$, because of its continuous wavelet $\psi^*$ may seem to 
  have an uncountable infinity of basis states but the right 
  hand side shows a countable infinity of basis states. We can 
  always do it. We can always use a measurable loop.

An $(n)$'s functional form is yet unknown.

Here, we have assumed that the Hamiltonian $H_n(t)$ is not a 
function of time, that is,

$H_n \neq H_n(t)$.

- Let us now write the original action in terms of these $a_n(t)$.

\[ S = \int dt \sum_r \psi^*(t, r) \left( \frac{\partial \psi(t, r)}{\partial t} - \hat{H}(\psi(t, r)) \right) \]

\[ = \int dt \sum_r \sum_k \psi^*(t, r) \left( \frac{\partial \psi(t, r)}{\partial t} - \hat{H}(\psi(t, r)) \right) \]

\[ = \int dt \sum_r \sum_k \psi^*(t, r) \left( \frac{\partial \psi(t, r)}{\partial t} - \hat{H}(\psi(t, r)) \right) \]

\[ = \int dt \sum_r \sum_k \psi^*(t, r) \left( \frac{\partial \psi(t, r)}{\partial t} - \hat{H}(\psi(t, r)) \right) \]

We now use the orthogonality of $\psi_k(t)$ to get:

\[ \int d^3r \psi_k^*(r) \psi_k(r) = \delta_{kk'} \]

\[ S = \int \sum_k \psi_k^*(t) \left( \frac{\partial \psi_k(t)}{\partial t} - \hat{H}_{kk'} \right) \]

We think this as a classical system with an infinite 
number of degrees of freedom.

\[ S_{\infty} = 0 \]

We thus get from the Euler-Lagrange's equations:

\[ \frac{\partial \psi_k(t)}{\partial t} - \hat{H}_{kk'} = 0 \]

and its complex conjugate:

\[ \frac{\partial \psi_k(t)}{\partial t} = -\frac{\partial \psi_k(t)}{\partial t} \]
Integrating, we get:

\[
\ln a_k(t) = -c_k t + \ln a_k(t=0)
\]

or \( a_k(t) = a_k(t=0) e^{-c_k t} \)

This is clearly a consequence of the equation of motion. We did not assume this form for the \( a_k(t) \) at the beginning.

\[
\delta E = \sum_k \int \delta a_k(t) \left\{ \frac{\partial}{\partial t} \frac{\delta a_k(t)}{\delta a_k(t)} + e_k \frac{\delta a_k(t)}{\delta a_k(t)} \right\} dt
\]

\[
= \sum_k \int \delta a_k(t) \left\{ \frac{\partial}{\partial t} \frac{\delta a_k(t)}{\delta a_k(t)} + e_k \frac{\delta a_k(t)}{\delta a_k(t)} \right\} dt
\]

[We work]

\[
= \sum_k \int \frac{\delta a_k(t)}{\delta a_k(t)} \delta a_k(t) dt
\]

Again, independently varying \( \delta a_k(t) \) and \( \delta a_k^*(t) \) leads us to the same equation.

The Lagrangian, \( L = \sum_k a_k(t) \left( \frac{\partial}{\partial t} a_k(t) - e_k a_k(t) \right) \)

[One correct way to arrive at it is to write it in terms of the real coordinates.]

Here the coordinates are \( a_k(t) \) 's.

Moreover, the conjugate momenta to \( a_k(t) \) are:

\[
\pi_k(t) = \frac{\partial}{\partial \frac{\partial L}{\partial \frac{\partial a_k(t)}{\partial t}}} = \frac{\partial}{\partial \frac{\partial L}{\partial \frac{\partial a_k^*(t)}{\partial t}}} = \frac{\partial}{\partial \frac{\partial L}{\partial \frac{\partial a_k(t)}{\partial t}}} = \frac{\partial}{\partial \frac{\partial L}{\partial \frac{\partial a_k^*(t)}{\partial t}}}
\]

We postulate the Poisson-bracketed relation as:

\[
\{ a_k(t), \pi_{k'}(t') \} = \delta_{kk'}
\]
The Hamiltonian, \( H = \sum_k \varepsilon_k a_k^\dagger a_k + \frac{1}{2} \sum_{k \neq \ell} \Delta_{k\ell} (a_k^\dagger a_\ell + a_\ell^\dagger a_k) \)

Now, we replace the \( \varepsilon \) by the \( \varepsilon^\dagger \) and we also replace the Poisson-Bracket \( \{a_k, a_\ell^\dagger\} \) by the quantum commutator \( [a_k, a_\ell^\dagger] \) and

\[
\frac{1}{\hbar} \{, \} \to \frac{1}{\hbar} [\, , \]

These are the rules.

We get:

\[
\frac{1}{\hbar} \{a_k, a_\ell^\dagger\} = \hbar \delta_{k\ell}
\]

and the Hamiltonian,

\[
\hat{H} = \sum_k \varepsilon_k a_k^\dagger a_k
\]

Now, the Poisson-Bracket, \( \{a_k, a_\ell^\dagger\} = 0 \) (as both are coordinates).

Assuming these, we get:

\[
\{a_k, a_\ell^\dagger\} = 0 \text{ and } \{a_k^\dagger, a_\ell^\dagger\} = 0 \text{ respectively.}
\]

Thus, we have recovered exactly the system that we found in the multi-particle mechanics.

Here the \( a_k^\dagger \) are encoding the same information as \( \psi \). Thus, these \( a_k^\dagger \) are the oscillation modes of \( \psi \).
N.B. - By construction, this automatically gives a Bose result.

For Fermions, we have to think of another way. At no point do we ever move into a boson picture. At no point do we get multiple bosons here, not multiple Fermions. Here, in our calculation, we never used any symmetry element.

To measure the position of an electron:

Whichever operator we choose, should be symmetric under the interchange of any two electron positions.

Let us choose the box-like step function,

$$\Theta(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1 & \text{for } y \geq 0 \end{cases}$$

Let us now consider a function,

$$\Theta(x) \Theta(y-x) \Theta(y) \Theta(y-y) \Theta(z) \Theta(z-y)$$

This function takes the value 1, when the 4th electron lies inside the box with edges $-1$, $0$, $1$. Otherwise it gives zero.

Then, we consider the following operator,

$$\sum_{i=1}^{n} \hat{a}_i \Theta(x) \Theta(y-x) \Theta(z) \Theta(z-y) \Theta(z) \Theta(z-y)$$

where now these $\hat{a}_i$, $\hat{b}_x$ and $\hat{b}_z$ are operators.

Let us consider a situation, where there is only one electron in the box, then, if we take the expectation value of this operator in a proper basis, we should get the position eigenvalue.
Let us now consider an array of $N$ electrons in space.

For simplicity, let us choose the electrons to be having the wave function,

$$\frac{1}{\sqrt{N!}} \psi_1(\mathbf{r}_1) \psi_2(\mathbf{r}_2) \cdots \psi_N(\mathbf{r}_N) \text{ all permutations of } \mathbf{r}_1, \ldots, \mathbf{r}_N.$$

This is an allowed wave function and is completely symmetrical under the interchange of any pair of $\mathbf{r}_i, \mathbf{r}_j$ for any $i, j = 1$ to $N$.

We may introduce a constraint.

$$\psi_i(\mathbf{r}) = 0 \text{ when } \mathbf{r} \text{ is inside the box for } i = 2, \ldots, N.$$  

$$\psi_1(\mathbf{r}) = 0 \text{ when } \mathbf{r} \text{ is outside the box.}$$

So, let our wave function be of the above form with these above constraints.

The above operator will measure the position.

We want to know what the expectation value of this operator is in this state.

The $i=1$ term (in the operator) will contribute for the first term in the wave function.

The $i=2$ term (i.e., $\psi_2$, associated with $\mathbf{r}_2$, now) will contribute for the second term in the wave function, and so on.

We have to choose $\psi_1(\mathbf{r})$ as the position eigenstate. Then now, we only know that $\psi_1(\mathbf{r})$ is when it is inside the box.

We can not ask for it collapsing into the position eigenstate.
of the "first" or the "second" electron, etc.

If we take the position eigenstates as $\Psi_1(\vec{r})$, $\Psi_2(\vec{r})$ etc., then it will remain a position eigenstate.

If the position eigenstate inside the box collapses, it does not mean that the position eigenstates outside the box also collapse.

For each box, we have a different operator as the $\hat{H}$ will change.

Here our universe is the whole system. The particle can be anywhere in space, the box (hypothetical) is just a subsystem.

If there are $N$ electrons in the system, we have to symmetrize over all the electrons' positions.

We can only restrict a condition on the wave function. We cannot put a constraint that to choose a wave function where the electrons are inside the box, the wave functions inside and outside the box should not overlap. They are either outside or inside.

We can distinguish a subsystem by restricting the wave function, not by restricting the identity of the particles.

- **Remark:** We took a system of $N$ identical, non-interacting bosons, each moving under a potential $V(\vec{r})$.

So, the corresponding Hamiltonian was the sum of the single particle Hamiltonians:

$$\hat{H}_{(N)} = \sum_{k=1}^{N} \hat{H}_k$$

where

$$\hat{H}_k = \frac{\hat{p}_k^2}{2m} + V(\vec{r}_k).$$
We then we expressed as:
\[ \hat{h}(\vec{r}) = \hat{e}_n \psi_n(\vec{r}), \]

Then we found that this system is equivalent to a system of an infinite number of harmonic oscillators with Hamiltonian
\[ \hat{H} = \sum_n \varepsilon_n \hat{a}_n^+ \hat{a}_n, \]

where \([\hat{a}_n, \hat{a}_n^+] = \delta_{mn}, [\hat{a}_m, \hat{a}_n] = 0; [\hat{a}_m^+, \hat{a}_n] = 0, \]

are the commutation relations.

We also found that this system allows us to consider different number of particles.

There is a number operator defined as:
\[ \hat{N} = \sum_n \hat{a}_n^+ \hat{a}_n \] and its eigenvalue is \( \hat{N} \).

So, this new system can describe quantum mechanical system with multiparticle.

When we found that this in turn is equivalent to the second quantization of Schrödinger equation for a single particle, we take the Schrödinger equation
\[ i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \hat{H} \psi(\vec{r},t), \]

where \( \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \vec{r}^2} + V(\vec{r}) \). Now is an ordinary differential operator.

We now forget what the \( \psi \)'s are and then treat them as classical fields, functions of \( \vec{r}, t \).

We now write, \( \psi(\vec{r},t) = \sum \hat{a}_n(t) \psi_n(\vec{r}) \).

The \( a_n(t) \)'s are like the dynamical coordinate variables.

The equation of motion then takes the form:
\[ \frac{\partial \hat{a}_n(t)}{\partial t} = -\varepsilon_n \hat{a}_n(t). \]
\[
\text{Equivalently, the action is } S = \int dt \left[ \dot{q}^* \cdot \dot{p} - \frac{1}{2} \nabla^2 q^* \cdot \nabla^2 q^* - i \hbar \frac{\partial}{\partial t} \right].
\]

We can think these \( q^* \)s as an infinitesimal number of variables like the \( q \)s.

We have, then, the conjugate momentum,
\[
\dot{q}_n = i \hbar \frac{\partial}{\partial q^*_n}.
\]

We have the classical Poisson bracket, \( \{ q_n, p_m \} = \delta_{nm} \), which led (in the quantum theory, after quantizing) to the commutator relations, \( [q_n, p_{m^*}] = i \hbar \delta_{nm} \), where these \( q_n \)s and \( p_{m^*} \)s are operators in the Heisenberg picture. These would evolve in time.

The system, \( \dot{q} \cdot \dot{p} + \nabla^2 q \cdot \nabla^2 q + i \hbar \frac{\partial}{\partial t} \) after quantizing becomes an operator \( \frac{\partial}{\partial t} \) equation (from the time evolution of the operator).

- **Steps for Second Quantization:**
  1. Take the Schrödinger equation for a single particle system.
     [This equation is not invariant under Special Relativity. For relativistic cases it will be replaced by the Klein-Gordon equation. The steps in the quantization procedure would remain the same though.]
  2. Regard the wave function as a classical field and the Schrödinger equation as the classical equation of motion.
  3. Then, from here, we construct the Lagrangian and the Hamiltonian which gives the equation of motion.
  4. We then quantize this classical field theory (classical system).
This procedure leads to a system which has the ability to describe multiple-particle systems which are mutually non-interacting.

If we follow these steps, we get a system of identical bosons, we do not get them for fermions.

- We first traded in the continuous wave function for the discrete one.
- We now do the manipulations in terms of the function $\Psi$ itself.
- To see how the different things in the harmonic oscillator formalism look in terms of the wave function formalism.

We have, 
\[ [a_+, a_m] = 0 = [a^\dagger, a_m^\dagger] \text{ and } [a_+, a_m^\dagger] = \delta_{m0}. \]

The commutation relations are equal to $i\hbar$.

- For example, in quantum mechanics, $[\hat{q}, \hat{p}] = i\hbar$, whereas $a_+ a$ are at the same time $\hbar$.

- New: 
\[ \psi(\vec{x}, t) = \sum_m a_m(t) \ U_m(\vec{x}) \]

\[ \begin{align*}
[\psi(\vec{x}, t), \psi(\vec{x}', t)] &= \left[ \sum_m a_m(t) \ U_m(\vec{x}) \right. \\
& \left. \sum_m a_m(t) \ U_m(\vec{x}') \right] \\
& = \sum_m \left[ a_m(t) \ U_m(\vec{x}) \right] \ U_m(\vec{x}') U_m(\vec{x}) = 0
\end{align*} \]

Again,
\[ \begin{align*}
[\psi^\dagger(\vec{x}, t), \psi^\dagger(\vec{x}', t)] &= \sum_m \left[ a_m^\dagger(t) \ U_m^\dagger(\vec{x}) \right] \ U_m^\dagger(\vec{x}') U_m^\dagger(\vec{x}) = 0
\end{align*} \]
Again, \[ \psi(x, t), \psi^*(x', t) \]

\[ = \sum m_n \left[ a_n(t), a_{n'}^+(t) \right] u_n(x) u_{n'}^*(x') \]

\[ = \sum m_n u_n(x) u_{n'}^*(x') = \sum u_n(x) u_{n'}^*(x') = \delta^{(3)}(x - x') \]

\[ \text{(Using completeness of the basis.)} \]

We could have considered \( \psi^*(x, t) \) as the coordinate, then \( \psi^*(x, t) \) would be considered as the conjugate momentum.

Now, \( \psi(x, t) = \sum a_n(t) u_n(x) \)

\[ \therefore \int \psi(x', t) \psi^*(x') \: d^3x' = \sum a_n(t) \int d^3x' u_n(x) u_{n'}^*(x') \]

\[ = \int d^3x' \sum a_n(t) u_n(x) u_{n'}^*(x') \]

\[ = \int d^3x' \sum a_n(t) u_n(x') \]

\[ = a_n(t) \]

\[ \therefore a_{n'}^+(t) = \int u_{n'}(x') \psi^*(x', t) \: d^3x' \]

Now, in terms of \( a_n \) and \( a_{n'}^+ \), the Hamiltonian is

\[ \hat{H} = \sum e_n a_{n'}^+ a_n \]

We now write,

\[ \hat{H} = \sum e_n \int d^3x d^3x' u_{n'}(x') u_{n}^*(x) \psi^+ (x', t) \psi(x, t) \]

\[ = \sum \int d^3x d^3x' u_{n'}(x') u_{n}^*(x) \psi^+ (x', t) \psi(x, t) \]
Now, $\hat{h}$ is a Hermitian operator, so integrating by parts we have:

$$\hat{h} = \sum \int d^3 \pi \int d^3 \pi' \ U_n(\pi') \ U^*_n(\pi) \ \hat{h} \ U^*(\pi', t) \ U_n(\pi).$$

The boundary terms going to zero, we get:

$$\hat{h} = \sum \int d^3 \pi \int d^3 \pi' \ U_n(\pi') \ U^*_n(\pi) \ \hat{h} \ U^*(\pi', t) \ U_n(\pi).$$

Again, $\sum U_n(\pi') U^*_n(\pi) = \delta(\pi' - \pi)$, we get:

$$\hat{h} = \int d^3 \pi \int d^3 \pi' \ U^*_n(\pi') \ \hat{h} \ U_n(\pi).$$

So $\hat{h} = \int d^3 \pi \ U^*_n(\pi') \ \hat{h} \ U_n(\pi).$

* N.B.: '$\hat{h}$ is an operator here, so '$\hat{h}^*$ is an operator. In quantum theory, the '$\hat{h}$' are the fields. We regard the $\hat{a}_n(t)$'s and the $\hat{a}_n^+(t)$'s as the operators, while the $U_n(\pi)$'s and $U^*_n(\pi)$'s are just some functions.

The best way to think of the $\hat{h}$ is the sum of the linear operators $\hat{a}_n(t)$ along with the functions $U_n(\pi)$.

- N.B.: Here '$\hat{h}$' is a differential operator, not an operator in the quantum theory, these act on the $U_n(\pi)$'s in the expansion of $\psi(\pi, t) = \sum a_n(t) \ U_n(\pi)$. The $\hat{a}_n(t)$'s are the operators, the $U_n(\pi)$'s are just ordinary functions.

The Hermitian space operators are the $a_n(t)$'s and the $a_n^+(t)$'s. $\hat{h}$ here is not a Hermitian space operator.

$$\hat{h} = \int d^3 \pi \ U^*_n(\pi') \ \hat{h} \ U_n(\pi)$$

Q: How to interpret this?

A: We first expand the $\psi$'s and then use the completeness of the $U_n$'s, using the $S$-function and then again look like...
Now, if we have a general one-body operator, how to
write it in terms of the fields?

Let us have a one-body operator in an N-particle
quantum mechanical system, viz.,
\[ \hat{\mathcal{B}} = \sum_{m,n} \mathcal{B}_{mn} a_m^\dagger a_n \] where \( \mathcal{B}_{mn} = \int d^3 \mathbf{r} \ U_{mn}(\mathbf{r}) U_{mn}^*(\mathbf{r}) \) the sum depends on the coordinates and momenta of one particle.

This operator maps into the harmonic oscillator picture:
\[ \hat{\mathcal{B}} = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ U_{mn}(\mathbf{r}^') U_{mn}^*(\mathbf{r}'') \ \psi_1(\mathbf{r}^',t) \]

Now we re-express,
\[ \hat{\mathcal{B}} = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \sum_{m,n} \mathcal{B}_{mn} \int d^3 \mathbf{r}^' d^3 \mathbf{r}'' \ \psi_1(\mathbf{r}^',t) \]

\[ = \int d^3 \mathbf{r}^' \ \psi_1(\mathbf{r}^',t) \]

\[ = \int d^3 \mathbf{r}^' \ \psi_1(\mathbf{r}^',t) \]
\[ \hat{B} = \int d^3 \pi \: \psi^\dagger(\vec{\pi}, t) \: h(\hat{\pi}, \hat{\psi}) \: \psi(\vec{\pi}, t) \]

**N.B.** If \( \psi \) was a Schrödinger wave function, then it is an expectation value of \( \hat{b}^\dagger \) in \( \psi, \psi^\dagger \) and we would get:

\[ B = \langle \hat{b}^\dagger \rangle \psi_\psi \]

So the general rule is: **In a quantum expectation value, we replace \( \psi^\dagger \) by \( \hat{b}^\dagger \), i.e., we change the classical fields to quantum operators and we retrieve our above result.**

**N.B.**

The number operator is given by:

\[ \hat{N} = \sum_a a^\dagger(t) a(t) \text{ whose eigenvalue is } \hat{N}^a \]

Now, we replace the \( a(t) \) and the \( a^\dagger(t) \).

\[ \hat{N} = \sum_a \int d^3 \rho d^3 \rho' \: u_n(\vec{\rho}/\vec{\rho}') u_n^\dagger(\vec{\rho}/\vec{\rho}') \: \psi^\dagger(\vec{\rho}', t) \: \psi(\vec{\rho}, t) \]

\[ = \int d^3 \rho d^3 \rho' \: s^{11}(\vec{\rho}/\vec{\rho}') \: \psi^\dagger(\vec{\rho}', t) \: \psi(\vec{\rho}, t) \]

\[ = \int d^3 \rho \: \psi^\dagger(t, t) \: \psi(t, t) \]

**N.B.** Let \( 10 \rangle \) be the ground state.

Then, \( a_n{10} = 0 \) for all \( n \).

The states are:

\[ (a^\dagger)^{m_1} (a)^{m_2} \cdots 10 \rangle \text{ in this harmonic oscillator representation. These are the states in the Hilbert space.} \]

These solutions, we do not think of states anymore, these are the explicit solutions for the harmonic oscillator.
example, we get the Hermite polynomials.

The Hilbert space is spanned by the states of the harmonic oscillator.

When we write, \( \hat{H} = \sum a_n^+ a_n \), the original Schrödinger equation gives us the \( a_n \)'s, these are not the Hilbert space states.

The Hilbert space in both the cases are completely different.

We could have written it as:

\[
\hat{H} = \int \left( \frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi(x,t) \, d^3r.
\]

The original single particle Hilbert space is not the Hilbert space of this system. Here the Hilbert space is much bigger.

If we start with a different basis, the new \( a_n \)'s will be linear combination of the old \( a_n \)'s.

So, instead of using the bases \( \{ u_n \} \), we could very well have used any other bases \( \{ v_n \} \) or \( \{ w_n \} \), where

\[
w_n = \sum_m S_{nm} \, u_m, \quad \text{where} \quad (S)_{nm} \text{ is a unitary matrix}.
\]

We can still carry out the whole formalism using this basis.

Now, the Hamiltonian in these new bases, will not be diagonal, some other operator might be diagonal in this new basis.
• If, instead of the energy eigenvalues, we use the position eigenvalues, then the axis will be replaced by the \( \Phi \).

The relation is:

\[ \psi_n(x,t) = \langle \Phi_n | \Phi(x,t) \rangle \]

\[ \sum_{n=0}^{\infty} \psi_n(x,t) \Phi_n = \Psi(x,t) \]

where \( \Phi_n \) is the energy eigenstate with eigenvalue \( E_n \) and \( \Phi(t) \) is the position eigenstate. We think the \( \Phi_n \)'s as numbers.

So, we can write the \( \psi_n(x,t) \) in terms of the position eigenstates.

In the classical domain, we obtain our quantum mechanics normalization, i.e., if we replace the quantum operators by the classical fields, we obtain:

\[ \int \psi^2 (x,t) \, dx = 1 \text{ for a single particle quantum system.} \]

For a second quantised system, these \( \psi \)'s are operators and can have any eigenvalues and can have multiparticle states.

• Two-body operators:

A two-body operator in the \( N \)-particle Hilbert space has the following form:

\[ \hat{\psi}_{12} = \sum_{\sigma=1}^{N} \sum_{\tau=1}^{N} \hat{\psi}_{\sigma\tau} \hat{\psi}_{\tau\sigma} \]

where each of the terms in the above depends on the coordinates and momenta of two particles.
In the harmonic oscillator representation, we have:

\[ \hat{\nu} = \sum_{m,n,p,q} \omega_{mnpq} \hat{a}^+_m \hat{a}^+_n \hat{a}^+_p \hat{a}^+_q, \]

where

\[ \omega_{mnpq} = \int d^3\eta d^3\eta' \ u_\eta^* (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta' (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)). \]

So, in terms of the \( \phi \)'s and \( \psi^+ \),'s,

\[ \hat{\nu} = \sum_{m,n,p,q} \int d^3\eta d^3\eta' d^3\eta'' d^3\eta''' \ u_\eta^* (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta' (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta''^* (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta''' (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)). \]

\[ = \sum_{m,n,p,q} \int d^3\eta d^3\eta' d^3\eta'' d^3\eta''' \ u_\eta^* (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta' (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta''^* (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)) \ u_\eta''' (\vec{p}_1, \hat{\sigma} (\vec{p}_1), \vec{p}_2, \hat{\sigma} (\vec{p}_2)). \]

So, now, even in the case of interactions, we can use this formalism.

- So, if the total N-particle Hamiltonian is

\[ \sum_{i=1}^N \hat{h}_i + \hat{v}_{NN}, \]

where the \( \hat{h}_i \) has some potential \( V \), and \( \hat{v}_{NN} \) is the interaction between the particles, for example, Coulomb interaction.
This above is the operator in particle mechanics language.

In the harmonic oscillator language on the other hand, the total Hamiltonian is:

\[ H_{\text{total}} = \hat{H} + \hat{V}. \]

Now, in terms of \( \psi, \psi^\dagger \)'s, we have:

\[ H_{\text{total}} = \int d^3x \, \psi^\dagger (\vec{r}, t) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right) \psi (\vec{r}, t) \]

\[ + \int d^3x \, d^3y, \, \psi^\dagger (\vec{r}, t) \psi^\dagger (\vec{y}, t) \psi (\vec{r}, t) \psi (\vec{y}, t) \]

\[ + \int d^3x \, d^3y \, \psi^\dagger (\vec{r}, t) \hat{\sigma} \psi (\vec{y}, t), \]

\[ \text{where:} \quad \hat{\sigma} \equiv \hat{p}_1 \hat{p}_2 \hat{p}_3 \]

N.B.: If we can find the exact eigenstates and the exact eigenvalues (i.e., diagonalize), then we can solve the problem exactly. The final Hamiltonian might not look like a harmonic oscillator form.

\[ \hat{H} = \sum a_n \hat{a}^\dagger a_n + \sum_{n_1 n_2} C_{n_1 n_2} \hat{a}_{n_1} \hat{a}_{n_2} \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger \]

If we can solve this exactly, we can solve in principle:

\[ \sum_{n} \hat{H}_n + \hat{V}_{\text{ext}}. \] If we can not solve exactly, we apply some kind of perturbation theory.

Try for a time-dependent Hamiltonian. Then, we cannot expand in the basis of the Hamiltonian.

We can actually choose our basis.

We can expand the wavefunction now in the form:

\[ \sum_{n} f_n(t) \hat{a}_n^\dagger \hat{a}_n, \] where now, \( f_n(t) \) is time-
dependent unlike the end.

Those am's and an's are not the same as the old ones. Here we get cross terms.
The formalism will still go through.

Once we have developed the second quantized theory and the particle number in our system is not conserved, then we add some other Hamiltonian term to the existing Hamiltonian like:

$$\sum_{m,n} c_{mnp} a_m^+ a_n^+ a_{p} + \sum_{m,n,p} c_{mnp}^* a_p^+ a_m a_n.$$ This part clearly does not preserve the particle number.

One-body operator:

$$\mathcal{B} = \sum_{m,n} \int d^3 r \, \Psi_+(\vec{r}, t) \, \Psi(\vec{r}, t) \, a_m^+ a_n$$

$$= \int \Psi_+(\vec{r}, t) \, \Psi^*(\vec{r}, t) \, \psi(\vec{r}, t) \, d^3 r$$

Now we try to prove the energy by writing:

$$\Psi(\vec{r}, t) = \sum_m a_m(t) \, \Psi_1(\vec{r})$$

and $$\Psi_+^*(\vec{r}, t) = \sum_m a_m^*(t) \, \Psi_1^*(\vec{r})$$

Plugging these in, we get:

$$\mathcal{B} = \sum_{m,n} \int d^3 r \, a_m^*(t) \, \Psi_1^*(\vec{r}) \, \Psi_1(\vec{r}) \, b_+(\vec{r}, t) \, a_{n}(t) \, \Psi_1(\vec{r})$$

$$= \sum_{m,n} a_{m}^+ \, a_{n} \, \int d^3 r \, \Psi_1^*(\vec{r}) \, \Psi_1(\vec{r}) \, b_+(\vec{r}, t) \, a_{n}(t) \, \Psi_1(\vec{r})$$
\[ A^n \mathbf{B} = \sum_{m,n} k_{mn} a^+_m a^n \] \[ (\because k_{mn} = \int d^3n \, u_m^*(\pi_n) \, \nu(\hat{\pi}_n, \hat{\mathbf{p}}_n) \, u_n(\pi_n) ) \]

- \text{Twist - Kontsevich expansion :}

\[ \hat{\nabla} = \sum_{m,n} V_{mn, k} a^+_m a^n a^{+T}_k a^+_{kT} \text{ where} \]

\[ V_{mn, k} = \int d^3n \, d^3p \, u_m^*(\pi_n) \, u_n^*(\pi_p) \, \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, u_p(\pi_p) \, u_{kT}(\pi_k) \]

- Plugging this in \( \hat{\nabla} \), we get :

\[ \hat{\nabla} = \sum_{m,n} \int d^3n \, d^3p \, u_m^*(\pi_n) \, u_n^*(\pi_p) \, \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, u_p(\pi_p) \, u_{kT}(\pi_k) \, a^+_m a^n a^{+T}_k a^+_{kT} \]

- Now, \( \sum_{m} a^+_m u_m^*(\pi_n) = \psi^{+T}(\pi_n, t) \) etc.

\[ \hat{\nabla} = \int d^3n \, d^3p \, \psi^{+T}(\pi_n, t) \, \psi^{+T}(\pi_p, t) \, \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, \psi(\pi_n, t) \, \psi(\pi_p, t) \]

- Now, going in the other way, we write,

\[ \psi(\pi_n, t) = \sum_{m} a^+_m u_m(\pi_n) \] etc. to get :

\[ \hat{\nabla} = \sum_{m,n} \int d^3n \, d^3p \, a^+_m(\pi_n) \, u^+_m(\pi_n) \, a^+_n(\pi_p) \, u^+_n(\pi_p) \, \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, u_p(\pi_p) \, u_{kT}(\pi_k) \]

\[ = \sum_{m,n,k} a^+_m(\pi_n) a^+_n(\pi_p) a^{+T}_k(\pi_k) a^{+T}_{kT}(\pi_k) \]

\[ \int d^3n \, d^3p \, u_m^*(\pi_n) \, u_n^*(\pi_p) \, \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, u_p(\pi_p) \, u_{kT}(\pi_k) \]

\[ = \sum_{m,n,k} V_{mn, k} a^+_m(\pi_n) a^+_n(\pi_p) a^{+T}_k(\pi_k) a^{+T}_{kT}(\pi_k) \]

\[ \hat{\nabla}(\hat{\pi}_n, \hat{\mathbf{p}}_n, \hat{\pi}_p, \hat{\mathbf{p}}_p) \, u_p(\pi_p) \, u_{kT}(\pi_k) \]
How to manipulate with the fields without their mode expansions, i.e., how to work directly with the fields?

Now, we have to deal with this continuum index \( \mathbf{n} \). To overcome this difficulty, we have to consider them as a discrete set of points, i.e., we have to take some kind of a cubic lattice.

In the \( \Psi(\mathbf{n}, \mathbf{r}) \), we let this \( \mathbf{n} \to \mathbf{w}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \); where the \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) are integers and \( \mathbf{w} \) is the lattice spacing.

We think of \( \Psi(\mathbf{n}, \mathbf{r}) \to \Psi(\mathbf{r}) \); where \( \mathbf{n} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \).

We will work with them and at the end let \( \mathbf{w} \to 0 \).

We now want to change from the continuum language to the discrete language.

So, now our integration,

\[
\int d^n r \to \mathbf{w}^3 \sum_{\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}}
\]

Taking the limit \( \mathbf{w} \to 0 \), gives us inside the integration.

We now consider \( \Psi(\mathbf{r}) \) a mere coordinate variable.

In this variable, we now calculate,

\[
\Psi(\mathbf{r}) = \delta_{\mathbf{x} \mathbf{y} \mathbf{z}} \delta_{\mathbf{r} \mathbf{r}}
\]

If \( \mathbf{r} = \mathbf{r} \), then this is 1, else it is 0.
We will see that this does not behave well as \( \omega \to 0 \).

Actually \[ \sum_{\{x_i, y_i, z_i\}} \frac{1}{\omega} \delta(x^2 + y^2 + z^2) \text{ diverges.} \]

Thus, to avoid these divergences, we define:

\[ \delta' = \frac{1}{\omega} \delta \]

Then \[ \delta'(x^2 + y^2 + z^2) = \frac{1}{\omega} \delta(x^2 + y^2 + z^2) = \frac{1}{\omega^3} \sum_{\{x_i, y_i, z_i\}} \delta(x_i^2 + y_i^2 + z_i^2) \]

We have taken \( \mathbf{r}' = \omega (x, y, z) \)

\( \mathbf{r}' = \omega (x', y', z') \)

If \( \mathbf{r}' \) and \( \mathbf{r} \) are different points on the lattice, then the above quantity is zero.

- **Properties:**
  - \( \delta'(r') = 0 \) if \( r' \neq r \)
  - \[ \sum_{\mathbf{r}'} \delta'(r') = \frac{1}{\omega^3} \sum_{\mathbf{r}} \delta(x^2 + y^2 + z^2) = 1. \]

Thus, we see that \( \delta'(r') \) has exactly the properties of \( \delta(r) \)

the Dirac-delta function when we take a continuum limit

(\( \omega \to 0 \)), i.e.

\[ \delta'(r') \to \delta^{(3)}(r' - r) \]

\( \delta'(r') \to 0 \)

[We do this one and for all. We will not discretize everywhere. Then we will work with the continuum resemblance.]
If we have,

\[ 2x \mathbf{\psi}^\dagger \mathbf{\psi}(t), \text{ we write } \mathbf{\psi} \text{ in the discretised form as:} \]

\[ 2x \mathbf{\psi}^\dagger \mathbf{\psi}(t+1) = \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t+1, \mathbf{\psi} \right) \]

\[ - \Delta \mathbf{\psi}^\dagger \mathbf{\psi}(t+1) = \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \]

\[ = \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t+1, \mathbf{\psi} \right) \]

\[ + \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \]

\[ = \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t+1, \mathbf{\psi} \right) \]

\[ + \int \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \left[ \mathbf{\psi} \mathbf{\psi}^\dagger \right] \left( t, \mathbf{\psi} \right) \]

Thus we see (in the Lagrangian or the Action) represent the \( \Delta \mathbf{\psi}^\dagger \mathbf{\psi} \) terms of this.

[We should recur the motional invariance.]

In the \( \hbar \) we have:

\[ \hbar^2 = \hbar x^2 + \hbar y^2 + \hbar z^2 \]

- Let \( \mathbf{\psi} \) be a function of \( \mathbf{\psi}(t) \) and \( \mathbf{\psi}(t+1) \) i.e. a function of the collection of such variables.

- For example:

\[ 1 = \int \mathbf{\psi} \mathbf{\psi}^\dagger \mathbf{\psi} \mathbf{\psi}^\dagger \left( \hbar x^2(\mathbf{\psi}, \mathbf{\psi}^\dagger) - \hbar x^2(\mathbf{\psi}, \mathbf{\psi}^\dagger) \right) \]

where

\[ \hbar^2 = \hbar x^2 + \hbar y^2 + \hbar z^2 \]

We can reinterpret \( 1 \) in terms of the discrete variables \( \mathbf{\psi}(t) \) and \( \mathbf{\psi}(t+1) \).

We can write \( \mathbf{\psi} = \mathbf{\psi}(t) \)
\[ L = \omega^3 \sum_{(x, y, z)} \psi^*(x, t) \left( i \dot{\psi} + \frac{\delta F}{\delta \psi} \right) \psi(x, t) \]

The variation of \( F \) can be written as:

\[ \delta F = \sum_{(x, y, z)} \left( \frac{\delta F}{\delta \psi} \psi + \frac{\delta F}{\delta \psi^*} \psi^* + \text{complex conjugate} \right) \]

Now, \( \frac{\delta F}{\delta \psi} = \omega^3 SF \) and \( \frac{\delta F}{\delta \psi^*} = \omega^3 \bar{SF} \).

We can also write \( \omega^3 \sum \) as \( \int d^3 \pi \).

Thus, multiplying and dividing each term on the R.H.S. by \( \omega^3 \), we get:

\[ SF = \int d^3 \pi \left( SF \psi(x, t) + \bar{SF} \bar{\psi}(x, t) + \text{complex conjugate} \right) \]

For functional derivatives, we can use the chain rule for ordinary derivatives.

The \( m^3 \) cancels (the normalization has it in the denominator while the integration does not have it in the numerator.)

\[ L = \int d^3 \pi \psi^*(x, t) \left( i \dot{\psi} + \frac{\delta F}{\delta \psi} \right) \psi(x, t) \]

\[ H = \sum_{x} \left( \frac{1}{m} \omega^3 \right) \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} - L \]

\[ = \int d^3 \pi \bar{SF} \psi(x, t) - L \]

\[ \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} = \int d^3 \pi SF \psi(x, t) \]
Again \( L = \int d^3\mathbf{r}' \, \psi^* (\mathbf{r}', t) \left( \partial_t \psi (\mathbf{r}', t) - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} (\mathbf{r}', t) \right) \).

We think of the Lagrangian as a function of coordinates and momenta (they being independent).

To find \( \frac{\partial L}{\partial \psi (\mathbf{r}, t)} = \int d^3\mathbf{r}' \, \psi^* (\mathbf{r}', t) \left( \partial_t \psi (\mathbf{r}', t) - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} (\mathbf{r}', t) \right) \frac{\partial}{\partial \psi (\mathbf{r}, t)} \)

Just as \( \psi (\mathbf{r}', t) = \phi^{(1)} (\mathbf{r}' - \mathbf{r}) \) here also,

\( \psi (\mathbf{r}, t) = \phi^{(1)} (\mathbf{r} - \mathbf{r}') \)

\( \psi^* (\mathbf{r}', t) = \phi^{(1)*} (\mathbf{r}' - \mathbf{r}) \)

\( \psi (\mathbf{r}', t) = \phi^{(1)} (\mathbf{r} - \mathbf{r}') \)

\( \psi^* (\mathbf{r}, t) = \phi^{(1)*} (\mathbf{r}' - \mathbf{r}) \)

\[ \therefore \frac{\partial L}{\partial \psi (\mathbf{r}, t)} = \int d^3\mathbf{r}' \, \psi^* (\mathbf{r}', t) \partial_t \phi^{(1)*} (\mathbf{r}' - \mathbf{r}) \]

\[ = \partial_t \psi^* (\mathbf{r}, t) \]

So, the Hamiltonian is given by:

\[ \int d^3\mathbf{r} \, \partial_t \phi^{(1)*} (\mathbf{r}) \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} (\mathbf{r}, t) - \int d^3\mathbf{r} \, \phi^{(1)*} (\mathbf{r}) \left( \partial_t \phi^{(1)} (\mathbf{r}, t) - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} (\mathbf{r}, t) \right) \]

\[ = \int d^3\mathbf{r} \, \phi^{(1)*} (\mathbf{r}) \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} (\mathbf{r}, t) \]

This is the same result we obtained by the mode expansion of the wave functions. [See page 46.]

At this stage, we are using classical field theory, so here we have "*" and not "*".

Another useful relation is that of the conjugate momentum.

In the discrete form, the momentum conjugate to \( \psi^* \) is:

\[ \frac{\partial}{\partial \psi^*} \]
\[ \langle \Psi(x), \Pi(x') \rangle_{\text{pq}} = \frac{1}{\sqrt{2\pi \hbar}} \int e^{-i\frac{p(x'-x)}{\hbar}} \Psi^*(x) \Pi(x') dx \]

It will be useful not for \( \langle \Pi \rangle \), but to use a normalized quantity.

So, we define, \( \Pi(x) = \frac{1}{\sqrt{\hbar}} p(x) \).

\[ \langle \Psi(x), \Psi(x') \rangle_{\text{pq}} = \frac{1}{\sqrt{2\pi \hbar}} \int e^{-i\frac{p(x'-x)}{\hbar}} \Psi^*(x) \Psi(x') dx \]

\[ = \frac{1}{\sqrt{2\pi \hbar}} \int e^{-i\frac{p(x'-x)}{\hbar}} \Psi^*(x) \Psi(x') dx \]

\[ \lim_{\hbar \to 0} \frac{\hbar}{\sqrt{2\pi \hbar}} \int e^{-i\frac{p(x'-x)}{\hbar}} \Psi^*(x) \Psi(x') dx \]

When we take the \( \hbar \to 0 \) limit, we get the Dirac delta function.

\[ \Pi(x) = \lim_{\hbar \to 0} \frac{1}{\sqrt{\hbar}} p(x) \]

[In this notation, we do not have to worry about the \( \psi \)'s.]

• We will, in field theory, directly define \( \Pi(x) \), and use this definition of the Poisson bracket, i.e.

\[ \langle \Psi(x), \Pi(x') \rangle_{\text{pq}} = \delta^{(3)}(x-x') \]

• In quantum theory, these \( \psi \)'s and \( \Pi \)'s are replaced by operators and we have to use the notation,

\[ [\Psi(x,t), \Pi(x',t)] = \hbar \delta^{(3)}(x-x') \]

Now, let us go back to the Schrödinger problem:

\[ L = \int d^3 x' \psi^*(x',t) \left( \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x',t)}{\partial x'^2} - \hat{V} \psi(x',t) \right) \]
\[ \Pi^{(\infty, t)} = \psi^*(\infty, t) \rightarrow \psi^*(\infty, t) \text{ in the quantum theory.} \]

[See page 57.]

We get then,

\[ [\psi(\infty, t), \psi^+(\infty', t)] \Delta^2 = \delta^{(1)}(\infty - \infty') \]

and

\[ [\psi(\infty, t), \psi(\infty', t)] = \delta^{(1)}(\infty - \infty'). \]

Thus, we recover the result that we obtained from the mode expansion of the \( \psi \) and the \( \psi^* \). [See page 58.]

**Note:** The \( \psi \)'s are discrete from the physical view point, whereas the \( \psi^* \)'s are not discrete from the physical view point because eventually we let \( \infty \rightarrow \infty' \).

The \( \psi \)'s and the \( \psi^* \)'s are still on the same footing. The \( n \) index represents the energy eigenstates. The \( z \) index represents the position eigenstates. [See page 59.]

**Identical Fermions:** We again start with the single particle Schrödinger equation,

let us (as in the case of bosons) consider \( N \) identical mutually non-interacting fermions each moving under a potential \( V(\vec{r}) \).

The Hamiltonian is:

\[ \sum_{i=1}^{N} \hat{h}_i \quad \text{where} \quad \hat{h}_i = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \vec{r}_i^2} + V(\vec{r}_i). \]

Let us (as in the case of bosons) suppose that the \( U_n(\vec{r})'s \) are eigenstates of \( \hat{h}(= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \vec{r}^2} + V(\vec{r})) \)

\[ \hat{h} U_n(\vec{r}) = \epsilon_n U_n(\vec{r}) \]

**Note:** As far as a single particle is concerned, there is no distinction between a boson and a fermion.
Now, to construct a basis in terms of the single particle wave functions.

We write,

\[ U_n(p_1, \ldots, p_N) = \frac{1}{N!} \sum_{\text{all permutations of } \{p_1, \ldots, p_N\}} \text{sign(perm)} \cdot U_{n_1}(p_{i_1}) \ldots U_{n_N}(p_{i_N}) \]

where "\text{sign(perm)}" is the number of exchanges involved in the permutation.

Q: How many permutations to go from \(1, \ldots, N\) to a given permutation? There are many ways "\text{sign(perm)}" is not fixed.

But whether "\text{sign(perm)}" is odd or even is fixed.

Here, we do not consider the spin property of the fermions and the bosons.

But given \(V(p)\) [the same], there is no way to distinguish between a fermion and a boson till there is one particle.

- Example: If \(N=3\), \(U_{n_1, n_2, n_3}(p_1, p_2, p_3)\)

\[ U_{n_1, n_2, n_3}(p_1, p_2, p_3) = \frac{1}{6} \left[ U_{n_1}(p_1)U_{n_2}(p_2)U_{n_3}(p_3) ight. \\
- U_{n_1}(p_3)U_{n_2}(p_1)U_{n_3}(p_2) \\
- U_{n_1}(p_2)U_{n_2}(p_3)U_{n_3}(p_1) \\
- U_{n_1}(p_2)U_{n_2}(p_1)U_{n_3}(p_3) \\
+ U_{n_1}(p_3)U_{n_2}(p_2)U_{n_3}(p_1) \\
+ U_{n_1}(p_3)U_{n_2}(p_1)U_{n_3}(p_2) \right]. \]
This is completely antisymmetric under the exchange of any two positions.

Additional feature: If any of the two $m_i$ are equal, then $(m_1, \ldots, m_n)$ vanishes.

i.e. if $m_i = m_j$ for any pair of $(i, j)$, then the state is zero.

Here also (like before), we can take an occupation number representation.

Here, $u_i$ appears $m_i$ times (say)

$u_i$ appears $m_i$ times (say)

Here another distinction will come as each $m_i = 0$ or 1.

Q: Can we find an equivalent representation with a harmonic oscillator?

Ans: It cannot be mapped to a harmonic oscillator, but it can be mapped to an equivalent system which is obtained and can not be related to a harmonic oscillator.

- The equivalent system: For every energy eigenstate $|m\rangle$, introduce a pair of operators $a_m, a_m^\dagger$ satisfying the following relations:

\[ [a_m, a_m^\dagger] = 1; \quad [a_m, a_n] = 0; \quad [a_m^\dagger, a_n^\dagger] = \delta_{mn}, \quad \text{where for any two operators } a, b, \text{ we define} \]
\[ \{ a, b \} = ab + ba \quad \text{(anticommutator).} \]

We are thus constructing an equivalent quantum system by hand.

Then, we would like to find the rules.

These $a$'s and $a^\dagger$'s are the creation and annihilation operators.
of a harmonic oscillator as in the latter case, they do not
anti-commute.

We now build a Hilbert space.

- The Hilbert space: - We define the vacuum (ground state),
  \( |0\rangle \) such that,
  \[ a_m^+ |0\rangle = 0 \quad \text{for every } n, \]
  then the basis states are of the form:
  first representation:
  \[ a_n^+ a_n^+ \ldots a_m^+ |0\rangle \quad (n,\ldots,m) \]
  second representation (occupation number representation):
  \[ (a_n^+)^{n} (a_m^+)^{m} \ldots |0\rangle \]

- claim: - We have an exact map between the quantum theory of
  both the states.
  We have to apply the symmetry property exactly.

If any two \( n \)'s are the same, then the state is zero.

proof: - \[ \langle a_n^+ \rangle \quad a_m^+ a_n^+ \]
\[ a_m^+ a_n^+ + a_n^+ a_m^+ = 0 \]
If \( m = n \), then
\[ 2 \langle a_n^+ \rangle = 0 \]
\[ \Rightarrow \langle a_n^+ \rangle = 0 \]
Thus, if two \( a_n^+ \)'s are the same, then the wave function
vanishes.

Since it passes the first property, i.e., if the \( m_a \) is
more than 1, then the state is zero.
We have to check the matrix elements of the operators in the two states must be the same.

The Hamiltonian in this description is:

$$\hat{H} = \sum_n e_n a_n^+ a_n.$$ 

More generally, if we have a one-body operator,

$$\hat{b}_{kl} = \sum_{l=1}^N b_{kl}(l, l', p_{l'}) [\text{each term depends on the coordinate and momentum of the } l'\text{ th particle}].$$

The equivalent representation is:

$$\hat{B} = \sum_{m,n} b_{mn} a_m^+ a_n \text{ where } b_{mn} = \int d^3r \bar{U}_{mn}^\dagger(l, l', p_{l'}) b_{kl}(l, l', p_{l'}) U_{mn}(l, l', p_{l'})$$

Now, to prove that:

$$\langle u_m | \hat{b}_{kl} | u_n \rangle = \langle m | \hat{B} | n \rangle.$$  

Proof: R.H.S.: \(\sum_{k=1}^N \sum_{l=1}^N \sum_{l'=1}^N \sum_{m=1}^N b_{kl} \langle 0 | a_{m_1} \cdots a_{m_{N-1}} \cdots \hat{a}_{m_l} a_{m_{l'}} \cdots \hat{a}_{m_{N-1}} | m_{l'} \rangle \int d^3r \bar{U}_{mn}^\dagger(l, l', p_{l'}) b_{kl}(l, l', p_{l'}) U_{mn}(l, l', p_{l'}) \).
\[ \text{R.H.S.} = \sum_{k,l=1}^{N} (-1)^{k+l} \delta(n, n') \delta(n, n'') \sum_{m=0}^{n} a_m \cdot \frac{\partial}{\partial a_{m'}} \cdot a_{m''} \cdot a_{m'} \cdot a_{m''} \cdot a_{m'} + \text{other terms} \]

where $a_m$ and $a_m^\dagger$ are explained on page 19.

Here, we have used the fact that $S_{m,n} a_m^\dagger = S_{m,n} K$.

From pages 20 and 21, we write:

\[ \text{R.H.S.} = \sum_{k,l=1}^{N} (-1)^{k+l} \delta(n, n') \delta(n, n'') \sum_{m=0}^{n} a_m \cdot \frac{\partial}{\partial a_{m'}} \cdot a_{m''} \cdot a_{m'} \cdot a_{m''} \cdot a_{m'} + \text{other terms} \]

Now, $\sum_{m=0}^{n} (a_m \cdot \frac{\partial}{\partial a_{m'}} \cdot a_{m''} \cdot a_{m'} \cdot a_{m''} \cdot a_{m'})$ is any constant function which is completely antisymmetric under $\tilde{n}_i \rightarrow \tilde{n}_j$ (for any pair of $i$, $j$) again since it is completely symmetric under the exchange of any two particles.

Thus, $\sum_{m=0}^{n} (a_m \cdot \frac{\partial}{\partial a_{m'}} \cdot a_{m''} \cdot a_{m'} \cdot a_{m''} \cdot a_{m'})$ is completely antisymmetric under $\tilde{n}_i \rightarrow \tilde{n}_j$ (for any pair of $i$, $j$).
Again, \( u_{\lambda_1, \lambda_2, \ldots, \lambda_n} (\pi_1, \ldots, \pi_n) = \frac{1}{n!} \sum_{\text{all permutations of } \pi_1, \ldots, \pi_n} \frac{\chi_\lambda(\pi_1) \cdots \chi_\lambda(\pi_n)}{\chi_\lambda(\pi)} \)

This term is also antisymmetric under \( \pi_i \leftrightarrow \pi_j \) (for any pair of \( i, j \))

So instead of calculating the \((N1)\) integrals separately, we perform only the first and multiply by \((N1) \times (-1)^{\pi_i \leftrightarrow \pi_j} \)

\[
\left( u_{\lambda_1} m_{\lambda_2} m_{\lambda_3} - u_{\lambda_2} m_{\lambda_1} m_{\lambda_3} \right) \left( u_{\lambda_1} m_{\lambda_2} m_{\lambda_3} - u_{\lambda_2} m_{\lambda_1} m_{\lambda_3} \right)
\]
For fermions, there is no field theory whose quantization will give us these rules.

The best we can do is that we take a classical field theory, then instead of replacing the bosonic bracket by \( \frac{1}{\hbar} [\cdot, \cdot] \), we do it by
\[
[\cdot, \cdot] = \{\cdot, \cdot\}_{\text{anticommutator bracket}}.
\]
If \( \{\cdot, \cdot\} \) is the anticommutator bracket, we will then be arriving at the

Thus, we must remember that, this is not a stand and rule and that there is not any known classical limit of the results we obtained for the fermions.

E.g., it has a Hilbert space, the linearity of the same function, ensuring that they form a quantum state.

Any quantum operator \( \hat{A} \) satisfies,
\[
\frac{\partial}{\partial t} \hat{A} = [\hat{A}, \hat{H}] + \frac{\partial}{\partial t} \hat{H}.
\]

Thus, here also the standard rules of quantum mechanics apply.

- Let us consider a harmonic oscillator, but, both fermions and bosons have any energy levels.

If we now replace the identity,
\[
[\hat{x}, \hat{p}] = i\hbar \text{ by}
\]
\[
[\hat{x}, \hat{p}] = i\hbar \quad \text{then we will not get something sensible.}
\]

Thus, the system will be different.

Thus, for both bosons and fermions, we will have to use,
\[
[\hat{x}, \hat{p}] = i\hbar.
\]

This is a reason, we are getting different results from a harmonic and a fermionic systems.

Suppose a fermion moves in a centripetal potential. We quantize
using commutation and not anti-commutation. The anti-commutation comes during the second quantisation.

Thus any result of quantising a classical limit, we have to replace $f, f^* \to \hat{f}, \hat{f}^*$. Fermions are not a result of first quantising a classical system.

- $\hat{f} \cdot \hat{f}^* = -\hat{f}^* \hat{f}$ is not natural. A spin-half object is anti-symmetric in $\hat{f}, \hat{f}^*$.

- If we take gaussian variables, it is not a classical theory anymore.

- Classical limit of harmonic oscillators are coherent states, $\exp \{\lambda a_0^{+} \}$. This is not possible for fermions. A proof (3.7) can be applied almost once.

21.1.2011

In (1.1), $D$, we can have particles like para-fermions. Here we can pump in some of the particles.

We generalise statistics by not generating the symmetry conditions, but by specifying the number of particles.

We know,

$$\frac{d}{dt} [\hat{A}, \hat{H}] = [\hat{A}, \dot{\hat{H}}].$$
If $[\delta, \delta] = 0$, then $\delta \neq \delta (v)$. We then formulate the whole problem in the Heisenberg picture.

- To find the relativistic generalisation of what we have been doing, one particle Schrödinger equation gives:

$$\frac{\partial^2}{\partial t^2} \psi = \left( \frac{-h^2}{2m} \nabla^2 + V(x) \right) \psi.$$

Having a potential breaks even the translational invariance. So, let us first consider few particles, i.e., $V(x) = 0$.

Thus, over the energy eigenstates become continuous.

We can always put the particle in a box and determine its momentum and hence its energy.

So, the energy eigenstates are of the form,

$$\frac{1}{\sqrt{(2\pi)^{3/2}}} e^{-i\kappa \cdot x}; \text{ where } (\kappa_x, \kappa_y, \kappa_z) = (h\kappa_x, h\kappa_y, h\kappa_z).$$

Therefore, we get:

$$\frac{-h^2}{2m} \frac{\nabla^2}{\partial x^2} \psi = \frac{1}{2m} \left( \kappa_x^2 + \kappa_y^2 + \kappa_z^2 \right) \psi.$$

Thus the energy eigenvalues are:

$$E = \frac{1}{2m} \left( \kappa_x^2 + \kappa_y^2 + \kappa_z^2 \right).$$

This does not have Lorentz invariance but only Galilean invariance.

We want to generalise Schrödinger equation, so that $E$ and $\psi$ follow Lorentz formula then we would quantise.
We generalise to the relativistic system:

We will use the following units (just to simplicity): \( h = 1, c = 1 \).

We want, \( E = \sqrt{p^2 + m^2} \) \[ actually, E = \sqrt{p^2 + m^2} \].

We still have the identification,

\[ \frac{c}{2x} \rightarrow \text{energy}, \]

\[ \frac{e^2}{2x} \rightarrow \text{momentum}. \]

We are looking for an operator (wave equation), such that

\( e^{-i (E \cdot \mathbf{p})} \)

is a solution when \( E = \sqrt{p^2 + m^2} \).

We have,

\[ e^{i \frac{c}{2x} \cdot (E \cdot \mathbf{p})} = E e^{i \frac{c}{2x} (E \cdot \mathbf{p})} \]

\[ \frac{c}{2x} = c = 1 \]  \[ \text{and} \]

\[ e^{-i \frac{e^2}{2x} \cdot (E \cdot \mathbf{p})} = E e^{-i \frac{e^2}{2x} (E \cdot \mathbf{p})} \]

What changes is the relation between \( E \) and \( p \), the operators will not change.

To get rid of the square root, we write,

\[ E^2 = p^2 + m^2 \] \[ (: h = 1, c = 1). \]

We write,

\[ \frac{\partial^2}{\partial t^2} \psi = -\nabla^2 \psi + m^2 \psi \] \[ (\text{using the previous two relations}). \]

\[ \Rightarrow \left[ \frac{\partial^2}{\partial x^2} + \nabla^2 - m^2 \right] \psi = 0 \]
\[ (\nabla - m^2) \psi = 0 \]

where \( \nabla = -i \partial_t + \nabla \) is the \( \nabla \)-

Here, we use the metric \((-1, 1, 1, 1)\).

The above is the Klein-Gordon equation.

Check: If we have, \( \psi = e^{-iEt + i\mathbf{p} \cdot \mathbf{r}} \) then

\[ (\nabla - m^2) \psi = 0 \]

\[ \Rightarrow (-\partial_t)^2 + (\nabla \cdot \mathbf{r})^2 - m^2 = 0 \]

\[ \Rightarrow E^2 = \mathbf{p}^2 + m^2 \]

\[ \Rightarrow E = \pm \sqrt{\mathbf{p}^2 + m^2} \]

Thus, irrespective of what sign we choose for \( E \), the Klein-Gordon equation is satisfied.

We have got more than what we had wanted!!!!!!

The spectrum, as we see, contains both positive and negative energy levels, i.e., particles with negative energy are allowed here, which we do not want.

Now, "\( \mathbf{p} \)" can be as large as we can think of, then there will not be any bound on the negative energy levels.

If it were bounded, we could have just shifted the sign of the energy. Here, we can not do that, because there is an infinity bound for the negative energy levels.

Thus, the first quantized equation is not manifestly positive.

But, we can still think of a second quantized this equation (as we did for the Schrödinger equation), then the equation will be manifestly positive.
In Schrödinger equation, the time evolution \( i \hbar \frac{\partial \Psi}{\partial t} \)
would even make a real wave-function evolve as a complex one.

On the contrary, the Klein-Gordon equation is real (because of \( \overline{\Phi} \overline{\Phi} \) and \( \Phi^2 \)) and a real wave-function always evolves as a real wave-function. 

Schrödinger equation is:
\[
\frac{\partial^2 \Psi}{\partial t^2} = -\frac{i}{\hbar^2} \Delta \Psi
\]

For a free particle.

We just give \( \Psi(t=0) \) and this gives the time evolution.
If \( \Psi(t=0) \) is real, it will become complex because of the \( i \).

In contrast, the Klein-Gordon equation is:
\[
\frac{\partial^2 \Phi}{\partial t^2} = \left( \frac{\partial^2}{\partial x^2} + m^2 \right) \Phi
\]

So, here, we will have to specify both \( \Psi(t=0) \) and \( \frac{\partial \Psi}{\partial t}(t=0) \) and let it evolve. As it is a second order equation in both \( t \) and \( x \), and has no \( \frac{\partial \Psi}{\partial t} \) hanging in the equation, so if \( \Psi(t=0) \) and \( \frac{\partial \Psi}{\partial t}(t=0) \) are real, so will they be at all times. It will never become complex.

By restricting to real, we are putting a condition on the Fourier modes; there is no \( i \).

We can solve the Klein-Gordon equation numerically.

We can add the complex conjugate of the \( \Psi \) to the one (as \( \Psi^* \)) to solution to the same Klein-Gordon equation.

But \( \Psi^* \) did not satisfy the same Schrödinger equation.
The equation will be:

$$-i \hbar \frac{\partial}{\partial t} \psi = \frac{\hbar^2}{2m} \nabla^2 \psi.$$

Thus, in the Schrödinger case, the energy eigenstates decay

cannot be real as the energy operator is $i \frac{\partial}{\partial t}.$

Here, in the Klein-Gordon case, we insist $\psi$ to be real.
If $\psi$ were a complex field, we could have taken its real and
complex parts separately.

$$\psi = \psi_r + i \psi_i.$$ Each of these will separately

satisfy the Klein-Gordon equation.

A complex field is equivalent to two real fields.

- **Notation:** \( x^0 = t, \) \( x^1 = x, \) \( x^2 = y, \) \( x^3 = z \)

and we will also write

$$\mathbf{\square} = \frac{\partial^2}{\partial x^0^2} + \nabla^2 = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu};$$

where \( \eta^{00} = -1; \) \( \eta^{ij} = \delta_{ij}; \) \( \eta^{0i} = \eta^{i0} = 0 \)

\( \) for \( i, j = 1, 2, 3. \)

- **N.B.:** We must note that the Klein-Gordon equation is not a proper

first quantised theory.

In the second quantised (relativistic) theory with appropriate

non-relativistic limit, we will get the second quantised non-relativistic

theory.

- **N.B.:** In Schrödinger’s equation, even if we take \( V(\mathbf{r}) \) as complex,

the relative difference between \( V^2 \) and \( V(\mathbf{r}) \) can never make this a
Now, we take the Klein–Gordon equation, treat it as a classical equation and then quantise it.

We try to write an action for this.

We write the action:

\[ S = \frac{1}{2} \int dt \, d^3x \, \left[ \phi \left[ \partial_0 - m^2 \right] \phi \right] \text{; here } \left( \frac{1}{2} \right) \text{ is the normalization factor.} \]

We replaced \( \psi \to \phi \) to distinguish Klein–Gordon field from Schrödinger field.

The normalisation factor does not affect the classical equation.

Now, \( S_S = \frac{1}{2} \int d^3x \, \left[ \phi \left[ \partial_0 - m^2 \right] \phi + \phi \left[ \partial_0 - m^2 \right] \phi \right] \).

We integrate the second term by parts:

\[
\begin{align*}
&\int dt \, d^3x \, \phi \left[ \partial_0 - m^2 \right] \phi \\
= &\int dt \, d^3x \, \phi \partial_0 \phi + \int dt \, d^3x \, m^2 \phi \\
= &\int d^3x \, \left[ \partial_0 \phi \phi \right] + \int d^3x \, dt \, \partial_0 \phi \phi + m^2 \int d^3x \, \phi \phi \\
= &\int d^3x \, \left[ \phi \left[ \partial_0 - m^2 \right] \phi \right] + \int dt \, d^3x \, \partial^2 \phi \phi \\
= &\int dt \, d^3x \, \left[ \partial^2 \phi \phi \right] - \int dt \, d^3x \, \partial^2 \phi \phi \\
&+ \int dt \, d^3x \, \phi \phi.
\end{align*}
\]
Thus, we write,
\[
S = \int dt \, d^3x \left\{ \frac{1}{2} \left[ (\partial - m^2) \phi \right] + \frac{1}{2} \phi \left( \partial - m^2 \right) \phi \right\}
\]

\[SS = 0 \] for arbitrary \( \phi \).

Thus, the integrand must vanish,

\[ \left. \phi \left( \partial - m^2 \right) \phi \right\vert = 0 \]

or \( \left( \partial - m^2 \right) \phi = 0 \)

\( (\partial^2 - m^2) \) is an invariant here.

Here, the Lagrangian is:

\[
L = \frac{1}{2} \int d^3x \phi \left[ \left( \partial - m^2 \right) \phi \right] - \frac{1}{2} \int d^3x \phi \left[ -\frac{\partial^2 \phi}{\partial t^2} + \partial^2 - m^2 \right] \phi
\]

\( \phi (x,t) \to x \) is just a label and we think of \( \phi \) as a
dynamical variable.

The Lagrangian prior depends on the second time
derivative of the coordinate.

Normally in classical physics, Lagrangian is a function of
coordinates, time and the first derivative (in time) of the
coordinates.

We could obviously add a first time derivative of
any arbitrary function to the Lagrangian without affecting
the equations of motion.
• But, if $L = L(\dot{q}, \ddot{q})$, we always do not have a Hamiltonian formulation.

The usual Lagrangian formulation usually does,

$$L = L(q, \dot{q}).$$

But now, we write the action in a different way.

$$\mathcal{S} = \frac{1}{2} \int dt \, d^3 x \, \frac{1}{2} \left[ \frac{\partial^2}{\partial t^2} - m^2 \right] \phi$$

$$= \frac{1}{2} \int dt \, d^3 x \, q \left[ -\frac{\phi'^2}{2} + V^2 - m^2 \right] \phi$$

We integrate the time-dependent parts.

We get:

$$\mathcal{S} = \frac{1}{2} \int dt \, d^3 x \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \int dt \, d^3 x + \frac{1}{2} \int dt \, d^3 x \left[ V^2 - m^2 \right] \phi \right]$$

To get a covariant loosing form, we can also integrate the $V^2$-part by parts. It is not necessary through.

• Then, we get:

$$\mathcal{S} = \frac{1}{2} \int dt \, d^3 x \left[ \frac{\partial}{\partial x^i} \left( \frac{\partial \phi}{\partial x^i} \right)^2 - \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^i} \right]$$

or

$$\mathcal{S} = \frac{1}{2} \int dt \, d^3 x \left[ -m^2 \frac{\partial^2}{\partial x^i} \frac{\partial^2}{\partial x^i} \phi \right. \left. \left( \frac{\partial \phi}{\partial x^i} \right)^2 - m^2 \phi^2 \right]$$

The conjugate momentum is defined as:

$$\Pi(x, t) = \delta L \frac{\delta L}{\delta (\phi(x, t))} = \hat{\phi}(x, t)$$

$$L = \frac{1}{2} \int d^3 x' \left[ \left( \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial x} \right) \right) - \left( \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial t} \right) \right) \right] - m^2 \phi(x, t) \phi(x, t)$$
\[ \begin{align*}
\delta \tilde{\mathcal{H}} & = \frac{1}{2} \int d^3 x \left( \frac{\partial}{\partial \mathbf{r}'} \left[ \frac{1}{\mathbf{q}'} \right] \cdot \mathbf{v} \right) \delta^{(3)} \left( \mathbf{r} - \mathbf{r}' \right) = \tilde{\mathcal{H}}(\mathbf{r}, t)
\end{align*} \]

\[ \begin{align*}
\Gamma: \quad \frac{\delta \tilde{\mathcal{H}}}{\delta \phi(\mathbf{r}, t)} & = \frac{\delta \tilde{\mathcal{H}}}{\delta \phi(\mathbf{r}', t)} \cdot \frac{\delta \tilde{\mathcal{H}}}{\delta \phi(\mathbf{r}, t)} \\
& = 2 \tilde{\mathcal{H}}(\mathbf{r}, t) \delta^{(3)}(\mathbf{r}' - \mathbf{r})
\end{align*} \]

Similarly, \((\partial^2 \tilde{\mathcal{H}})^2\) can be written as:

\[ \frac{1}{\omega} \left( \phi_{x+1, y-1, z} - \phi_{x-1, y+1, z} \right)^2 + \ldots \]

We will have to write a commutation relation between \(\tilde{\Pi}(\mathbf{r}, t, \phi(\mathbf{r}, t))\).

Note: If we choose a different definition of \(\Pi(\mathbf{r}, t)\), we might not get a manifestly Lorentz invariant.

So, the Hamiltonian is:

\[ \begin{align*}
\tilde{\mathcal{H}} & = \sum \frac{2 \mathbf{q}}{2 \hbar} \cdot \mathbf{\dot{q}} - 2 \\
& = \frac{1}{2} \int d^3 x \left[ \tilde{\Pi}(\mathbf{r}, t) \tilde{\phi}(\mathbf{r}, t) \right] - \frac{1}{2} \int d^3 x \left[ \tilde{\phi}^2(\mathbf{r}, t) - (\partial \mathbf{\phi})^2 - m^2 \mathbf{\phi}^2 \right] \\
& = \frac{1}{2} \int d^3 x \left[ \tilde{\Pi}(\mathbf{r}, t) + (\partial \mathbf{\phi})^2 + m^2 \mathbf{\phi}^2 \right]
\end{align*} \]

- Here, we note that each term is positive definite.
- The momentum operator has to be a function of space.
- Translational symmetry.
In (1), the indices do not match up, so we cannot identify them directly as Lorentz four vectors.

N.B. This Hamiltonian is an analogue of what we got for the Schrödinger problem.

\[ H_{	ext{el}} = \int d^3x \left[ \frac{-\hbar^2}{2m} \nabla^2 + V(x) \right] \psi \]

The \( \psi^* \) term can be interpreted by parts, and the \( V(x) \psi \) can be considered as an analogue of \( m^2 \psi^* \psi \).

The Klein-Gordon equation gave negative energy when we considered the same equation. But here, we are considering something different.

- If we take "\( \psi \)" to be real, it can not be an eigensate of \( \frac{-\hbar^2}{2m} \), i.e., it can not be an energy eigensate if it is real.

But, here, we are considering the Hamiltonian of the field theory.

- Previously (in the Schrödinger case), we expanded the \( \psi \) in terms of the \( a \)'s and we simplified the Hamiltonian.

- Here, we do a similar thing. We go to a different basis, the plane wave basis.

- We introduce new variables,

\[ \Phi(x, t) = \int e^{-i \frac{\pi}{2} t} \Phi(x, t) d^3x / (2\pi)^{3/2} \]
\[ \Phi(\mathbf{r}, t) = \int e^{-i \mathbf{r} \cdot \mathbf{r}'} \Pi(\mathbf{r}', t) \, d^3r' \]

These are the Fourier transforms of \( \Phi(\mathbf{r}, t) \) and \( \Pi(\mathbf{r}, t) \).

We must note that, we have not touched time at all. Then, if we know \( \Phi(\mathbf{r}, t) \) and \( \Pi(\mathbf{r}, t) \), we can re-construct \( \Phi(\mathbf{r}, t) \) and \( \Pi(\mathbf{r}, t) \) and vice versa.

This is just an identical (alternate) set of variables. (It is just like instead of using \( \Phi, \Pi \), we use a linear combination of them.)

The Hamiltonian will look simpler in these variables.

We have (from definition),
\[
\{ \Phi(\mathbf{r}, t), \Phi(\mathbf{r}', t) \}_\text{PB} = 0
\]

And
\[
\{ \Pi(\mathbf{r}, t), \Pi(\mathbf{r}', t) \}_\text{PB} = 0
\]

and
\[
\{ \Phi(\mathbf{r}, t), \Pi(\mathbf{r}', t) \}_\text{PB} = \delta^{(3)}(\mathbf{r} - \mathbf{r}')
\]

Using these relations we can very well calculate the Poisson Brackets of \( \Phi(\mathbf{r}, t), \Pi(\mathbf{r}, t) \).

Now,
\[
\{ \Phi(\mathbf{r}, t), \Phi(\mathbf{r}', t) \}_\text{PB} = \int d^3x \, d^3x' \, e^{-i \mathbf{r} \cdot \mathbf{x}'} e^{-i \mathbf{r}' \cdot \mathbf{x}} \{ \Phi(\mathbf{x}), \Phi(\mathbf{x}') \}_\text{PB}
\]

\[
= 0
\]

Similarly,
\[
\{ \Pi(\mathbf{r}, t), \Pi(\mathbf{r}', t) \}_\text{PB} = 0
\]

And
\[
\{ \Phi(\mathbf{r}, t), \Pi(\mathbf{r}', t) \}_\text{PB} = \int d^3x \, d^3x' \, e^{-i \mathbf{r} \cdot \mathbf{x}'} e^{-i \mathbf{r}' \cdot \mathbf{x}} \{ \Phi(\mathbf{x}), \Pi(\mathbf{x}', \mathbf{t}) \}_\text{PB}
\]

\[
= \int d^3x \, d^3x' \, e^{-i \mathbf{r} \cdot \mathbf{x}'} e^{-i \mathbf{r}' \cdot \mathbf{x}} \delta^{(3)}(\mathbf{r} - \mathbf{r}')
\]
\[ \mathcal{S}[\mathbf{\Phi}(\mathbf{r},t), \mathcal{P}(\mathbf{p}^\prime,t)] = \int \frac{d^3x}{(2\pi)^3} e^{-i\mathbf{p} \cdot \mathbf{x}} \mathcal{S} \]

It is in the momentum space.

This is actually the statement of momentum conservation.

Here we have a $\mathcal{S}^{(3)}(\mathbf{r} + \mathbf{p}^\prime)$, we will see later how we get a $\mathcal{S}^{(3)}(\mathbf{r} - \mathbf{p}^\prime)$.

To show that,

\[ H = \frac{1}{2} \int d^3x \left[ \mathcal{P}(\mathbf{r},t) \mathcal{P}(-\mathbf{r},t) + (\mathbf{r}^2 + m^2) \mathcal{F}(\mathbf{r},t) \mathcal{F}(-\mathbf{r},t) \right] \]

We start with,

\[ H = \frac{1}{2} \int d^3x \left[ \mathcal{F}(\mathbf{r},t) + (\mathbf{r} \cdot \mathbf{\Phi}(\mathbf{r},t))^2 + m^2 \mathcal{F}(\mathbf{r},t) \right] \]

and we use,

\[ \mathcal{\dot{F}}(\mathbf{r},t) = \mathcal{P}(\mathbf{r},t) = \int \frac{d^3\mathbf{x}}{(2\pi)^3/2} e^{i\mathbf{x} \cdot \mathbf{r}} \mathcal{P}(\mathbf{r},t) \]

\[ \text{and} \quad \mathcal{\dot{F}}(\mathbf{r},t) = \int \frac{d^3\mathbf{x}}{(2\pi)^3/2} e^{i\mathbf{x} \cdot \mathbf{r}^\prime} \mathcal{F}(\mathbf{r},t) \]

\[ \therefore \quad \text{we wrote,} \]

\[ H = \frac{1}{2} \int d^3x \int d^3x' \mathcal{P}(\mathbf{r},t) \mathcal{P}(\mathbf{r}',t) \mathcal{F}(\mathbf{r}',t) \mathcal{F}(\mathbf{r}'',t) \]

Writing $\mathbf{r}' = -\mathbf{r}$, we get $d^3x = -d^3y$.

\[ \therefore \quad H = -\frac{1}{2} \int d^3x d^3x' \left[ \mathcal{P}(\mathbf{r},t) \mathcal{P}(\mathbf{r}',t) - \mathbf{r} \cdot \mathbf{\Phi}(\mathbf{r},t) \mathcal{F}(\mathbf{r}',t) \right] \mathcal{S}^{(3)}(\mathbf{r} + \mathbf{r}') \]
writing \( \vec{v}' = -\vec{v}'' \), we get \(-d^3\vec{v} = d^3\vec{v}''\).

We get:

\[
H = \frac{1}{2} \int d^3\vec{v} d^3\vec{v}'' \left[ \int (\vec{v}', \vec{v}'' \cdot \vec{v}') \left( \frac{\vec{v}'' \cdot \vec{v}}{\gamma_1 m^2} \right) - \gamma_1 m^2 \right] f(\vec{v}', \vec{v}'')
\]

so \( H = \frac{1}{2} \int d^3\vec{v} \left[ \int (\vec{v}', \vec{v}'' \cdot \vec{v}') \left( \frac{\vec{v}'' \cdot \vec{v}}{\gamma_1 m^2} \right) + \gamma_1 m^2 \right] f(\vec{v}', \vec{v}'')
\]

Now we had required \( \phi, \Pi \) to be real but not \( \phi, \Pi \) to be real, because they have on \( \vec{v}'' \) in the exponential.

But, we must note that, they can not be arbitrary complex variables.

We have:

\[
\phi(\vec{v}, t) = \int \frac{d^3\vec{v}}{(2\pi)^3/2} e^{-i\vec{v} \cdot \vec{v}} \tilde{\phi}(\vec{v}, t)
\]

and \( \Pi(\vec{v}, t) = \int \frac{d^3\vec{v}}{(2\pi)^3/2} e^{i\vec{v} \cdot \vec{v}} \tilde{\Pi}(\vec{v}, t) \).

Now if \( \tilde{\phi} \) is an arbitrary complex function of \( \vec{v}, t \),

then \( \phi(\vec{v}, t) \) is not necessarily real.

We need to put some conditions such that \( (\phi, \Pi) \) are real.

Our condition is:

\[
(\tilde{\phi}(\vec{v}, t))^* = \tilde{\phi}(-\vec{v}, t)
\]

Proof: \(-\phi(\vec{v}, t) = \int e^{i\vec{v} \cdot \vec{v}} \phi(\vec{v}, t) \frac{d^3\vec{v}}{(2\pi)^3/2} = \tilde{\phi}(-\vec{v}, t)\)

Similarly, \( (\tilde{\Pi}(\vec{v}, t))^* = \tilde{\Pi}(-\vec{v}, t) \).
Now, if we calculate

\[ \int \frac{d^3x}{(2\pi)^3} \, e^{-i\mathbf{p} \cdot \mathbf{r}} \, \tilde{\mathbf{r}}(\mathbf{r},t) \tilde{\mathbf{r}}(\mathbf{r}',t) \]

we have

\[ = \int \frac{d^3x}{(2\pi)^3} \, e^{-i\mathbf{p} \cdot \mathbf{r}} \, \tilde{\mathbf{r}}(\mathbf{r},t) \tilde{\mathbf{r}}(\mathbf{r}',t) \]

\[ = \int \frac{d^3x}{(2\pi)^3} \, e^{-i\mathbf{p} \cdot \mathbf{r}} \, s^{(2)}(\mathbf{r} - \mathbf{r}') \]

\[ = \int \frac{d^3x}{(2\pi)^3} \, e^{-i(\mathbf{p} + \mathbf{p}' \cdot \mathbf{r})} \]

\[ = s^{(2)}(\mathbf{r} - \mathbf{r}') \]

We will next take a classical system and the variables \( \tilde{\mathbf{r}}(\mathbf{r},t) \) and \( \tilde{\mathbf{r}}(\mathbf{r},t) \) and the conditions \( (\tilde{\mathbf{r}}(\mathbf{r},t))^* = \tilde{\mathbf{r}}(-\mathbf{r},t) \) and \( (\tilde{\mathbf{r}}(\mathbf{r},t))^* = \tilde{\mathbf{r}}(-\mathbf{r},t) \), and replace the \( \tilde{\mathbf{r}} \)'s by the \( \mathbf{T} \)'s and replace \( s \) by \( \frac{1}{\alpha \mathbf{e} \cdot \mathbf{J} \cdot \alpha} \), and we consider these \( \tilde{\mathbf{r}} \)'s and \( \mathbf{T} \)'s as operators and try to find the spectrum.

If there was no \( (-\mathbf{r}) \), then we would have had it like

\[ \mathbf{T}^2 + (\mathbf{p} + m^2)^2. \]

N.B.: In the expression \( \mathbf{T}^2 + \frac{1}{2} \mu \mathbf{p}^2 \), where \( \mu \) is the

stiffness parameter, so the expression is not exactly symmetric under

\( \mathbf{r} \rightarrow \mathbf{r}' \).

We can write the terms as \( \mathbf{T} \) instead of \( \int d^3\mu \) by

discarding. Then each term can be thought of as a harmonic

oscillator.

But, this is not necessary.
Let we have two terms,

\[ H = \frac{1}{2} \left( \tilde{p}_1^2 + \tilde{q}_1 \tilde{q}_1' \right) + \frac{1}{2} \left( \tilde{p}_2^2 + \tilde{q}_2 \tilde{q}_2' \right). \]

Now, we take \( \tilde{q}_1 = \tilde{q}_1 + \tilde{q}_2, \tilde{q}_1' = \tilde{q}_1' - \tilde{q}_2' \), \( \tilde{q}_2 = \tilde{q}_1' - \tilde{p}_1 \).

We then get,

\[ \tilde{q}_1' = \frac{1}{2} (\tilde{q}_1 + \tilde{q}_2) \]

\[ \tilde{q}_2 = \frac{1}{2} (\tilde{q}_1' - \tilde{q}_2) \]

\[ \tilde{q}_2' = \frac{1}{2} (\tilde{q}_1' + \tilde{q}_2) \]

We then can write the Hamiltonian in terms of these new variables,

\[ H = \frac{1}{2} \left[ \frac{1}{4} \left( \tilde{p}_1'^2 + \tilde{q}_1'^2 + 2 \tilde{p}_1' \tilde{q}_1' \right) + \frac{1}{2} \left( \tilde{q}_1'^2 + \tilde{q}_2'^2 \right) \right] \]

Now if \( \tilde{q}_2' = \tilde{q}_1' \), then,

\[ H = \frac{1}{2} \left[ \frac{1}{4} \left( \tilde{p}_1'^2 + \tilde{q}_1'^2 \right) \right] \]

so, now also, we can interpret \( \tilde{q}_1, \tilde{q}_1', \tilde{p}_1' \) of the coordinates and the momenta.

To find the conjugate to \( \tilde{q}_1, \tilde{q}_1' \).

We have,

\[ L = \frac{d}{dt} \left[ \frac{1}{2} \left( \tilde{p}_1'^2 + \tilde{q}_1'^2 \right) \right] - \frac{1}{2} \tilde{q}_1'^2 - m^2 \tilde{q}_1'^2 \]
\[ \mathcal{L} = \int \frac{d^3 \mathbf{v}}{(2\pi)^3} \left\{ \frac{\bar{\mathcal{P}}(\mathbf{v}, t) \mathcal{P}(\mathbf{v}', t) - \frac{1}{2} \varepsilon(\mathbf{v}) \varepsilon(\mathbf{v}') + m^2}{\varepsilon(\mathbf{v} + \mathbf{v}') - \varepsilon(\mathbf{v}') - \varepsilon(\mathbf{v})} \right\} \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) \]

\[ - \frac{1}{\varepsilon(\mathbf{v} + \mathbf{v}') - \varepsilon(\mathbf{v}') - \varepsilon(\mathbf{v})} \left( \int \frac{d^3 \mathbf{v}'}{(2\pi)^3} \left\{ \bar{\mathcal{P}}(\mathbf{v}, t) \mathcal{P}(\mathbf{v}', t) + \left[ \frac{1}{2} \varepsilon(\mathbf{v}) + m^2 \right] \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) \right\} \right) \]

In the last step, we replaced \( \mathbf{v}' \) by \( -\mathbf{v}' \).

Thus, \( d^3 \mathbf{v}' = d^3 \mathbf{v} '' \).

\[ \mathcal{L} = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \bar{\mathcal{P}}(\mathbf{v}'' + t) \mathcal{P}(\mathbf{v}'', t) - \left( \frac{1}{2} \varepsilon + m^2 \right) \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]

\[ \therefore \quad \mathcal{S}(\mathbf{v}, t) = \bar{\mathcal{P}}(\mathbf{v}, t) \]

Conjugate momentum to \( \mathcal{S}(\mathbf{v}, t) \) is:

\[ \mathcal{S}(\mathbf{v}, t) = \mathcal{S}(\mathbf{v}, t) \]

\[ \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \bar{\mathcal{P}}(\mathbf{v}'' + t) \mathcal{P}(\mathbf{v}'', t) \]

\[ = \bar{\mathcal{P}}(-\mathbf{v}'', t) \mathcal{P}(\mathbf{v}'', t) \]

\[ \therefore \quad \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) \]

\[ = \bar{\mathcal{P}}(-\mathbf{v}'', t) \mathcal{P}(\mathbf{v}'', t) \]

\[ = \bar{\mathcal{P}}(-\mathbf{v}'', t) \mathcal{P}(\mathbf{v}'', t) \]

\[ \chi = \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon(\mathbf{v}'' + t) - \varepsilon(\mathbf{v}'', t) \right] \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]

\[ \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon(\mathbf{v}'' + t) - \varepsilon(\mathbf{v}'', t) \right] \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]

\[ \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon(\mathbf{v}'' + t) - \varepsilon(\mathbf{v}'', t) \right] \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]

\[ \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon(\mathbf{v}'' + t) - \varepsilon(\mathbf{v}'', t) \right] \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]

\[ \mathcal{S}(\mathbf{v}, t) \mathcal{S}(\mathbf{v}', t) = \int \frac{d^3 \mathbf{v}''}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon(\mathbf{v}'' + t) - \varepsilon(\mathbf{v}'', t) \right] \mathcal{S}(\mathbf{v}'' + t) \mathcal{S}(-\mathbf{v}'', t) \]
\[ \forall \int -\nabla^2 + m^2 \psi(\mathbf{r}, t) = \int \frac{d^3 \mathbf{r}'}{(2\pi)^{3/2}} \int \frac{d^3 \mathbf{r}''}{(2\pi)^{3/2}} e^{-i\mathbf{r}' \cdot \mathbf{r}''} \psi(\mathbf{r}', t) \psi(\mathbf{r}'', t) \]

This is clearly a non-local theory.

Like, if we add terms like

\[ (\psi(\mathbf{r}, t))^2 \]

non-vanishing interactions, then we get a non-local theory.

\[ \text{But, } \left[ D - m^2 \right] \phi + \lambda \phi = 0 \text{ still goes as a local theory.} \]

In the non-linear case, it will not superpose, but in the linear case, if there is a potential,

\[ [D - m^2 + V(\mathbf{r})] \phi = 0 \]

\[ \left[ -\frac{\partial^2}{\partial \mathbf{r}^2} + \nabla^2 - m^2 + V(\mathbf{r}) \right] \phi = 0. \]

Then, we will have to work the eigenfunctions of \( \{ \nabla^2 - m^2 + V(\mathbf{r}) \} \) operator.

\[ \text{27.1.2011} \]

For the Klein-Gordon field, the action is

\[ S = \int d^4x \frac{1}{2} \left[ (\partial \phi)^2 - (\nabla \phi)^2 - m^2 \phi^2 \right] \]

We then treated \( \phi \) with its Fourier transform, we then introduced,

\[ \hat{\phi}(\mathbf{r}, t) = \int \frac{d^3 \mathbf{r}'}{(2\pi)^{3/2}} e^{-i\mathbf{r} \cdot \mathbf{r}'} \phi(\mathbf{r}', t). \]
We defined, \( \Pi'(\mathbf{r},t) = \frac{8}{\hbar} ; \quad \tilde{\Pi}'(\mathbf{r},t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i \mathbf{p} \cdot \mathbf{r}} \Pi(\mathbf{x},t) \)

\[ = \frac{8}{\hbar} \tilde{\Pi}'(\mathbf{r},t) \]

Given this, the Hamiltonian,

\[ H = \frac{1}{2} \int d^3x \left[ \tilde{\Pi}'(\mathbf{r},t) \tilde{\Pi}'(-\mathbf{r},t) + (\mathbf{p}^2 + m^2) \tilde{\phi}(\mathbf{r},t) \tilde{\phi}'(\mathbf{r},t) \right] \]

and we also had the Poisson-Bracket relations,

\[ \{ \tilde{\phi}(\mathbf{r},t), \tilde{\phi}'(\mathbf{r}',t) \}_{PB} = 0, \quad \{ \tilde{\phi}'(\mathbf{r},t), \tilde{\phi}'(\mathbf{r}',t) \}_{PB} = 0, \]

\[ \{ \tilde{\phi}(\mathbf{r},t), \tilde{\phi}'(\mathbf{r}',t) \}_{PB} = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \]

We also had the relations,

\[ (\tilde{\phi}(\mathbf{r},t))^* = \tilde{\phi}'(-\mathbf{r},t) \]

and \( (\tilde{\Pi}'(\mathbf{r},t))^* = \tilde{\Pi}'(-\mathbf{r},t) \)

We now try to quantise this theory.

We now consider \( \tilde{\phi}, \tilde{\Pi}' \) as operators and their complex conjugate as the \( \dagger \) 's,

\[ (\tilde{\phi}(\mathbf{r},t))^\dagger = \tilde{\phi}'(-\mathbf{r},t) \]

and \( (\tilde{\Pi}'(\mathbf{r},t))^\dagger = \tilde{\Pi}'(-\mathbf{r},t) \)

and \( [\tilde{\phi}(\mathbf{r},t), \tilde{\Pi}'(\mathbf{r}',t)] \equiv i \delta^{(3)}(\mathbf{r} - \mathbf{r}') \) \( [\text{taking } \hbar = 1] \),

the Hamiltonian now becomes an operator.
We define new operators which will have the same properties as the creation, annihilation operators.

We define:

\[ a(\vec{r}, t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega \frac{1}{\sqrt{2}} \Phi(\vec{r}, t) + \omega \frac{\sqrt{2}}{\sqrt{2}} \Pi(\vec{r}, t) \\ \omega \frac{1}{\sqrt{2}} \end{pmatrix} \]

where \[ \omega^2 = \sqrt{\vec{p}^2 + m^2} \]

\[ a^+(\vec{r}', t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega \frac{1}{\sqrt{2}} \Phi^+(\vec{r}', t) - \omega \frac{\sqrt{2}}{\sqrt{2}} \Pi^+(\vec{r}, t) \\ \omega \frac{1}{\sqrt{2}} \end{pmatrix} \]

\[ = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega \frac{1}{\sqrt{2}} \Phi(-\vec{r}', t) - \omega \frac{\sqrt{2}}{\sqrt{2}} \Pi(-\vec{r}', t) \\ \omega \frac{1}{\sqrt{2}} \end{pmatrix} \]

\[ \therefore [a(\vec{r}, t), a^+(\vec{r}', t)] = \frac{-\omega^2}{\sqrt{2}} \frac{-\omega}{\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \Phi(\vec{r}, t), \Pi(-\vec{r}', t) \right] \]

\[ + \frac{\omega}{\sqrt{2}} \frac{\omega}{\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \Pi(\vec{r}, t), \Phi(-\vec{r}', t) \right] \]

\[ = \frac{\omega}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{\omega}{\sqrt{2}} \frac{-\omega}{\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \Phi(\vec{r}, t), \Pi(-\vec{r}', t) \right] \]

\[ = \frac{1}{\sqrt{2}} \left( \frac{\omega^2}{\sqrt{2}} \right) \frac{\omega}{\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \Phi(\vec{r}, t), \Pi(-\vec{r}', t) \right] \]

\[ = \frac{1}{\sqrt{2}} \left( \frac{\omega^2}{\sqrt{2}} \right) \delta^{(3)}(\vec{r} - \vec{r}') \]

\[ = \frac{1}{\sqrt{2}} \left( \frac{\omega^2}{\sqrt{2}} \right) \delta^{(3)}(\vec{r} - \vec{r}') \]

Similarly, \[ [a(\vec{r}, t), a(\vec{r}', t)] \]

\[ \therefore \]
Similarly, \(a^+(\mathbf{r}, t), a^+(\mathbf{r}', t)\) = 0.

Now, \[
\hat{N} = \frac{i}{\hbar} \int d^3 \mathbf{r} \left[ \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) \right]
\]

Now, we want to get \(\bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t)\)
and \(\bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t)\), so we multiply,

\[a(\mathbf{r}, t) a^+(\mathbf{r}', t)\]

\[
\therefore a(\mathbf{r}, t) a^+(\mathbf{r}, t) = \frac{\alpha}{\hbar} \left[ \omega \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) \right]
\]

Now we want to remove the last two terms.

Now, \[
a^+(\mathbf{r}, t) a(\mathbf{r}, t) = \frac{\alpha}{\hbar} \left[ \omega \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) \right]
\]

Now, \[
[\bar{\psi}^{(1)}(-\mathbf{r}, t), \bar{\psi}^{(1)}(\mathbf{r}, t)] = \epsilon \delta^{(3)}(\mathbf{r} - \mathbf{r}')
\]

\[
\therefore \bar{\psi}^{(1)}(-\mathbf{r}, t) \bar{\psi}^{(1)}(\mathbf{r}, t) = \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \epsilon \delta^{(3)}(\mathbf{r} = \mathbf{r'})
\]

\[
a^+(\mathbf{r}, t) a(\mathbf{r}', t) a^+(\mathbf{r}, t) a(\mathbf{r}', t)
\]

\[
\frac{\alpha}{\hbar} \left[ \omega \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) \right]
\]

\[
= \frac{\alpha}{\hbar} \left[ \omega \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) \right]
\]

\[
\therefore \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) - \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t)
\]

\[
\omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t)
\]

\[
\therefore \bar{\psi}^{(1)}(\mathbf{r}, t) \bar{\psi}^{(1)}(-\mathbf{r}, t) + \omega \bar{\psi}^{(2)}(\mathbf{r}, t) \bar{\psi}^{(2)}(-\mathbf{r}, t)
\]
\[ \hat{H} = \frac{1}{2} \int d^3k \omega^2 \left[ a(\vec{k}, t) a^+(\vec{k}, t) + a^+(\vec{k}, t) a(\vec{k}, t) \right] \]

Now, in the second term, we do \( \vec{k} \rightarrow -\vec{k} \), then we get:

\[ \int d^3k a^+(\vec{k}, t) a(-\vec{k}, t) \rightarrow -\int d^3k a^+(\vec{k}, t) a(\vec{k}, t) \]

Thus, the second term is invariant under \( \vec{k} \rightarrow -\vec{k} \).

Thus, \( \hat{H} = \frac{1}{2} \int d^3k \omega^2 \left[ a(\vec{k}, t) a^+(\vec{k}, t) + a^+(\vec{k}, t) a(\vec{k}, t) \right] \)

Now, we have:

\[ [a(\vec{k}, t), a^+(\vec{k}', t)] = \delta^{(3)}(\vec{k} - \vec{k}') = \delta^{(3)}(0) \]

\[ \therefore a(\vec{k}, t)a^+(\vec{k'}, t) = \delta^{(3)}(0) + a^+(\vec{k}, t)a(\vec{k}, t) \]

Thus, \( \hat{H} = \int d^3k \omega^2 a^+(\vec{k}, t) a(\vec{k}, t) + \frac{1}{2} \left( \int d^3k \omega^2 \right) \delta^{(3)}(0) \)

Now, the second term is an infinite constant. This overall constant is irrelevant when we are interested in calculating the difference in energy between two states. The overall energy is completely

...
The term, \( \int d^3x \sqrt{-g} \), increases without bound with increasing \( V \).

Again, \( \int d^3x (\kappa = 0) \) is another infinity because the spacetime has infinite volume.

If we take a box of finite volume, then the divergence could be gotten rid of.

These kinds of divergences are known as ultraviolet (UV) or short-distance divergences. This occurs for very high energy/momentum (i.e., at very small distances).

We do not know whether this formula is valid for very high energies.

But, we can always put a cut-off.

We must note that, even in classical mechanics, the zero of the potential energy does not have any meaning. A zero level is always fixed and differences in energies are considered with respect to that level.

*N.B.* Energy is a source of gravity. So, whenever we encounter gravity, we cannot leave out this second term. Thus, we face problems quantizing gravity.

But, for our case at hand, we can choose a certain zero level.

In every problem, we should decide on what we should subtract. We should measure the difference in energies of the two levels.
we will never be able to measure the infinite term.

But, if we have independent oscillators with interactions, then what we subtract does not only depend on the oscillators but also on their interactions with each other. (Take, if we add a $\frac{k}{x^2}$ term to the simple harmonic oscillator, then the zero level changes.)

---

**Note:** If we take a system with no particles and say that it has zero energy and now if we add a particle with negative energy and take it to $-\infty$, then this is not allowed.

*The individual energies should not be unbounded from below.*

Thus, the Klein-Gordon equation, so the first quantised form, does not make much sense.

There we had negative infinity (in the first quantised form).

But here (i.e., in the second quantised form) we have a positive infinity only. Thus, if we fix a zero level, everything is specified.

This way, we are fixing the zero-energy level. We can fix an $E_0$ that energy will be for no particles.

Then, adding particles is unambiguous.

---

**Note:** In the classical system, $N$ (the number of particles) is large.

Now, in the quantum system, if we did not have the $\frac{k}{x^2}$, then there would have been no difference in the classical limit.
Thus, it is a purely quantum mechanical term that we are referring to.

Here, this entire term is an analogous term, which is a quantum effect.

Here also, we can not measure simultaneously, $\mathbf{q}, \mathbf{p}$ or $\mathbf{q}, \mathbf{p}$, as they do not commute.

There are infinite numbers of $\mathbf{q}$'s and $\mathbf{p}$'s at infinite number of space points.

The difference between the two points can be as small as possible but still, there can be an infinite number of points in between these two points. This is the ultraviolet divergence at small volumes.

If we consider a lattice, and if its volume is $1 \text{ cm}^3$, then there are finite number of points. But if we take this volume to infinity, then the number of points also tend to infinity.

- Summary:
  1. One infinity is due to the infinite volume in space,
  2. Another infinity is due to an infinite number of points between two points in space.

Thus, we drop the constant (infinitely) term and take the Hamiltonian,

$$\hat{H} = \int d^3 \mathbf{r} \, \psi^\dagger(\mathbf{r}) \Delta m^2 \psi(\mathbf{r})$$

Then, we would like to build the eigenstates (at time $t=0$).
For every \( \mathbf{p} \), it is like a harmonic oscillator.

So, the ground state is denoted by \( 10 \) such that

for all \( \mathbf{p} \), \( a(\mathbf{p}, t=0) = a(\mathbf{p}) \)

\( a(\mathbf{p}) 10 = 0 \).

But at any time \( t \), we will get the same spectrum.

Thus, \( a(\mathbf{p}, t) 10 = 0 \).

Thus, by construction, we have measured the ground state energy to be zero because,

\[
\hat{\mathcal{H}} 10 = \int d^3 \mathbf{k} \omega_\mathbf{k} a^+ (\mathbf{k}, t) a (\mathbf{k}, t) 10 = 0.
\]

The first excited state will be \( a^+ (\mathbf{p}, t) 10 \).

We will have,

\[
\hat{\mathcal{H}} a^+ (\mathbf{p}, t) 10 = \omega_\mathbf{p} a^+ (\mathbf{p}, t) 10.
\]

Proof:

\[
\hat{\mathcal{H}} a^+ (\mathbf{p}, t) 10 = \int d^3 \mathbf{k} \omega_\mathbf{k} a^+ (\mathbf{k}, t) a (\mathbf{k}, t) 10 + a^+ (\mathbf{p}, t) a (\mathbf{p}, t) 10
\]

\[
= \int d^3 \mathbf{k} \omega_\mathbf{k} a^+ (\mathbf{k}, t) 10 \mathcal{L}(\mathbf{k}^2),
\]

\[
\omega_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}.
\]

Since, single particle momentum states can be there with

energy values, \( \omega_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2} \).

\[\text{Note: Now, if we can measure the momentum eigenvalues,}\]
\[\text{viz., if we can get the momentum eigenstates (by constructing}\]
\[\text{the momentum operator), then we can say that the states}\]
\[\text{are (the above one) single particle momentum states.}\]
\[ \hat{a}(\mathbf{\tilde{w}}_1, t) \hat{a}^\dagger(\mathbf{\tilde{w}}_2, t) |0\rangle \]
\[ \frac{1}{\sqrt{2}} \hat{a}(\mathbf{\tilde{w}}_1, t) \hat{a}^\dagger(\mathbf{\tilde{w}}_2, t) |0\rangle \]
\[ = \int d^3 \mathbf{w} \omega^{-1} a^\dagger(\mathbf{w}, t) a(\mathbf{w}, t) a^\dagger(\mathbf{\tilde{w}}_1, t) a^\dagger(\mathbf{\tilde{w}}_2, t) |0\rangle \]
\[ = \int d^3 \mathbf{w} \omega^{-1} a^\dagger(\mathbf{w}, t) \left[ a^\dagger(\mathbf{\tilde{w}}_1, t) a(\mathbf{\tilde{w}}_1, t) a^\dagger(\mathbf{\tilde{w}}_2, t) a(\mathbf{\tilde{w}}_2, t) \right] |0\rangle \]
\[ = \omega^{-1} a^\dagger(\mathbf{\tilde{w}}_1, t) a^\dagger(\mathbf{\tilde{w}}_2, t) \]
\[ + \int d^3 \mathbf{w} \omega^{-1} a^\dagger(\mathbf{w}, t) a^\dagger(\mathbf{\tilde{w}}_1, t) a^\dagger(\mathbf{\tilde{w}}_2, t) a(\mathbf{\tilde{w}}_1, t) a(\mathbf{\tilde{w}}_2, t) \]
\[ = (\omega^{-1} + \omega^{-1}) a^\dagger(\mathbf{\tilde{w}}_1, t) a^\dagger(\mathbf{\tilde{w}}_2, t) |0\rangle \]
\[ \therefore [a^\dagger(\mathbf{\tilde{w}}_1, t), a^\dagger(\mathbf{\tilde{w}}_2, t)] = 0. \]

Thus, this denotes a two-particle state given by two vectors with the total energy
\[ \sqrt{\mathbf{\tilde{w}}^2 + m^2} + \sqrt{\mathbf{\tilde{w}}^2 + m^2} \]
where \( \mathbf{\tilde{w}} \) is the momentum of one particle and \( \mathbf{\tilde{w}} \) is that of the other particle.

One-particle level can be considered by the continuous parameter
\[ \mathbf{\tilde{w}} \] of the momentum eigenvalue (to be found later) in this case is \( \mathbf{\tilde{w}} + \mathbf{\tilde{w}} \).

Now even if \( \mathbf{\tilde{w}} + \mathbf{\tilde{w}} = 0 \), the total energy is not zero as there is a \( \sqrt{\mathbf{\tilde{w}}^2 + m^2} \) and a \( \sqrt{\mathbf{\tilde{w}}^2 + m^2} \) which is characteristic of two particles.
We can compare the second quantized Klein-Gordon Hamiltonian with the second quantized Schrödinger Hamiltonian.

In the Klein-Gordon case,

$$\hat{H}_K = \int d^3 x \sqrt{\hat{\nabla}^2 + m^2} \ a^+(\mathbf{r},t) \ a(\mathbf{r},t).$$

In the Schrödinger case,

$$\hat{H}_S = \sum_n \varepsilon_n \ a_n^+ a_n; \text{ where the } \varepsilon_n \text{ 's are the single-particle energy eigenvalues.}$$

Here, the above Hamiltonian is for four particles.

Now, if we have a Schrödinger equation without a potential V, then the ε_n's for four particles would have become continuous and we have

$$\hat{H}_S = \int d^3 x \left( \frac{\hat{\nabla}^2}{2m} \right) a^+(\mathbf{r},t) a(\mathbf{r},t).$$

The equal-time commutation relation would be

$$[a(\mathbf{r}), a(\mathbf{r}')] = 0,$$

$$[a^+(\mathbf{r}), a(\mathbf{r}')] = 0,$$

and $$[a(\mathbf{r}), a^+(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \text{ (in both cases).}$$

In $\hat{H}_K$, $\sqrt{\hat{\nabla}^2 + m^2}$ is actually $\sqrt{\mathbf{p}^2 + mc^2}.$

Now, for large $m$ or small $|\mathbf{p}|$, we have

$$\sqrt{\mathbf{p}^2 + m^2} \approx m + \frac{\mathbf{p}^2}{2m} + \cdots$$

($m$ is a constant term.)
Thus, we see that in the non-relativistic limit, the equations of the two Hamiltonians, \( H_1 \) and \( H_2 \), are similar.

\[ \text{N.B. But we must note that at the first quantised level, the} \]
\[ \text{Klein-Gordon equation did not have a perfectly physical} \]
\[ \text{interpretation (because of infinite lower bound in energy)} \]
\[ \text{but the Schrödinger equation did have a perfectly physical interpretation.} \]

Thus, though the starting point, the Schrödinger equation
\[ \text{was a single derivative in time and the Klein-Gordon equation} \]
\[ \text{involved a double derivative in time, yet both yielded almost} \]
\[ \text{similar results. The latter is the relativistic version of the former.} \]

Now, for \( v = 0 \), the Schrödinger equation reads:
\[ \frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = -\frac{1}{2m} \psi^2 (x = 1). \]

Now, if we take \( \psi = e^{-\frac{R \cdot x}{2m}} \) to be a solution.
\[ \frac{1}{2m} \frac{\partial^2 \psi}{\partial t^2} = -\frac{1}{2m} \frac{\partial \psi}{\partial t} e^{-\frac{R \cdot x}{2m}} = \frac{1}{2m} e^{-\frac{R \cdot x}{2m}}. \]

Now, we want to find some operator which acting on
\[ e^{-\frac{R \cdot x}{2m}} \] would give \( e^{-\frac{R \cdot x}{2m}} \). Is it altogether possible?

We use the Fourier transform of \( \psi(R, t) \) as:
\[ \dot{\psi}(R, t) = \frac{1}{(2\pi)^{3/2}} \int e^{iR \cdot x} \psi(x, t) \, dx. \]
We have:
\[ \hat{A} = \int d^3x \, \hat{\Psi}^*(\vec{r}, t) \, \hat{\Psi}(\vec{r}, t) \]

\[ = \frac{-i}{2m} \int d^3x \, \int d^3x' \, \frac{e^{i \vec{r} \cdot \vec{r}'}}{(2\pi)^{3/2}} \, \hat{\Psi}^*(\vec{r}', t) \hat{\Psi}(\vec{r}, t) \]

\[ = \frac{1}{2m} \int d^3x \, \int d^3x' \, \frac{e^{i \vec{r} \cdot \vec{r}'}}{(2\pi)^{3/2}} \, \hat{\Psi}^*(\vec{r}', t) \hat{\Psi}(\vec{r}, t) \]

\[ = \int d^3x \, \int d^3x' \, \delta(\vec{r} - \vec{r}') \, \hat{\Psi}^*(\vec{r}', t) \hat{\Psi}(\vec{r}, t) \]

\[ \propto \hat{A} = \int d^3x \, \hat{\Psi}^*(\vec{r}, t) \frac{e^{i \vec{r} \cdot \vec{r}'}}{2m} \hat{\Psi}(\vec{r}, t) \]

We would like to find a \( \hat{\text{curl}} \), such that:

\[ \hat{\text{curl}} \, e^{i \vec{r} \cdot \vec{r}'} = \frac{e^{i \vec{r} \cdot \vec{r}'}}{(2\pi)^{3/2}} \]

Then we could have used the same Schrödinger equation.

But, alas, we can not find a simple \( \hat{\text{curl}} \).

There is linearity in \( \hat{\Psi} \), not in momentum.

We have to find a local differential operator.

Definition of \( \hat{\text{curl}} \):- Given a function \( \psi(\vec{x}, t) \), \( \hat{\text{curl}} \) should give another function.

Way about:- If we take the function \( \psi(\vec{x}, t) \), take its Fourier transform, multiply it by \( \sqrt{2\pi} + m^2 \) and take its inverse Fourier transform.

We define \( \hat{\text{curl}} \psi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3x \, \frac{e^{i \vec{r} \cdot \vec{r}'}}{(2\pi)^{3/2}} \, \hat{\psi}(\vec{r}', t) \)
Then, we will get:

\[ S = \int d^3x \, \psi^*(x, t) \left( \frac{\partial^2}{\partial t^2} - \hat{H}_0 \right) \psi(x, t) \]
and get the same \( S \), i.e.

\[ \hat{A} = \int d^3x \, \sqrt{\text{det}\, m} \, \psi^*(x, t) m(x, t) \psi(x, t). \]

Local operator:- Depends on \( \psi^* \) and its derivative at all \( x \).

But, clearly \( m \) itself is not a local operator.

Only, if the operator is a polynomial in momentum, it is local.

else, it is not so.

We now want to know about the interaction.

In the Klein–Gordon case, the action

\[ S = \frac{i}{2} \int (\nabla \psi)^2 - (\overline{\psi} \gamma^0 \psi)^2 - m^2 \psi^2 \, d^4x. \]

We have to know \( \psi \) and its derivative at the same point.

We may think of each \( \psi \) as a coordinate variable.

If we couple two oscillators widely separated by two points in space, then it is a non-local scenario.

We may add \( \hat{A} \) and \( \left[ \psi^*(x, t) \, \psi^*(x, t) \, \psi(x, t) \, \psi(x, t) \right] \) which looks local, i.e., at the same space point. But actually, this is not local.

Because this formalism is not manifestly local, we have:

\[ \hat{A} + \left[ \psi^*(x, t) \, \psi^*(x, t) \, \psi(x, t) \, \psi(x, t) \right]. \]
It has a product of one \( \psi^* (x, t) \) at one point and \( \psi \) at a completely different space-point. We do not know what kind of interactions will occur at a distance.

Thus, in our theory, we should add every term to the Lagrangian which are at the same space point.

Here, we need the information at all points.

Any such Hamiltonian has a conserved particle number.

The number operator is,
\[
\hat{N} = \int d^3x \ a^+(\vec{r}) \ a(\vec{r}) \quad \text{and in the Schrödinger problem, it is:}
\]
\[
\hat{N} = \int d^3x \ \psi^* (x, t) \ \psi (x, t).
\]

We could have maintained the conservation by demanding that every term has the same number of \( \psi \)'s and \( \psi^* \)'s.

We can write the \( \hat{N} \) in terms of the \( \hat{Q}, \hat{P} \) in the Klein - Gordon case, then the expression for the number operator (local) for \( \hat{Q}, \hat{P} \) is complicated.

- One, so, typically, local interactions in the Klein - Gordon theory do not conserve \( \hat{N} \).

- Example: If we add to the action
\[
\Delta S = \int d^4x \ \left( \frac{\partial^2}{\partial \phi^2} \right)^4 \],

then the extra term in the Lagrangian would be:
\[
\Delta L = \int d^3x \ \left( \frac{\partial^2}{\partial \phi^2} \right)^4 \quad \text{and the extra term in the Hamiltonian would be:}
\]
\[
\Delta H = -\int d^3x \ \left( \frac{\partial^2}{\partial \phi^2} \right)^4.
\]
We can write \( p(x,t) \) as a linear combination of \( a(t) \) and \( a(t)^{+} \).

Thus, we will have terms with equal number of \( a(t) \) and \( a(t)^{+} \) as well as unequal number of \( a(t) \) and \( a(t)^{+} \). For example, we will have terms like:

\[ a(t)a(t), a(t)^{+}a(t)^{+}, \text{ etc.} \]

So, local interaction written in terms of \( a(t)^{+}a(t) \) does not conserve the number of particles.

So, in this theory, with high energy, we can get more particles.

But if someone does not bother about locality but only wants particle conservation, then he can work with \( \chi_{t}^{+}\chi_{t} \). One could then add to the original action some terms with equal number of \( \chi_{t} \) and \( \chi_{t}^{+}\). Then we must use the Schrödinger representation.

But in the relativistic case, we want locality. We do not want the transmission of signal faster than the speed of light.

So, we cannot conserve particle number.

The number operator commutes with the Lorentz transformation operator, which can be described by giving the system a Lorentz boost.

We cannot assign a charge to a given event. For local operators, we have to integrate over.

In relativity, if we have sufficient energy, we can create more particles.

We can always carry a particle and its antiparticle.

Relativity does not forbid it.

Two-particle states have a different kind of dispersion relation.
An idea second quantization, we get the correct spectrum.

For quadratic Lagrangians, we will get particle representation.

When there are cubic and/or higher order terms in the Lagrangian, we use perturbation theory. We can see scattering among particles.

In the first case, we solve a first quantized case and in the second quantization case, one always start with the action for a single particle.

In Schrödinger picture, the time derivative is of order 1. Therefore, we get the same spectrum for both the first and the second quantized cases. But, there is no guarantee that it will happen for all.

But, on the other hand, in the Klein-Gordon picture, in the first quantized theory, we get the energy spectrum as

$$\omega = \pm \sqrt{k^2 + m^2},$$

but in the second quantized theory, we only get the positive energy spectrum.

For fermions, we will assume that all the negative energy states have been filled up by the fermions.

Negative energy states are completely filled up.

We can always put a fermion in the positive energy level. (There are single-particle states.)
On the other hand, if we remove a negative energy electron (excite it so that it goes and occupies a positive energy level), then a "hole" is created. Thus the total energy is the ground state energy.

But for holes, we can never fill the negative energy state completely, as we can always put more holes to any energy level.

- We have \( \left[ \phi \left( \vec{r}, t \right), \Pi \left( \vec{p}, t \right) \right] = \delta^{(3)} \left( \vec{r} - \vec{p} \right) \).
  - we can think of \( \Pi \left( \vec{p}, t \right) = - \frac{\delta}{\delta \phi \left( \vec{r}, t \right)} \) just like in quantum mechanics, where \( \phi = - \frac{\delta}{\delta x} \).

Now, one imagines that there is a functional \( F \left[ \phi \right] \).

then

\[
\left[ \phi \left( \vec{r}, t \right), \Pi \left( \vec{p}, t \right) \right] F \left[ \phi \right] = - \int \left[ \phi \left( \vec{r}, t \right), \frac{\delta}{\delta \phi \left( \vec{r}, t \right)} \right] F \left[ \phi \right] \\
= - \int \left[ \phi \left( \vec{r}, t \right) \delta F \left[ \phi \right] - \delta \phi \left( \vec{r}, t \right) \frac{\delta F \left[ \phi \right]}{\delta \phi \left( \vec{r}, t \right)} \right] \\
= - \int \left[ \phi \left( \vec{r}, t \right) \frac{\delta F \left[ \phi \right]}{\delta \phi \left( \vec{r}, t \right)} - \frac{\delta \phi \left( \vec{r}, t \right)}{8 \pi \delta (\vec{r} - \vec{p})} \right] \\
= - \int \frac{\delta \phi \left( \vec{r}, t \right)}{8 \pi \delta (\vec{r} - \vec{p})} \delta F \left[ \phi \right]
\]

on \( \left[ \phi \left( \vec{r}, t \right), \Pi \left( \vec{p}, t \right) \right] F \left[ \phi \right] = - \frac{\delta F \left[ \phi \right]}{8 \pi \delta (\vec{r} - \vec{p})}. \)
Symmetries and conservation laws (Noether's theorem):

Let us begin with some general system.

Suppose we have a classical field theory with fields \( \phi_1(x), \phi_2(x), \ldots \) (i.e., multiple fields).

These fields can either be scalar fields, or components of vector fields. In our study, we will not make any assumptions.

The action will be denoted as a functional,

\[ S = \int \mathcal{L} \, dx, \]

where \( X = (x, t) \).

**Definition:** A function gives a number, if we specify the argument.

A functional gives a number, if we specify the function, i.e., \( \phi_1(x), \phi_2(x), \ldots \).

**Example:** The Klein-Gordon action functional is:

\[ S = \int \left( \frac{1}{2} \nabla \phi \cdot \nabla \phi - \frac{1}{2} m^2 \phi^2 \right) dx. \]

We now talk about a transformation of the fields.

The notion of a transformation is that, given a set of functions \( \phi_1(x), \phi_2(x), \ldots, \phi_n(x) \), there is a rule which transforms these functions to \( \tilde{\phi}_1(x), \tilde{\phi}_2(x), \ldots, \tilde{\phi}_n(x) \).

**Note:** A transformation is a rule to generate new functions from a given set of functions.
The rule should be such that if we have one set of functions,
then we can find the new set of functions.

Just like in quantum mechanics one can have \( N \) coordinates
for \( N \)-particles. A classical field theory can have multiple
variables of fields.

The transformation is generally written as:

\[ \tilde{\Phi}(x) = F_\Phi [\phi(x, y)]. \]

Here \( F_\Phi \) denotes a functional of all these functions.
This is the most general form of the transformation.

\( \Phi \) is a function of \( \phi \) for every point \( y \) and is
a functional of all the fields, \( \tilde{\Phi} \).

Thus, we have to know all the \( \phi \)'s at all the space-time
points.

We define a symmetry as when the transformation keeps the
action unchanged. Thus, then the transformation generates a
symmetry under:

\[ S [\tilde{\phi}(x, y)] = S [\phi(x, y)] \text{ for all possible choices} \]

of \( \tilde{\phi}(x, y) \).

We have to put certain boundary conditions on the \( \phi(x, y) \).

For a symmetry, the value of the action for these two
sets of fields have to be the same.
Example: Space-time translation.

It is defined as:

\[ \Phi_n(x) = \Phi_n(x+a); \quad \text{where } a \text{ is a constant 4-vector.} \]

This is a transformation.

Now, we ask the question whether it is a symmetry or not. The answer obviously depends on what definition we choose.

Let us for example choose the five scalar field,

\[ \phi(x) = \frac{1}{(2\pi)^2} \int d^4x \left[ -\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]; \quad \text{where } \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Now, to check whether this is a symmetry under the above transformation, we calculate,

\[ S[\Phi] = \frac{1}{2} \int d^4x \left[ -\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right] \]

Next, we define \( x^\mu = x^\mu + a^\mu \).

Then, \[ \begin{align*}
\frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial a^\mu}, \\
\frac{\partial}{\partial x^0} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial a^0}, \\
\frac{\partial}{\partial x^1} &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial a^1}, \\
\frac{\partial}{\partial x^2} &= \frac{\partial}{\partial x^2} + \frac{\partial}{\partial a^2}, \\
\frac{\partial}{\partial x^3} &= \frac{\partial}{\partial x^3} + \frac{\partial}{\partial a^3},
\end{align*} \]

and \[ d^4x = dx^0 dx^1 dx^2 dx^3 = dy^0 dy^1 dy^2 dy^3 = d^4y. \]

(\( a^\mu \) is a constant four vector).

The range of integration \((-\infty, +\infty)\) remains the same (assumed).
Then, by the above assumption, we get:

\[
\mathcal{S}[\phi] = \int d^4y \left[ -\eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi(x)^2 \right]
\]

Now, \( y \) is a dummy integration variable which we can

very well write as \( x \).

Thus, we get:

\[
\mathcal{S}[\phi] = \int d^4x \left[ -\eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi(x)^2 \right]
\]

\[
= \mathcal{S}[\phi].
\]

Thus, the above transformation is a symmetry in this case.

\[\therefore\]

We must note that adding a total time derivative to the action, does not change it. This is already allowed

in this formalism. When a total derivative is added, it always

integrate to zero at the boundary.

We will assume that the fields fall off sufficiently

just at the boundary.

But, for the above action, a constant transformation,

\[\phi \rightarrow \phi = \phi + \epsilon \quad (\epsilon \text{ constant}),\]

will not be a symmetry because by no means the term,

\[m^2 \phi^2 + \epsilon^2\]

will give us \( \mathcal{S}[\phi] = \mathcal{S}[\phi] \).

The condition that the symmetry holds is irrespective of

the initial field configuration.

\[\therefore\]

If the initial fields \( \phi(x) \) satisfy the equations of

motion, then so does the \( \phi(x) + \epsilon \).
That is, whatever variation we take in \( \delta \phi(x, t) \), if we take in 
\( \delta \phi(x, 4t) \), we will get the same equations of motion.

But, we should never use the equations of motion to prove symmetry.

We will check for symmetry for \( \phi \) very close to \( \phi_0 \), where we will have only first order variation terms.

Example: Let us consider the Schrödinger field theory.

The action is:
\[
S = \int d^4x \left[ \partial^\mu \phi(x, t) \partial_\mu \phi(x, t) - \frac{\hbar^2}{2m} \phi(x, t) \right],
\]
where we have taken \( \hbar = \frac{1}{2m} \) and
\[
\hbar = \frac{1}{2m} (\nabla^2 + V(x)) \text{ as a differential operator}.
\]

Now, we ask if the transformation,

(i) \( \phi(x, t) = \psi(x + \alpha, t + \alpha^\prime) \) is a symmetry or not.

Here is clearly not a symmetry because if we write,
\[
\psi(x, t) = \phi(x + \alpha, t + \alpha^\prime),
\]
the form of the potential,
\[
V(x) = V(\psi(x, t) - \alpha^\prime)
\]
will change.

But for time independent potentials, the transformation,
\[
\psi(x, t) = \psi(x + \alpha, t + \alpha^\prime)
\]
is a symmetry.

(ii) We can multiply \( \psi \) by some phase \( e^{i\theta} \).

Then \( \psi(x, t) = e^{i\theta} \phi(x, t) \) [\( \theta \) constant].

When \( S[\psi(x, t)] = \int d^4x \left[ \partial^\mu \psi(x, t) \partial_\mu \psi(x, t) - \frac{\hbar^2}{2m} \psi(x, t) \right] \)
\[
= S[\phi(x, t)].
\]
The exponential terms cancel out.

Thus, \( \Phi(x,t) = e^{ix} \phi(x) \) is a symmetry in this case.

All of the above are examples of continuous symmetry. There are always some parameters \((a, \xi)\) which would be continuous.

Analogous to the phase shift, in the scalar case,

\( \delta = \xi \) is a symmetry. This is termed a discrete symmetry.

As the scalar field action has all quadratic terms, just by looking into it, we can say that \( \Phi(x,t) = \phi(x) \)

is a symmetry.

- **Infinitesimal Symmetry**: We can choose \( \phi \) close to \( \phi \).

So, from continuous symmetry, we can go to an infinitesimal symmetry by taking the parameters to be infinitesimal.

- **Example**: We know, \( \hat{\phi}(x) = \phi(x+a) \). An infinitesimal symmetry means that we want to choose \( a^\mu \) such that

\[ \hat{\phi}(x) = \phi(x+a) \approx \phi(x) \]

Property: we imagine \( a^\mu \) to be small (infinitesimal), such that

\[ \phi(x) = \phi(x) + a^\mu \partial_\mu \phi + O(a^2) \]

Here, we consider bounded function and as long as \( \partial_\mu \phi \) is continuous in the limit \( a \rightarrow 0 \), we get:

\[ \hat{\phi}(x) \rightarrow \phi(x) \]
Example: In the case

\[ \Psi(x, t) = e^{-\alpha x} + (x^2), \]

as the limit \( \alpha \to 0 \), we get

\[ \Psi(x, t) = (1 + \alpha x) \Psi(x, t) + O(\alpha^2) + \text{higher order terms}, \]

\[ = \Psi(x, t) + \alpha x \Psi(x, t) + O(\alpha^2) + \ldots. \]

Now, we suppose that we have an infinitesimal transformation with some set of parameters \( \varepsilon^1, \varepsilon^2, \ldots, \varepsilon^k \) which are infinitesimal under which \( \Psi \approx \tilde{\Psi} \).

Suppose, at this stage, we do not know whether it is a symmetry (i.e., the transformation) or not.

We calculate, \( S[\Psi] - S[\tilde{\Psi}] \), irrespective of whether it is a symmetry or not. The difference is very small.

We get,

\[ S[\Psi] - S[\tilde{\Psi}] = \sum_k \varepsilon_k \tilde{\Psi}_k + O(\varepsilon^2), \]

where the \( \tilde{\Psi}_k \) are some functionals of \( \tilde{\Psi} \), and we ignore the \( O(\varepsilon^2) \) terms.

We define this (i.e., the infinitesimal transformation) to be a symmetry if all the \( \tilde{\Psi}_k \) are individually zero.

Example: Again, if we consider,

\[ S \left[ \frac{\partial}{\partial t} \Psi(x, x) \Psi(x, t)^2 \right] + \frac{\partial}{\partial x} \tilde{\Psi}(x, x) \] and the transformation,

\[ \tilde{\Psi}(x, t) = \Psi(x, t) + \varepsilon \Psi(x, t) \] [we take \( \varepsilon \) as \( \varepsilon \) here],

then,

\[ S[\Psi] - S[\tilde{\Psi}] = O(\varepsilon^2). \]
Proof: \[ S[\Psi] = [dt \int d^3x \Psi^*(\x, t) \bar{\Psi} - e - e^* (\x, t) \bar{\Psi} + \frac{\bar{\Psi}^2}{2m} - V] \]
\[ = \int dt \int d^3x \Psi^*(\x, t) \frac{\partial}{\partial t} \Psi + \int d^3x \left( -e \bar{\Psi} + \frac{\bar{\Psi}^2}{2m} - V \right) + \int dt \int d^3x \Psi^*(\x, t) \bar{\Psi} + \int d^3x \left( -e \bar{\Psi} + \frac{\bar{\Psi}^2}{2m} - V \right) \]
\[ + e^2 \int d^3x \int d^3y \Psi^*(\x, t) \bar{\Psi} \Psi^*(\y, t) \bar{\Psi} \]
\[ = S[\Phi] - S[\Psi] = e^2 S[\Psi] = O(e^2). \]

Hence proved.

This means that the corresponding \( \Phi \)'s are zeroes and they form a symmetry.

Often we may not know off-hand what the transformation is like. So, we deal with the infinitesimal transformation and check whether the first order variation is zero or not.

- N.B. If we use the equations of motion (assuming that \( \Phi \) satisfies the equations of motion) and get \( S = 0 \) for all \( \Phi \), then this is an empty statement as we have already assumed that \( \Phi \) satisfies the equations of motion.

- We now take some infinitesimal symmetry transformation, \( S[\Phi] = \mathcal{S}[\Phi] - \mathcal{S}[\Psi] = e H \int d^3x \left( \frac{\Phi^2}{2m} - V \right) \), where these \( H \)'s are functionals.

- Suppose we have checked that the above infinitesimal transformation is a symmetry, \( S[\Phi] = 0 \).
Now, we define a new transformation, \( \hat{\phi}_\varepsilon(y) \) such that,

\[
\hat{\phi}_\varepsilon(y) - \phi_\varepsilon(y) = \varepsilon(y) H\varepsilon(\varepsilon(x,y)).
\]
That is, we have replaced the infinitesimal parameter by an infinitesimal function.

This is not going to be a symmetry in general.

We will calculate this,

\[
\varepsilon \{ \hat{\phi}_\varepsilon(y) \} - \varepsilon \{ \phi_\varepsilon(y) \} = \int_a^{\varepsilon} \varepsilon(x) J^M \{ \varepsilon\varepsilon(y) \}, x
\]
for some functional \( J^M \).

The only reason that it may fail to be zero is because when we calculate the derivative acting on \( \varepsilon(y) \), there do not give zero because \( \varepsilon(y) \varepsilon \) is not constant in the \( \varepsilon \) and we previously had. Then, we know that the variation is zero.

In this case, we will get terms where there will be at least one derivative acting on \( \varepsilon \). If we have more derivatives acting on \( \varepsilon \), we integrate by parts, and we get only one derivative on \( \varepsilon \).

We will assume that the boundary terms, i.e., \( \Phi \to 0 \) and \( \partial u \to 0 \),

If \( \phi_\varepsilon(y) \) satisfies the equations of motion, then,

\[
\int_a^{\varepsilon} \varepsilon(x) J^M = 0 \text{ for any } \varepsilon(x).
\]

Integrating by parts and demanding that \( J^M \) vanishes at the boundary, we will get:
(d^a x \epsilon \times \xi) \text{det} J^a = 0 \quad \text{for any } \xi(x),

which will in turn imply that

$$2n \text{ det} J^a = 0.$$  

**Exercise:** For the Chern-Simons field theory, the action is

$$S = \int d^3 x \left[ \frac{\kappa}{2 \pi} \left( \sqrt{g} - \nu \right) \psi^*(r, t) \right].$$

Previously, we had the following transformation,

$$\psi'(r, t) = \psi(r, t) + \alpha \psi(r, t),$$

which generated a symmetry (see pages 112-113).

Now, we consider the transformation,

$$\psi'(r, t) = \psi(r, t) + \epsilon(r, t) \psi(r, t).$$

We should first check that the terms without the derivative of \( \psi(x) \) must cancel.

Check:

$$S[\psi] = \int dt \left[ \frac{\kappa}{2 \pi} \left( \sqrt{g} - \nu \right) \psi^*(r, t) \right] \left[ \frac{\alpha}{2} \phi + \frac{1}{2m} \nabla^2 - \nu \right].$$

$$= \int dt \left( \sqrt{g} - \nu \right) \psi^*(r, t) \left[ \frac{\alpha}{2} \phi + \frac{1}{2m} \nabla^2 - \nu \right]$$

$$= \int dt \left( \sqrt{g} - \nu \right) \psi^*(r, t) \left[ \phi \left( \frac{\alpha}{2} \right) - \nu \right]$$

$$+ \frac{1}{2} \frac{\nabla^2 (\psi^* \psi + \psi^* \psi) - \psi^* \psi \nabla^2 \psi - \psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi}{2}$$

$$+ \frac{1}{2} \left( \psi^* \psi \nabla^2 \psi^* - \psi \nabla^2 \psi \psi^* + \psi^* \nabla^2 \psi \psi + \psi \nabla^2 \psi^* \psi^* \right)$$

$$+ \kappa \left( \epsilon \right) \text{ (neglect)}.$$
\[ \mathcal{L} - \mathcal{E} = - \left( \frac{1}{2m} \right) \int d^3x \left\{ - \frac{1}{2} \left[ \cdots \right] \psi^* \psi \right\} \\
+ \frac{1}{2m} \int d^3x \left[ \cdots \right] (\nabla \psi) \psi^* \\
+ \left( \frac{1}{2m} \right) \int d^3x \left[ \cdots \right] \psi^* \psi \\
- \left( \frac{1}{2m} \right) \int d^3x \left[ \cdots \right] \nabla \psi \psi^* \\
- \left( \frac{1}{2m} \right) \int d^3x \left[ \cdots \right] \psi^* \nabla \psi \\
\]

The surface integral vanishes and we get:

\[ S(\Phi) - S(\psi) = - \left( \frac{1}{2m} \right) \int d^3x \left\{ \left[ \psi^* \nabla \psi + \psi \nabla \psi^* \right] \right\} \\
- \left( \frac{1}{2m} \right) \int d^3x \left\{ \nabla \psi \psi^* \right\} \\
= \left( \frac{1}{2m} \right) \int d^4x \left\{ \left( - \psi^* + \psi \right) \right\} \\
\]

where \( J^\mu = \left( - \psi^* + \psi \right) \). \\

\[ J^0 = \mathcal{H}(\psi, \bar{\psi}) = -\mathcal{H}(\psi, \bar{\psi}) \\
\]

and \( J^\mu = \frac{1}{2m} \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] \).

Now, \( J_m = 2 \mathcal{J}^\mu = 2 J^0 + 2 J^\mu \\
\]

\[ = - 2 \left\{ \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] \right\} + \frac{1}{2m} \nabla \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] \\
= - \psi \frac{\partial \psi}{\partial x} + ( - \psi^* \frac{\partial \psi^*}{\partial x}) + \frac{1}{2m} \nabla \psi \nabla \psi^* + \frac{1}{2m} \nabla \psi \nabla \psi^* \\
\]

\[ = \left[ - \frac{1}{2m} \nabla \psi \nabla \psi^* + ( - \psi^* \frac{\partial \psi^*}{\partial x}) + \psi \left( - \frac{1}{2m} \nabla \psi \nabla \psi^* \right) \right] \\
\]
Now, the Schrödinger equation is:

\[ \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left[ -\frac{1}{2m} \nabla^2 + V \right] \psi \]

and its conjugate equation is:

\[ -\frac{\partial \psi^*}{\partial t} = -\frac{i}{\hbar} \left[ -\frac{1}{2m} \nabla^2 + V \right] \psi^* \]

Thus, \[ \Im \mathcal{J}_M = -\frac{\partial}{\partial t} \psi^* (\psi V \psi^* + \psi V^* \psi) \]

\[ = -\psi^* \frac{\partial \psi}{\partial t} - \frac{1}{2m} \left( \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} \right) \]

\[ = 0. \]

Thus, the charges are conserved. Thus goes the conserved charge for this theory.

Now, the time derivative \[ \mathcal{J}^0 = -\frac{\partial}{\partial t} \]

and the space derivatives, \[ \mathcal{J}^i = \frac{1}{m} (\frac{\partial}{\partial x_i} \psi + \frac{\partial}{\partial x_i} \psi^*) \]

are conserved.

16.2.2011

We assumed that our theory has certain symmetries.

We assumed that there are many fields \( \phi_1, \phi_2, \ldots \) and the corresponding action is:

\[ S \left[ \phi_1, \phi_2, \ldots \right] \]

We then defined a transformation of the field,

\[ \tilde{\phi}_n (x) = F_n \left[ \phi_1 (x), \phi_2 (x), \ldots \right] \]

The \( x \) in there as we need to define \( \tilde{\phi}_n (x) \) at every location except \( x \).

We declared the above transformation to be a symmetry, if:

\[ S \left[ \tilde{\phi}_n (x) \right] = S \left[ \phi_1 (x), \phi_2 (x), \ldots \right] \]

i.e., the action evaluated in the new field configuration = the action evaluated at the old field configuration.
We then introduced the continuous symmetry, where the transformation was labelled by some continuous parameters \( \alpha, \beta, \ldots \).

Often, it is a useful convention that if we set the \( \alpha_i = 0 \), then \( f_\alpha = f_0 \).

This is known as the origin of the parameter space.

Example: (ii) Translation: Here, the transformation rule was:

\[ f(x) = f(x + a) \] where \( a \) is a constant \( a = (a_1, a_2, a_3) \).

If \( a^2 = 0 \), then \( f(x) = f(x) \).

(ii) In the Schrödinger field theory, we defined the transformation (found to be a symmetry) as:

\[ \varphi(x, t) = e^{i\alpha} \varphi(x, t) \]

Now, setting \( \alpha = 0 \), we get:

\[ \varphi(x, t) = \varphi(x, t) \]

Of course, we could have written:

\[ \varphi(x, t) = e^{i(\alpha + 2\pi)} \varphi(x, t) \] for example, which again would have been a symmetry. In that case, \( \alpha + 2\pi \Rightarrow \varphi(x, t) = \varphi(x, t) \).

Then, we introduced the notion of infinitesimal transformation, where the transformation parameter is not zero, but is infinitesimally small (i.e., small \( \alpha \)), so that:

\[ f_\alpha(x) \approx f_\alpha(x) \] (very close to exact equality).
In this case, we could write the transformation as:

\[ \Phi_n(x) = \Phi_n(x) + \epsilon f_n \left[ \left\{ \Phi_n(y) \right\}, x \right] + \Theta(\epsilon^2) + \cdots ; \]

where \( f_n \) is a functional and \( \epsilon \) is an infinitesimal parameter.

Based on infinitesimal transformations, we derived a relation between symmetries and conservation laws.

Let us imagine that the above transformation is a symmetry, we then make \( \epsilon \) as a function of \( \Phi \) (space-time), \( \epsilon(x) \), \( \Phi(x) \) then is no longer a symmetry in general.

\[ \delta \Phi = \delta \Phi \neq 0 \]

\[ = \int d^4x \, \phi(x) \, J^\mu(x), \]

where \( J^\mu(x) \) is a functional of the field and a function \( \delta \Phi^{-1} \).

The reason for this general form is that, it is linear in \( \epsilon \).

If \( \epsilon \) is a constant, the difference obviously vanishes.

The non-zero result is due to the derivative of \( \epsilon \).

Now, the above relation implies (if throughout we assume that the fields fall off at \( \epsilon \), that,

\[ \int d^4x \, \epsilon(x) \, \partial_\mu J^\mu(x) = 0, \]

If the equations of motion hold, then, thus for any \( \phi \), to first order,

\[ \delta \Phi = \delta \Phi \approx 0, \]

or

\[ \delta \Phi - \delta \Phi = 0 (\epsilon^2) \]

\[ \Rightarrow \partial_\mu J^\mu = 0, \text{ i.e., when the equation of motion holds.} \]
There are 4 components of $\pi^\mu$, we do not assume Lorentz invariance.

Let us now consider the Schrödinger problem:

The action, $S = \int dt d^3x \psi^\dagger(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 - V(x) \right] \psi(x)$.

The symmetry condition is:

$\psi = e^{i\chi} \psi$.

For infinitesimal symmetries $\chi$, $\delta = \frac{\delta}{\delta \chi}$.

Then, $\psi = \psi + \delta \psi$.

Now, we consider the transformation,

$\psi(x) = \psi(x) + \delta \psi(x) \psi(x)$ and calculate $S[\psi]$.

Then, $S[\psi] = \int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \psi(x)$.

$S[\psi] = \int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \frac{\partial}{\partial \chi} \psi(x) + \int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \delta \psi(x)$.

Now, we get:

$S[\psi] - S[\psi^\dagger] = \int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \delta \psi(x)$.

$\int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \delta \psi(x)$.

$+ \int dt d^3x \left[ \psi^\dagger(x) \frac{\partial}{\partial t} + \frac{i}{2\hbar} \nabla^2 - V(x) \right] \psi(x) \delta \psi(x)$.
Thus, we get:

\[ S_{\Psi} - S_{\Phi} = \int dt d^3x \left[ \frac{-\psi^*(\Phi) \partial \psi(\Phi)}{\Phi} \right. \]

\[ + \left. \frac{1}{2m} \psi(\Phi) \partial^2 \psi(\Phi) \right] \]

\[ + \left. \frac{1}{2m} \psi^*(\Phi) \partial \Phi \psi(\Phi) \right] \] .

Now, the second term from \( S_{\Psi} \) is:

\[ \frac{1}{2m} \int dt d^3x \ \psi^* \partial^2 \psi \]

\[ - \frac{1}{2m} \int dt d^3x \ \Phi \partial (\psi^* \partial \psi) \]

Now, the first term can be converted to a surface integral which vanishes at the boundary.

Thus, we get:

\[ S_{\Psi} - S_{\phi} = \int dt d^3x \left[ \psi^*(\Phi) \Phi \partial \psi(\Phi) \right. \]

\[ + \left. \frac{1}{2m} \psi(\Phi) \partial^2 \psi(\Phi) \right] \]

\[ - \frac{1}{2m} \int d^3x \left[ \partial \psi(\Phi) \right. \]

\[ \left. + \partial \psi^* \partial \Phi \right] \]

\[ \left. + \frac{1}{2m} \partial \Phi \partial (\psi^* \partial \psi) \right] \]

\[ = \int dt d^3x \left[ \frac{1}{2m} \partial \psi(\Phi) \partial \Phi \right. \]

\[ \left. + \psi^*(\Phi) \partial \psi(\Phi) \right. \]

\[ \left. + \frac{1}{2m} \partial \Phi \partial (\psi^* \partial \psi) \right] \]

\[ - \left. \int d^3x \left[ \frac{1}{2m} \partial \psi(\Phi) \partial \Phi \right. \]

\[ \left. + \psi^*(\Phi) \partial \psi(\Phi) \right. \]

\[ \left. + \frac{1}{2m} \partial \Phi \partial (\psi^* \partial \psi) \right] \]

\[ = \int dt d^3x \left[ \psi^* \partial \psi(\Phi) \right. \]

\[ \left. + \psi^*(\Phi) \partial \psi(\Phi) \right. \]

\[ \left. + \frac{1}{2m} \partial \Phi \partial (\psi^* \partial \psi) \right] \]

\[ \right] \]

\[ \text{where } J^0(x) = \psi^* \psi(x) \]

\[ \text{and } J^\pm(x) = \frac{1}{2m} \left[ \psi^* \partial \psi(x) \right. \]

\[ \left. - (\partial \psi(x) \psi^*) \right] \] .

This is a conserved current in the Schrödinger picture.

[Subject in Footnote 16-17.]
\[ 2 \mu J^E = 0. \]
\[ \Rightarrow \int d^3x \cdot 2 \mu J^E = 0. \]
\[ \Rightarrow \int d^3x \, \partial \phi = \int d^3x \, \partial \phi = 0. \]

Now, the second term vanishes if the fields \( \rightarrow 0 \) at infinity:
\[ \Rightarrow \phi = \int d^3x \, \phi \text{ is conserved.} \]

Conserved charge calculated at a fixed time slice.

The conserved charge remains the same provided the fields evolve with the equation of motion.

[The conservation laws hold when the equations of motion are satisfied.]

It can be different in different frames though.

\bullet Symmetry of Maxwell's source four equations:
\[ s = \int d^4x \, (-E^\mu - B^\mu). \]

But this action is not symmetric under \( E^\mu \rightarrow -E^\mu \).

\[ \nabla \cdot B = 0 \]
\[ \text{and} \, \nabla \cdot E = 0 \text{ (if we change density).} \]

These equations are symmetric under the rotation of \( E^\mu \) and \( B^\mu \).

There are cases where even the action principle is not known.

If we can find such an action, we can then use Noether's procedure else not.

\bullet For simplicity, we now take the Klein-Gordon field \( \phi^\mu \):
\[ s = \int d^4x \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \phi^2 - \frac{1}{2} m^2 \phi^2 \right]. \]
we now consider a symmetry,

\[
\Phi(x) = \Phi(x+a) \quad (i.e. \text{ translation})
\]

To show that this is a symmetry, we just shift the integration variable from \(x\) to \(y = x+a\),

we now take \(\mu = \mu(y) \quad (\text{infinite-well})\),

then \(\Phi(y) = \Phi(x) + \mu \Delta \Phi(x)\)

they reduce for a four-variant change,

For the Schrödinger equation, we had one parameter \(\alpha\) and

"there are no \(\text{ four-variant change}\),

Now, we assume \(\mu(x)\). \(\mu(x)\)

\[
S[\Phi] = \int d^4x \left[ -\frac{1}{2} \mu^2 \Phi''(x) + \mu \Phi(x) \right]
\]

\[
- \int d^4x \left( \Phi(x) + \mu \Phi(x) \right)
\]

\[
S[\Phi] = \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
- \int d^4x \left( \Phi(x) + \mu \Phi(x) \right)
\]

\[
= \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]

\[
= \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]

\[
= \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]

\[
= \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]

\[
= \int d^4x \left[ -\frac{1}{2} \Phi''(x) + \Phi(x) \right]
\]

\[
+ \mu \int d^4x \left[ \Phi(x) \right]
\]
Now, we can write

\[ q(x) = q(x) \]

\[ = \frac{1}{2} \mu (q(x))^2 \]

and

\[ \eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \]

\[ = \eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \]

Thus, this term can be written as:

\[ \frac{1}{2} \mu (q(x))^2 \]

Here, we integrate by parts the last two terms and let the boundary terms go to zero at the boundary. Then, we get:

\[ S(q) - S(q) = \int d^4 x \left[ -\eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \right] \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

\[ = \int d^4 x \left[ -\eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \right] \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

\[ = \int d^4 x \left[ -\eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \right] \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

\[ = \int d^4 x \left[ -\eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \right] \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

\[ = \int d^4 x \left[ -\eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \right] \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

Thus, the index shows that there are four different conserved currents:

\[ \frac{1}{2} \mu \nabla \cdot (\eta^S \nabla \Phi(x)) \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

\[ \eta^S \nabla \cdot (\eta^S \nabla \Phi(x)) \]

\[ + \frac{1}{2} \eta^S \mu (q(x))^2 \]

For each component there is one conserved current. Each conserved current again has four components.
The continuity equation then is:

\[ \frac{\partial n(x, t)}{\partial t} + \nabla \cdot \{ n(x, t) \mathbf{v}(x) \} = 0. \]

This will be four such equations, one for each \( \mathbf{r} \).

When we write \( \mathbf{v}(x) = 0 \), then \( n(x, t) \) should be zero at the boundary.

\( \mathbf{H} \) depends on \( \mathbf{r} \) and \( t \). Equating we are considering.

\( \mathbf{H}(\mathbf{r}, t) \) = time translation.

\( \mathbf{H}(\mathbf{r}, t) \) = \( x^2 \) translation, and so on.

To know that:

\[ 3 \Gamma \left[ \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \right] = 0. \]

Proof:

\[ \frac{\partial}{\partial x} \left[ \frac{1}{2} \nabla \phi \cdot \nabla \phi \right] = \frac{1}{2} \left[ \nabla \phi \cdot \nabla \phi \right] \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \]

\[ + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right) \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right). \]

\[ = \frac{1}{2} \left[ \nabla \phi \cdot \nabla \phi \right] \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \]

\[ + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \]

\[ = \frac{1}{2} \left[ \nabla \phi \cdot \nabla \phi \right] \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \]

\[ = \left( \nabla \phi \cdot \nabla \phi \right) \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \nabla \phi \frac{\partial \phi}{\partial x}. \]

[ The Klein - Gordon equation is:

\[ \partial^2 \phi(x, t) = \frac{\partial^2 \phi(x, t)}{\partial t^2} + m^2 \phi(x, t). \]

We get:

\[ \partial^2 \phi(x, t) = 1 \left( \partial^2 \phi(x, t) \right) \frac{\partial \phi}{\partial x} \]

\[ + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \]

\[ + m^2 \left[ \partial^2 \phi(x, t) \right] \phi(x, t) \]

\[ - \partial \phi(x, t) \frac{\partial \phi}{\partial x}. \]
Now, the first term is:

\[
\frac{1}{2} \partial^2 q(x) (\partial \partial q(x))
\]

\[
- \frac{1}{2} \partial q(x) (\partial \partial^2 q(x))
\]

Now, the first and the second terms are equal.

We get:

\[
\partial \partial^T J^{\mu} = \partial \partial q(x) (\partial \partial \partial q(x))
\]

Now, the second term is:

\[
\partial \partial q(x) (\partial \partial \partial q(x))
\]

\[
- \partial \partial q(x) (\partial \partial^2 q(x))
\]

\[
\partial \partial J^{\mu} = 0. \text{ [Here proved].}
\]

The energy-momentum tensor is defined as:

\[
T^{\mu \nu} = - \eta^{\mu \nu} J^{\mu} \quad \text{["\,\, \text{convention}".}
\]

Thus, we get from

\[
\partial \partial J^{\mu} = 0
\]

\[
\partial \partial (- \eta^{\mu \nu} J^{\nu}) = 0
\]

\[
\partial \partial T^{\mu \nu} = 0;
\]

and \( T^{\mu \nu} = - \frac{1}{2} \eta^{\mu \nu} \eta^{\rho \sigma} \partial \partial q(x) \partial \partial \phi(x) \)

\[
= - \frac{1}{2} \eta^{\mu \nu} \eta^{\rho \sigma} [(\partial q(x))^2
\]

\[
+ \eta^{\rho \sigma} \eta^{\mu \nu} \partial \partial q(x) \partial \partial \phi(x)].
\]

The source terms are defined as:

\[
p^\nu = \int T^{\nu 0} \, d^3x \quad \text{and} \quad p^0 = \int T^{0 0} \, d^3x, \text{ where } 0 \text{ is the index of } J^{\nu}.
\[ p^2 = \int \eta^{\mu \nu} \eta^{\alpha \beta} \phi_{\mu}(x) \phi_{\beta}(x) \eta^{\kappa \lambda} \phi_{\kappa}(x) \phi_{\lambda}(x) \, d^4x \]

\[ = \int \left[ -\eta^{\mu \nu} \eta^{\alpha \beta} \frac{\partial}{\partial x^\mu} \phi_{\mu}(x) \frac{\partial}{\partial x^\nu} \phi_{\nu}(x) - \frac{1}{2} \eta^{\mu \nu} m^2 \eta^{\alpha \beta} \left( \phi_{\theta}^\dagger(x) \phi_{\phi}(x) \right)^2 \right. \\
\left. + \eta^{\mu \nu} \eta^{\alpha \beta} \frac{\partial}{\partial x^\mu} \phi_{\mu}(x) \frac{\partial}{\partial x^\nu} \phi_{\nu}(x) \right] \, d^4x \]

Now add to the first two terms zero.

In the second term, the non-linear contribution comes

from \( \phi = 0 \) and \( M^2 \). Thus,

\[ p^2 = - \int \partial_{(x)} \phi(x) \partial_{(x)} \phi(x) \, d^4x \]

and \( \phi^2 = \int \phi(x) \, d^4x = \int \left[ -\frac{1}{2} \eta^{\mu \nu} \eta^{\alpha \beta} \frac{\partial}{\partial x^\mu} \phi_{\mu}(x) \frac{\partial}{\partial x^\nu} \phi_{\nu}(x) \right. \\
\left. - \frac{1}{2} \eta^{\mu \nu} m^2 \eta^{\alpha \beta} \left( \phi_{\theta}^\dagger(x) \phi_{\phi}(x) \right)^2 \right. \\
\left. + \eta^{\mu \nu} \eta^{\alpha \beta} \frac{\partial}{\partial x^\mu} \phi_{\mu}(x) \frac{\partial}{\partial x^\nu} \phi_{\nu}(x) \right] \, d^4x \]

Now, the last two terms give:

\[ \frac{1}{2} \eta^{\mu \nu} \frac{\partial}{\partial x^\mu} \phi_{\mu}(x) \frac{\partial}{\partial x^\nu} \phi_{\nu}(x) \]

and the first term is:

\[ -\frac{1}{2} \eta^{\mu \nu} \eta^{\alpha \beta} \frac{\partial}{\partial x^\mu} \phi_{\alpha}(x) \frac{\partial}{\partial x^\nu} \phi_{\beta}(x) \]

Again, we get:

\[ p^2 = \frac{1}{2} \int d^4x \left[ \left( \phi_{\theta}^\dagger(x) \phi_{\phi}(x) \right)^2 + (\vec{\phi} \cdot \vec{\phi})^2 + m^2 \eta^{\mu \nu} \right] \\
\quad + \int \left( \partial_{(x)} \phi(x) \right)^2 \]

\[ = \frac{1}{2} \int d^4x \left[ \left( \phi_{\theta}^\dagger(x) \phi_{\phi}(x) \right)^2 + (\vec{\phi} \cdot \vec{\phi})^2 + m^2 \eta^{\mu \nu} \right] = \mathcal{N} \]

Now, coming back to \( p^2 = - \int \partial_{(x)} \phi(x) \partial_{(x)} \phi(x) \, d^4x \)

\[ = \int \left[ \left( \phi_{\theta}^\dagger(x) \phi_{\phi}(x) \right)^2 + (\vec{\phi} \cdot \vec{\phi})^2 + m^2 \eta^{\mu \nu} \right] \, d^4x \]

\[ = \mathcal{N} \]
\[ p^2 = - \int d^3k \int d^3k' \left( \phi(k) \phi(k') S(k+k') \phi(k',t) \phi(k,t) \right) \]

\[ = - \left( d^3k \phi(k) \phi(k') \phi(-k',t) \right) \]

Now in terms of \( a \) and \( a^\dagger \), we have:

\[ \frac{1}{\sqrt{2}} \phi(k',t) + \frac{1}{\sqrt{2}} \phi(-k',t) = a^\dagger(k,t) \]

and

\[ \frac{1}{\sqrt{2}} \phi(k',t) - \frac{1}{\sqrt{2}} \phi(-k',t) = a(k,t) \]

\[ \Rightarrow \frac{1}{\sqrt{2}} \phi(k',t) = a(k,t) + a^\dagger(-k,t) \]

\[ \Rightarrow \phi(k',t) = \frac{1}{\sqrt{2}} \left[ a(k,t) + a^\dagger(-k,t) \right] \]

\[ \Rightarrow \frac{1}{\sqrt{2}} \phi(-k',t) = \frac{1}{\sqrt{2}} \left[ a(-k,t) + a^\dagger(k,t) \right] \]

Thus, putting these in the expression of \( p^2 \), we get:

\[ p^2 = -\frac{1}{2} \int d^3k \int d^3k' \left[ a(k,t) + a^\dagger(-k,t) \right] \left[ a(-k,t) + a^\dagger(k,t) \right] \]

\[ -\frac{1}{2} \int d^3k \int d^3k' \left[ a(k,t) a(-k,t) - a(k,t) a^\dagger(k,t) \right. \]

\[ \left. - a^\dagger(-k,t) a^\dagger(k,t) \right] \]

We may apply these on the single particle states, which will produce an eigenvalue, which is \( \Delta^2 \).

\[ \Delta^2 = \left( \frac{1}{\sqrt{2}} \int \phi^* \phi \right)^2 \]

We square the term:

\[ \Delta = -\frac{1}{2} \int d^3k \int d^3k' \left[ a(k,t) a(-k,t) \right] \]

\[ \Rightarrow \text{We now transform } \phi \rightarrow \bar{\phi}, \text{ then, we get:} \]

\[ \cdots \]
\[ r = -\frac{1}{2} \int_{-\infty}^{\infty} [a(-\vec{r},t)a(\vec{r},t)] \] 

\[ = \frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot [a(\vec{r},t)a(-\vec{r},t)] \] 

\[ = -\mathbf{I} \]

\[ = 0. \]

Thus, the term is anti-symmetric under \( \vec{r} \rightarrow -\vec{r} \) and hence

Thus, integral is zero.

Similarly

\[ \int_{-\infty}^{\infty} d^3 \mathbf{K} \cdot a^+(\vec{r},t)a^+(\vec{r},t) \]

\[ \rightarrow \int_{-\infty}^{\infty} \frac{1}{2} \mathbf{K} \cdot a^+(\vec{r},t)a^+(\vec{r},t) \]

\[ = -\frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot a^+(\vec{r},t)a(\vec{r},t) \] 

\[ = 0. \]

Hence this integral also vanishes.

\[ \cdot \quad p^z = -\frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot [-a(\vec{r},t)a^+(\vec{r},t) + a^+(\vec{r},t)a(-\vec{r},t)] \]

Again, in the last term, we change \( \vec{r} \rightarrow -\vec{r} \).

\[ \cdot \quad p^z = \frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot [a(\vec{r},t)a^+(\vec{r},t) + a^+(\vec{r},t)a(\vec{r},t)] \]

Now,

\[ [a(\vec{r},t),a^+(\vec{r}',t)] = \delta^{(3)}(\vec{r} - \vec{r}') \]

\[ a(\vec{r},t)a^+(\vec{r},t) = a^+(\vec{r},t)a(\vec{r},t) + \delta^{(3)}(0) \]

\[ \cdot \quad p^z = \frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot a^+(\vec{r},t)a(\vec{r},t) + \frac{1}{2} \int_{-\infty}^{\infty} d^3 \mathbf{K} \mathbf{K} \cdot \delta^{(3)}(0) \]
Now, \( \mathfrak{g}^{(3)}(0) = \frac{1}{(2\pi)^3} \int d^3x \, e^{-i \cdot (x-x')} \)

If we now set \( x' \to 0 \), we get:

\[ \mathfrak{g}^{(3)}(0) = \frac{1}{(2\pi)^3} \int d^3x = \frac{1}{V} \]

where \( V \) is the volume of space.

The constant infinity \( \mathfrak{g} \) due to the infinite volume.

We can regularize this by considering the field theory to be in a finite box \( V \), momentum will then be a discrete sum.

In this integral,

\[ \int d^3k \, k_i \to \int \frac{d^3k}{(2\pi)^3} \]

Now, instead of \( \to +\infty \), we let \( dk_x, dk_y, dk_z \) run from \(-V\) to \( V\) and we get:

\[ \int_{-V}^{V} dk_x \int_{-V}^{V} dk_y \int_{-V}^{V} dk_z k_i = 0 \quad (i = x, y, z) \]

The vacuum itself carries some momentum which we cannot measure.

The vacuum can also carry some energy (the ground state energy), but we can only measure the difference in the two energy levels.

We can always measure some quantity with respect to that carried by the vacuum.

We cannot fix any actual origin of momentum or energy.
When we consider gravity, it matters whether the vacuum carries momentum or energy.

\[ F = \frac{1}{2} \mathbf{v}^2 + m^2 \] is actually a difference of the single particle energy state to that of the vacuum.

So, we eventually drop the \( \frac{1}{2} \int d^3k \delta^{(3)}(0) \) integral which gives us an infinite constant.

- **N.B. 1:** We take the Quantum Field Theory in a finite box and put some periodic boundary conditions on it. Then, we will get a Kronecker delta with a factor of \( (L/2\pi)^3 \) for every direction, which will give us \( V/(2\pi)^3 \) with the delta function.

- **N.B. 2:** In the presence of gravity, log somehow regularizing for momentum, we can achieve regularity for energy because of the \( \mathbf{k}^2 \) term. We can not regularize that.

Thus, we end up at:

\[ \hat{P}_c = \int d^3k \mathbf{k} \left[ a^\dagger(\mathbf{k},t) a(\mathbf{k},t) \right] \]

Thus, by design, we get:

\[ \hat{P}_c |0\rangle = 0 \quad (\because a(\mathbf{k}, t) |0\rangle = 0 \text{ for all } \mathbf{k}) \]

Now, we apply it on a single particle state, to get:

\[ \hat{P}_c a^\dagger(\mathbf{r}_1, t) |0\rangle = \int d^3k' \mathbf{k}' \left[ a^\dagger(\mathbf{k}_2', t) a(\mathbf{k}_2', t) a^\dagger(\mathbf{r}_1, t) a(\mathbf{r}_1, t) \right] |0\rangle = \int d^3k' \mathbf{k}' \left[ a^\dagger(\mathbf{r}_1', t) a^\dagger(\mathbf{r}_2, t) a(\mathbf{r}_2, t) + 5^{(3)}(\mathbf{r} - \mathbf{r'}) |10\rangle \right] \]
The first term vanishes \( \delta R(t, t + 10) = 0 \), we thus get:

\[
\hat{P}_p a^+(\mathbf{R}, t) \langle 10 \rangle = \int d^3 \mathbf{k} \hat{\mathbf{K}} a^+(\mathbf{R}', t) / \sqrt{3} (\mathbf{R} - \mathbf{R}') \langle 10 \rangle \\
= \hat{P}_p a^+(\mathbf{R}, t) \langle 10 \rangle
\]

Thus, \( \hat{\mathbf{K}}_i \) is the momentum carried by the state \( a^+(\mathbf{R}, t) \langle 10 \rangle \) in the \( i \)th direction.

Previously, we had seen that the single particle state carries an energy, \( E_{1/2} = \sqrt{R^2 + m^2} \) (see page 76).

For any other relation, we will not have any particle interpretation.

The particles emerge once we quantify the state obey the relations between the energy and the momentum as that by a single particle.

So far we have seen that in the Noether's theorem, a continuous symmetry \( \Rightarrow \) a conservation law. This tells us precisely how to find the conserved current \( J^\mu \).

We would now like to prove the converse, i.e., given a conservation law, there is a symmetry.

We will prove it in a somewhat limited context: we will assume the Lagrangian to be a function of the fields and their first order space derivatives. (Just like the Hamiltonian depends on \( R, \dot{R} \)).

We assume that \( L \) depends on \( \phi, \dot{\phi} \) and \( \phi', \dot{\phi}' \).

For example, in proving the original statement, the action was not assumed to be limited.

Conversely, we will not involve any first order space derivatives,
we can always discretize them.

For numerical notation, it is best to have both.

Let us consider a finite number of coordinate variables and their time derivatives.

To simplify notation, we consider a classical mechanics with coordinates $x_1, x_2, \ldots, x_n$ and velocities $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$.

The Lagrangian then is:

$$L = \frac{1}{2} \sum_{i=1}^{n} m i \dot{x}_i^2 - V(x_1, x_2, \ldots, x_n).$$

We think this index "$i$" as a continuous variable so we can take their Fourier momentum mode.

Let us assume that $\hbar = 0$.

[Note: We can obviously include it and then our calculation will change a bit.]

In the Lagrangian formulation, we take these $\{q_i, p_i\}$ as independent variables.

To go to the Hamiltonian formulation, we define the conjugate momentum as:

$$\dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} = -\sum_{n} \frac{\partial V(q, p_n)}{\partial q_i}.$$

The total number of variables is $2n$.

We can also define the action. The action is a functional of the trajectory ($L$ and $H$ are functions of the $2n$ coordinates).

The action $S$ provides a number. We have to specify the trajectory and hence all the following.
\[ \mathbf{v}(t) = \frac{d}{dt} \mathbf{r}(t), \quad \frac{d\mathbf{v}}{dt}(t) \text{ is the degree of the trajectory at every point.} \]

Consider a trajectory, the velocity at each point is finite.
But, when the Lagrangian is concerned, we think \( \{ q, \dot{q} \} \) and \( \{ q, \ddot{q} \} \) as independent variables.

Let us suppose that \( \mathcal{L} (\{ q, \dot{q} ; s, \dot{s}, t \}) \) be a conserved charge. This means that \( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial q} = 0 \) when the equations of motion hold.

\[ \frac{d}{dt} q \text{ might not be zero.} \]

Writing explicitly, we get:

\[ \frac{d}{dt} q = \sum \frac{\partial}{\partial \dot{q}_n} \frac{\partial \mathcal{L}}{\partial q_n} \frac{d\dot{q}_n}{dt} + \frac{\partial \mathcal{L}}{\partial q} = 0. \]

The Hamilton's equations tell us that:

\[ \frac{d\dot{q}_n}{dt} = -\frac{\partial \mathcal{H}}{\partial q_n}; \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial \dot{q}_n} \frac{d\dot{q}_n}{dt}. \]

\[ \frac{d}{dt} q = \sum \left[ \frac{\partial}{\partial \dot{q}_n} \frac{\partial \mathcal{H}}{\partial q_n} \frac{d\dot{q}_n}{dt} + \frac{\partial \mathcal{H}}{\partial q} \right] = \sum \left[ \frac{\partial}{\partial \dot{q}_n} \mathcal{H} \right] \frac{d\dot{q}_n}{dt} + \frac{\partial \mathcal{H}}{\partial q}. \]

The equations of motion take as \( \frac{d\dot{q}_n}{dt} \) and \( \frac{d\mathcal{H}}{dt} \) in terms of function of \( (q, \dot{q}) \).

\[ \therefore \frac{d\mathcal{H}}{dt} = f(q, \dot{q}, t) = 0; \quad (\because \mathcal{H} = \mathcal{H}(q, \dot{q}) ). \]
Thus, \( f(x,y,t) \) has to vanish identically.

It does not depend on the equation of motion.

This must be zero, without the equation of motion.

Thus, if \( d\alpha \neq 0 \), then, we can not make it zero by using the equation of motion.

A non-trivial function \( \alpha \) may not make this vanish.

In that case \( \alpha \) is not continuous at that point.

\( \text{N.B.} \): We may complete \( \alpha \) by taking in dependence of \( V_n \), \( \alpha_n \) etc.

\( \text{We will now show that } V_n \to V_n + E \{ \beta \} \) \( \rho \)

\( \text{c.e. } \alpha_n = \alpha_n + E \left[ \frac{300 \sqrt{n} - 200 \sqrt{n}}{\sqrt{n}} \right] \)

\( = \sqrt{n} - E \left[ \frac{300 \sqrt{n} - 200 \sqrt{n}}{\sqrt{n}} \right] \)

\( \text{is a symmetry of } \)

\( \text{c.e. } \) \( \) does not change to

the first order in \( \alpha \). Here, we can not use the equation of motion.
\[
S = \int \frac{1}{2} \sum_{n} m_n \dot{v}_n^2(t) dt
\]

\[
L = \frac{1}{2} \sum_{n} m_n \dot{v}_n^2(t) - H.
\]

\[
\int \left( \left( \sum_{n} m_n \dot{v}_n^2(t) + \frac{1}{2} \sum_{n} \phi_n \phi_n \right) - \sum_{n} \left( \frac{1}{2} m_n \dot{v}_n^2(t) + \frac{1}{2} \phi_n \phi_n \right) \right) dt
\]

\[
\sum_{n} \left( \frac{1}{2} m_n \dot{v}_n^2(t) - \frac{1}{2} \phi_n \phi_n \right)
\]

Now, \( \dot{v}_n \) is not taken as \( \frac{d v_n}{dt} \).

We are not using the equations of motion here, but just the Legendre transformation between \( L \) and \( H \).

[Note: We want the \( H \) because we write \( L \) in terms of \( \dot{v}_n \).]

\[
\int \left( \sum_{n} \left( \frac{1}{2} m_n \dot{v}_n^2(t) + \frac{1}{2} \phi_n \phi_n \right) - \sum_{n} \left( \frac{1}{2} m_n \dot{v}_n^2(t) + \frac{1}{2} \phi_n \phi_n \right) \right) dt
\]

The action is to be evaluated at \( v_n(t) \) and \( \dot{v}_n(t) \).

On the trajectory, we get:

\[
v_n = \frac{d}{dt} (v_n), \text{ because on the trajectory,}
\]

\[
\dot{v}_n = \frac{dv_n}{dt} \text{ (by definition)}
\]

Hence,

\[
\int \left( \frac{1}{2} \sum_{n} m_n \dot{v}_n^2(t) - \frac{1}{2} \sum_{n} \phi_n \phi_n \right) dt
\]
\[ L = \frac{1}{2} \sum_{n} k_{n} \dot{q}_{n}^{2} \] is an abstract function. It is useful not to think of it as the Lagrangian as of now.

\[ H = \sum_{n} k_{n} q_{n} \] with \( k_{n} = \frac{2}{\omega_{n}} \) - definition of Legendre transformation.

Given \( L \), we can calculate \( H \) and vice versa.

Given \( H \), we can calculate \( k_{n} \). We can invert this relation to obtain \( \dot{q}_{n} = \pm H \). Here, remember that we used the equations of motion, another form we said

\[ \dot{q}_{n} = \frac{\partial H}{\partial \dot{q}_{n}} \cdot \]

\[ [ \text{We can not use the Euler-Lagrange equations unless we put } \dot{q} \text{, we can not define the action}. \]

At every given point on the trajectory, we have to calculate each of these quantities \( k_{n}, \dot{q}_{n}, H \) etc.

\[ \therefore \left[ \sum_{n} k_{n} \dot{q}_{n}(t) \right] - \left[ \sum_{n} k_{n} \dot{q}_{n}(t) \right] \]

\[ = \int dt \left\{ - \frac{dk_{n}}{dt} \dot{q}_{n} - 2H \dot{q}_{n} \right\} \]

\[ + \int dt \sum_{n} \frac{1}{2} \left( \frac{d\dot{q}_{n}}{dt} k_{n} \right) \]

We drop the second term at the boundary.

\[ \therefore \text{we drop} \sum_{n} \frac{1}{2} \delta \left( \dot{q}_{n} k_{n} \right) \text{at the boundary} \]

Now, we use \( \sum_{n} k_{n} = - \frac{dE}{dt} \).

\[ \therefore \text{we get:} \]

\[ \left[ \sum_{n} k_{n} \dot{q}_{n}(t) \right] - \left[ \sum_{n} k_{n} \dot{q}_{n}(t) \right] \]

\[ = \int dt \sum_{n} \left\{ \frac{dE}{dt} \dot{q}_{n} \frac{d}{dt} + \frac{dE}{dt} \frac{dH}{dt} \right\} \]

Thus, we have not used anywhere that \( \omega_{n} \) is constant. Now, we use the fact that \( \omega_{n} \) is constant.
we then have,

\[ \sum_{i=1}^{n} \left( \frac{\partial \gamma_i}{\partial \theta} \cdot \frac{d\gamma_i}{dt} + \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} \right) + \Delta \theta = 0 \]

\[ \frac{d\gamma_i}{dt} = 0 \]

\[ \Delta \theta = \frac{d\gamma_i}{dt} \]

We have to evaluate the term inside the double along the trajectory \( \gamma \), on the trajectory,

\[ \Delta \theta = \frac{d\gamma_i}{dt} \]

\[ \int \left( \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} + \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} \right) dt \]

\[ \int \left( \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} + \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} \right) dt \]

Here also we have used the Euler-Lagrange equation.

\[ \int \left( \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} + \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} \right) dt \]

\[ \int \left( \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} + \frac{\partial \gamma_i}{\partial \gamma} \cdot \frac{d\gamma_i}{dt} \right) dt \]

Thus, \( \frac{d\gamma_i}{dt} = \Delta \theta \) is a symmetry of \( \gamma \).

**Note:** \( \frac{d\gamma_i}{dt} \) is not zero along the trajectory.

\[ \frac{1}{2} m v^2 \]

\[ \frac{1}{2} m v^2 \] is different at different points.
$Q = mV^2$ is the canonical charge.

\[ \frac{dx}{dt} = m \frac{d}{dt} = m \frac{d^2x}{dt^2}. \]

For an arbitrary trajectory, the trajectory vanishes only for fixed points at $t_1$, $t_2$.

The boundary terms are zero.

For an arbitrary trajectory, we can choose it to spell off at two points.

By invariance of the action, we may it to be invariant.

In the boundary terms, it will then not affect the equations of motion.

Actually, we do not care about the boundary terms.

For derivation of the field equations, the boundary terms are irrelevant.

We might as well take trajectories where the boundary terms are given.

If there are not give, then also our equations of motion will hold.

$S = 0$ between these two trajectories.