Any function = Even function + Odd function

Examples :-

1) θ_1 is an odd function (one non-zero coeff. & is odd)

2) $\theta_1\theta_2$ is an even function (one non-zero coeff. & is even)

2) $\sin\theta_1 = \theta_1 - \theta_1^3/3! + \theta_1^5/5! - - -$

$= \theta_1 \longrightarrow$ odd fn. $\because \theta_1^2 = 0$

first pretend it is Bosonic & then consider the fact that θ_1 is a grassmann variable

3) $e^{i\theta_1} = 1 + i\theta_1 - \underbrace{\tfrac{1}{2}\theta_1^2 + \cdots}_{0}$

$= 1 + i\theta_1$

first expand in usual Taylor series expn.

Differentiation :- (It is just a formal operation)

① $\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}$

② $\frac{\partial}{\partial\theta_i}(FG) = \frac{\partial F}{\partial\theta_i}G + (-1)^F F\frac{\partial G}{\partial\theta_i}$

$(-1)^F = \begin{cases} 1 & \text{if } F \text{ is even} \\ -1 & \text{if } F \text{ is odd} \end{cases}$

this is not really (-1) raised to the power F

Example :-

① $\frac{\partial}{\partial\theta_1}(\theta_1\theta_2) = \theta_2 - (-1)\theta_1(0)$

$= \theta_2$

② $\frac{\partial}{\partial\theta_1}(\theta_2\theta_1) = (0)\theta_1 + (-1)\theta_2(1)$

$= -\theta_2$

2 ways

$\partial/\partial\theta_1(\theta_2\theta_1) = \frac{\partial\theta_2}{\partial\theta_1}\theta_1 - \theta_2\frac{\partial\theta_1}{\partial\theta_1} = -\theta_2$

$\partial/\partial\theta_1(-\theta_1\theta_2) = -\theta_2$

Results

① $$\frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} F = -\frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} F \quad \text{for any } F.$$

Proof :-

Define $\vec{\theta}' = (\theta_1, \ldots, \cancel{\theta_i}, \ldots, \cancel{\theta_j}, \ldots, \theta_n)$

$$F = A(\vec{\theta}') + \theta_i B(\vec{\theta}') + \theta_j C(\vec{\theta}') + \theta_i \theta_j D(\vec{\theta}')$$

Doing a Taylor series expn. only in θ_i & θ_j's, but of course the rest of the dependence will be there

$$\therefore \frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} F = -D(\vec{\theta}')$$

$$\& \quad \frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} F = D(\vec{\theta}')$$

This shows that

$$\frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} F = -\frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} F$$

$$\Rightarrow \frac{\partial^2}{\partial \theta_i^2} F = 0$$

Simply bcos Taylor series expn. can't contain more than one power of θ

② Consider two functions F & G.

then $FG = GF$ if either F or G is even.

(If either of them is even, you can move through the even no. of θ_i's without picking up a '$-$' sign) ~~F through~~ every term of F through " " " G or vice versa

$FG = -GF$ if both F & G are odd

③ Chain rule of differentiation :-

Suppose we have a function

$$F(x) = \sum f_n x^n$$

$\hookrightarrow$ ordinary numbers

Suppose $G(\vec{\theta})$ is a function of the grassman variables $\theta_1, \theta_2, \ldots, \theta_n$.

$$F(G(\vec{\theta})) = \sum_n f_n (G(\vec{\theta}))^n$$

$$\frac{\partial}{\partial\theta_i} F(G(\vec{\theta}))$$

$$= \sum_n f_n \left[\frac{\partial G(\vec{\theta})}{\partial\theta_i} G(\vec{\theta})^{n-1} + (-1)^G G(\vec{\theta}) \frac{\partial G(\vec{\theta})}{\partial\theta_i} G(\vec{\theta})^{n-2} + \cdots + \{(-1)^G\}^{n-1} G(\vec{\theta})^{n-1} \frac{\partial G(\vec{\theta})}{\partial\theta_i} \right]$$

Suppose G is even.

(As long as G is even, the fact that $\partial G/\partial\theta_i$ is odd doesn't matter bcos you can pull $\partial G/\partial\theta_i$ through to the left without picking up a '$-$' sign)

$\hookrightarrow$ Then, $$\frac{\partial}{\partial\theta_i} F(G(\vec{\theta})) = \frac{\partial G(\vec{\theta})}{\partial\theta_i} \sum_n n f_n (G(\vec{\theta}))^{n-1}$$

$$F'(x) = \sum_n n f_n x^{n-1} \longrightarrow \text{defn of } F'(x)$$

$$\frac{\partial}{\partial \theta_i} F(G(\vec{\theta})) = \frac{\partial G(\vec{\theta})}{\partial \theta_i} F'(G(\vec{\theta}))$$

$\longrightarrow$ Chain rule of differentiation

---

e.g. $\rightarrow$ (1) $$\frac{\partial}{\partial \theta_1}\left(\cos(\underbrace{\theta_1 \theta_2}_{G(\vec{\theta})})\right) = \theta_2 \sin(\theta_1 \theta_2)$$

LHS

$$\frac{\partial}{\partial \theta_1}\{1\} = 0$$

RHS

$$\theta_2 (\theta_1 \theta_2) = 0$$

---

(2) $$\frac{\partial}{\partial \theta_1} e^{i\theta_1 \theta_2} = i\theta_2 e^{i\theta_1 \theta_2}$$

LHS

$$\frac{\partial}{\partial \theta_1}(1 + i\theta_1 \theta_2) = i\theta_2$$

RHS

$$i\theta_2 (1 + i\theta_1 \theta_2) = i\theta_2$$

---

(3) $$\frac{\partial}{\partial \theta_1}(\cos\theta_1) = \sin\theta_1 \Rightarrow \text{LHS} \neq \text{RHS}$$

$$\overset{\shortparallel}{\frac{\partial}{\partial \theta_1}(1) = 0} \qquad \overset{\shortparallel}{\theta_1}$$

This gives wrong answer bcos θ_1 is an odd fn.

④ $\frac{\partial}{\partial\theta_1} e^{iA_{ij}\theta_i\theta_j}$

$$= \left(iA_{ij}\,\theta_j\,\delta_{i1} - iA_{ij}\,\theta_{\cancel{x}}\,\delta_{j1}\right)e^{iA_{ij}\theta_i\theta_j}$$

$$= \left(iA_{1j}\theta_j \cancel{} - i\overset{=(-A_{1i})}{A_{i1}}\theta_i\right)e^{iA_{ij}\theta_i\theta_j}$$

$$= 2iA_{1j}\theta_j\, e^{iA_{ij}\theta_i\theta_j}$$

$$= 2iA_{1i}\theta^i\, e^{iA_{ij}\theta_{ij}}$$

Another way was is to expand →

$$\sum_k \frac{1}{k!}\left(iA_{ij}\theta_i\theta_j\right)^k$$

Integration over grassman variables

(We won't try to define definite int. or indefinite int. where the limit of int. is the grassman variable itself — we will consider the case where the limit is taken at $\pm\infty$)

→ $d\theta$ should dim. opp. to that of θ

$$\int d\theta_i\, F(\vec{\theta}) = \frac{\partial}{\partial\theta_i}F(\vec{\theta}) \longrightarrow \text{definition}$$

$$\int d\theta_i\,\frac{\partial}{\partial\theta_i}\left(F(\vec{\theta})\right) = \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_i}F(\vec{\theta}) = 0$$

$$\int d\theta_i\,\frac{\partial}{\partial\theta_i}(FG) = 0$$

$$\frac{\partial F}{\partial\theta_i}G + (-1)^F F\frac{\partial G}{\partial\theta_i}$$

Int. of Total deriv. gives zero as we reqd. in our Bosonic th. for int. by parts for the surface int. to vanish

This is the only kind of integral we need for path int. — Here we don't have any notion of fields vanishing at ∞ — just we want to reproduce the ans. we got for the int. for Bosonic theory

$$\Rightarrow \int d\theta_i\,\frac{\partial F}{\partial\theta_i}G = -(-1)^F\int d\theta_i\, F\frac{\partial G}{\partial\theta_i} \longrightarrow \text{Integration by parts}$$

Shifting integration variables

For a real variable x,

$$\int_{-\infty}^{\infty} F(x)\,dx = \int_{-\infty}^{\infty} F(x+c)\,dx$$

c → independent of x

Is it true for grassman variables?

OR → in other words

~~Define~~ :- $\vec{\theta}' = (\theta_2, \theta_3, \ldots, \theta_n)$

$$\int d\theta_1\, F(\theta_1, \theta_2, \ldots, \theta_n) \overset{?}{=} \int d\theta_1\, F(\theta_1 + f(\vec{\theta}'), \theta_2, \ldots, \theta_n)$$

$$F(\theta_1, \ldots, \theta_n) = A(\vec{\theta}') + \theta_1 B(\vec{\theta}')$$

$$\cancel{F(\theta_1)}\; F(\theta_1 + f(\vec{\theta}'), \theta_2, \ldots, \theta_n) = A(\vec{\theta}') + (\theta_1 + f(\vec{\theta}')) B(\vec{\theta}')$$

Now, $\int d\theta_1 F(\theta_1, \theta_2, \ldots, \theta_n)$

$$= \frac{\partial}{\partial\theta_1} F(\theta_1, \theta_2, \ldots, \theta_n)$$

$$= B(\vec{\theta}')$$

Again, $\int d\theta_1\, F(\theta_1 + f(\vec{\theta}'), \theta_2, \ldots, \theta_n)$

$$= \frac{\partial}{\partial\theta_1} F(\theta_1 + f(\vec{\theta}'), \theta_2, \ldots, \theta_n)$$

$$= B(\vec{\theta}')$$

(∴ under a shift, grassman int. behaves ~~as~~ like ordinary integration)

1/9/06 [By defining the int. for gras. variables in this way, we wanted make sure that those prop. are satisfied

this ~~corresponds~~ applies to the ~~r~~edefining of field bcos by changing the defn. of fields you don't expect to get diff. types of ans. e.g. $\to$ $\phi = x + x^2$]

$$\boxed{\int d\theta\, F(\theta) = \int d\theta\, F(\theta + c)}$$

c $\to$ indep. of θ but dep. on other gras. variables

$$\int d\theta\, F(\lambda\theta) = ?$$

$x \to$ ordinary variable

$$\int_{-\infty}^{\infty} dx\, F(\lambda x) \underset{y=\lambda x}{=} \int_{-\infty}^{\infty} \frac{dy}{y} F(y)$$

$$\underset{y \to x}{=} \frac{1}{\lambda} \int_{-\infty}^{\infty} dx\, F(x)$$

$$F(\theta) = A + \theta B$$

A, B $\to$ θ independent but could depend on other grassman variables

$$\therefore \int d\theta\, F(\theta) = B$$

$$\int d\theta\, F(\lambda\theta) = \int d\theta\, (A + \lambda\theta B) = \lambda B = \lambda \int d\theta\, F(\theta)$$

λ $\to$ ordinary no.

$$\int d\theta\, F(\theta\theta_1) = B\theta_1$$

θ_1 $\to$ grassman var.

$$\boxed{\int d\theta\, F(\lambda\theta) = \lambda \int d\theta\, F(\theta)}$$

→ exactly opp. to what we get for an ordinary variable

Formally we write

$$d(\lambda\theta) = \frac{1}{\lambda}\, d\theta$$

$$\frac{1}{A^{(0)} + A_i^{(1)}\theta_i + \cdots} = \frac{1}{A^{(0)}}\left(1 + \frac{1}{A^{(0)}} A_i^{(1)} \theta_i + \cdots\right)^{-1}$$

this makes sense only if $A^{(0)} \neq 0$

Now Taylor series expand this

→ Using this <u>formal</u> relationship, we can write

$$\int d\theta\, F(\lambda\theta) = \int \lambda\, d(\underbrace{\lambda\theta}_{\theta'})\, F(\underbrace{\lambda\theta}_{\theta'})$$

$$= \lambda \int d\theta'\, F(\theta') = \lambda \int d\theta\, F(\theta)$$

Suppose we consider a fn. of $\theta_1, \ldots, \theta_m, \vec{\theta}'$

Total set of variables $\theta_1, \ldots, \theta_n$

where $\vec{\theta}' = (\theta_{m+1}, \ldots, \theta_n)$ & $m < n$

We have a fn. $F(\theta_1, \ldots, \theta_m, \vec{\theta}')$

$$\int d\theta_m d\theta_{m-1} \cdots d\theta_2 d\theta_1 F(\theta_1, \ldots, \theta_m, \vec{\theta}')$$

Order of int. is $\theta_1, \ldots \theta_n$.

$$\Updownarrow ?$$

$$\int d\theta_m \, d\theta_{m-1} \ldots d\theta_1 \, F\left(\sum_{j=1}^{m} A_{1j}\theta^j, \sum_{j=1}^{m} A_{2j}\theta^j, \ldots, \sum_{j=1}^{m} A_{mj}\theta^j, \vec{\theta}'\right)$$

Int. is over only a part of all the Grass. variables

We have instead taken a linear combn. of the θ_i's — replacing the argument by a linear combn.

Consider

$$\int dx_1 \ldots dx_m \, F(x_1, \ldots, x_m, \vec{y})$$

where $\vec{y} = (x_{m+1}, \ldots, x_n)$

$$\Updownarrow ? \quad \hookrightarrow \text{BOSONIC CASE}$$

$$\int dx_1 \ldots dx_m F\Big(\underbrace{\sum_{i=1}^{n} A_{1i} x_i}_{= z_1}, \underbrace{\sum_{i=1}^{m} A_{2i} x_i}_{= z_2}, \ldots, \underbrace{\sum_{i=1}^{m} A_{mi} x_i}_{= z_m}, \vec{y}\Big)$$

$$\therefore \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$dz_1 \, dz_2 \ldots dz_m = \pm(\det A) \, dx_1 \, dx_2 \ldots dx_m$$

$$\pm(\det A)^{-1} \int dz_1 \ldots dz_m \, F(z_1, \ldots, z_m, \vec{y}) = \pm(\det A)^{-1} \int dx_1 \ldots dx_m \, F(x_1, \ldots, x_m, \vec{y})$$

Each x_i runs from $-\infty$ to ∞

(We $\overset{\text{might}}{\wedge}$ expect that ~~perhaps~~ it is) $\to$

$$\pm (\det A) \int d\theta_m \cdots d\theta_1 \, F(\theta_1, \ldots, \theta_m, \vec{\theta}')$$

for the Grassman variables

Proof :-

$$F(\theta_1, \theta_2, \ldots, \theta_m, \vec{\theta}')$$

$$= B^{(0)}(\vec{\theta}') + \sum_{i=1}^{m} \theta_i B_i^{(1)}(\vec{\theta}') + \sum_{i=1}^{m} \sum_{j=1}^{m} \theta_i \theta_j B^{(2)}_{ij}(\vec{\theta}')$$

$$+ \cdots + \underbrace{\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_m=1}^{m} \theta_{i_1}, \ldots, \theta_{i_m} B^{(m)}_{i_1 \ldots i_m}(\vec{\theta}')}_{m! \, \theta_1 \ldots \theta_m B^{(m)}_{1 \ldots m}(\vec{\theta}')}$$

$\hookrightarrow$ $m!$ non-zero terms are there in this summation

$$\boxed{\text{B's are totally antisym. in the corr. indices}} \quad B^{(m)}_{i_1 \ldots i_m} = B^{(m)}_{1 \ldots m} \underbrace{\epsilon_{i_1 \ldots i_m}}$$

total antisymmetric tensor with $\epsilon_{1 \ldots m} = 1$

[only the last term of the expn. will contribute to the int. bcz $\int d\theta \, F(\theta) = \frac{\partial}{\partial \theta} F(\theta)$]

[One of the terms is $B^{(m)}_{1 \ldots m} \theta_1 \ldots \theta_m$ & another is $B^{(m)}_{2134 \ldots m} \theta_2 \theta_1 \theta_3 \ldots \theta_m = B^{(m)}_{1234 \ldots m} \theta_1 \theta_2 \ldots \theta_m$]

RHS :- $\pm (\det A) \left[B^{(m)}_{1 \ldots m} \times m! \right]$

i.e., $\pm (\det A) \int d\theta_m \ldots d\theta_1 \, F(\theta_1, \ldots, \theta_m, \vec{\theta}')$

$$= \pm (\det A) \left[B^{(m)}_{1 \ldots m} \times m! \right]$$

Note :- $\int d\theta_m \, d\theta_{m-1} \cdots d\theta_1 \, \theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}$

$$= \epsilon_{i_1 \cdots i_m}$$

$$\therefore \int d\theta_m \cdots d\theta_1 \, F = \epsilon_{i_1 \cdots i_m} B^{(m)}_{i_1 \cdots i_m}(\vec{\theta}')$$

$$= \epsilon_{i_1 \cdots i_m} B^{(m)}_{1 \cdots m} \epsilon_{i_1 \cdots i_m}$$

$$= m! \, B^{(m)}_{1 \cdots m}$$

Bcos there are 2 ϵ's, the sign doesn't matter - it is always + & you can permute $i_1 \cdots i_m$ in $m!$ ways

There are $m!$ terms & each of them gives us 1

Now, LHS :-

$$\int d\theta_m \cdots d\theta_1 \, F\left(\sum_{j=1}^{m} A_{1j} \theta^j, \sum_{j=1}^{m} A_{2j} \theta^j, \cdots, \sum_{j=1}^{m} A_{mj} \theta^j, \vec{\theta}'\right)$$

Degree change doesn't change by linear transfs. & hence, again, only the last term in the expn. of F will contribute.

$$= \int d\theta_m \cdots d\theta_1 \, m! \left(\sum_{i_1=1}^{m} A_{1 i_1} \theta_{i_1}\right)\left(\sum_{i_2=1}^{m} \left(A_{2 i_2} \theta_{i_2}\right)\right) \cdots \left(\sum_{i_m=1}^{m} A_{m i_m} \theta_{i_m}\right) B^{(m)}_{1 \cdots m}(\vec{\theta}')$$

using the fact that the last term of the expn. has the form $m! \, \theta_1 \cdots \theta_m B^{(m)}_{1 \cdots m}(\vec{\theta}')$

A's can be nos. or Grassmann variables

$$= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_m=1}^{m} m! \, A_{1 i_1} A_{2 i_2} \cdots A_{m i_m} \underbrace{\int d\theta_1 \cdots d\theta_m \, \theta_{i_1} \cdots \theta_{i_m}}_{\epsilon_{i_1 \cdots i_m}} B^{(m)}_{1 \cdots m}(\vec{\theta}')$$

$$= (\det A) \, m! \, B^{(m)}_{1 \cdots m}(\vec{\theta}')$$

In Bosonic theory you have $|\det A|$, but here it is just $(\det A)$.

In Bosonic th., you have to worry about the limits & that gives $|\det A|$

– But here you don't have limits

If $X_i = \sum_{j=1}^{m} A_{ij} \theta_j$,

then $dX_1 \ldots dX_m = (\det A)^{-1} d\theta_1 \ldots d\theta_m$

this is again a formal relation, but it encodes what we have proved

$$\text{L.H.S.} = \int d\theta_m \ldots d\theta_1 \, F(X_1, \ldots, X_m, \bar{\theta}')$$

$$= (\det A) \int dX_m \ldots dX_1 \, F(X_1, \ldots, X_m, \bar{\theta}')$$

$$= (\det A) \int d\theta_m \ldots d\theta_1 \, F(\theta_1, \ldots, \theta_m, \bar{\theta}')$$

δ – fn

For a Bosonic variable,

$$\int dx \, \delta(x) f(x) = f(0)$$

We want, for a grassman variable also,

$$\int d\theta \, \delta(\theta) f(\theta) = f(0)$$

What is $\delta(\theta)$?

$$f(\theta) = a + \theta b$$

$$f(0) = a$$

Choose $\delta(\theta) = \theta$.

Then $\int d\theta \; \delta(\theta) f(\theta) = \int d\theta \; \theta a = a$

Product of δ-fns

$$\int d\theta_m \ldots d\theta_1 \, \delta(\theta_1) \ldots \delta(\theta_m) \, f(\theta_1, \ldots, \theta_m)$$

$$= f(0, 0, \ldots, 0)$$

$$\Rightarrow \delta(\theta_1) \ldots \delta(\theta_m) = \theta_1 \theta_2 \ldots \theta_m$$

For Bosonic variables:

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

For fermionic variables:

$$\delta(a\theta) = a\theta = a\,\delta(\theta)$$

Change of variables works the opp. way for fermions — $\delta(a\theta)$ must compensate for this

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For Bosons

$$\delta\left(\sum_{i=1}^{m} A_{1i} x^i\right) \delta\left(\sum_{i=1}^{m} A_{2i} x^i\right) \ldots \delta\left(\sum_{i=1}^{m} A_{mi} x^i\right)$$

$$= \delta(x^1)\,\delta(x^2) \ldots \delta(x^m) \frac{1}{|\det A|}$$

For Fermions

$$\delta\left(\sum_{i=1}^{m} A_{1i} \theta^i\right) \delta\left(\sum_{i=1}^{m} A_{2i} \theta^i\right) \ldots \delta\left(\sum_{i=1}^{m} A_{mi} \theta^i\right)$$

$$= (\det A)\, \delta(\theta_1) \ldots \delta(\theta_m)$$

write $\epsilon \ldots$ & contract

Complex Grassman variables $\theta_1, \dots, \theta_n$

$$\theta_i = \frac{1}{\sqrt{2}}\left(\phi_i + i\,\psi_i\right)$$

$$\theta_i^+ = \frac{1}{\sqrt{2}}\left(\phi_i - i\,\psi_i\right)$$

where $(\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n)$

are $2n$ <u>real</u> grassman variables.

(We know how to deal with a linear combn of real variables & nowhere have we assumed that the coeff. have to be real)

$$\therefore\ d\theta_1\, d\theta_1^+ = \left\{\det\begin{pmatrix} \frac{1}{\sqrt{2}} & i/\sqrt{2} \\ +1/\sqrt{2} & \frac{-i}{\sqrt{2}} \end{pmatrix}\right\}^{-1} d\phi_1\, d\psi_1$$

$$= i\, d\phi_1\, d\psi_1$$

$$\boxed{\begin{pmatrix} \theta_1 \\ \theta_1^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}}$$

For every complex gras. var., we introduce a pair of real gras. variables

– Alternate interpretation

Treat $\theta_1, \dots, \theta_n, \theta_1^+, \dots, \theta_n^+$ as independent grassman variables. without worrying about whether they are real or complex

Then, $\int d\theta_1\, \theta_1 = 1$, $\int d\theta_1^+\, \theta_1^+ = 1$,

$$\int d\theta_1\, \theta_1^+ = 0$$

[We can derive all these relations by considering $\theta_i = \frac{1}{\sqrt{2}}(\phi_i + i\psi_i)$
& $\theta_i^+ = \frac{1}{\sqrt{2}}(\phi_i - i\psi_i)$]

Ex: Show that

(a) $\int \prod_{i=1}^{n} d\theta_i\, d\theta_i^+ \exp(\theta_i^+ A_{ij} \theta_j) = (\det A)$ upto a phase

anti-Hermitian or Herm. or even any arbit matrix

related to the ordering

Real variables (n even)

$$\int \left(\prod_{i=1}^{n} d\theta_i\right) \exp(\theta_i A_{ij} \theta_j) = \sqrt{\det A}\; \times 2^{n/2}$$ up to a phase

antisymmetric

Int. vanishes
int. vanishes

(b) If $(\theta_1, \theta_2, \dots, \theta_n, \chi_1, \dots, \chi_n)$ is a set of 2n grassman variables then

$$\int \prod_{i=1}^{n} (d\theta_i\, d\chi_i) \exp(\theta_i A_{ij} \chi_j)$$

$= \det A$ up to a sign

don't have any particular sym. bcoz θ's & χ's are indep. variables

$$(O_1 O_2 \dots O_n)^\dagger = O_n^\dagger \, O_{n-1}^\dagger \dots O_1^\dagger$$ → in opp. order

(Taking complex conjugate of Grass. var.)

This defn. is more natural

(could also have defined c.c. by taking in the same order)

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complex Grassmann variables

$$\Psi_\alpha(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \tilde{\Psi}_\alpha(-k)$$

$$\Psi_\alpha^\dagger(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \tilde{\Psi}_\alpha^\dagger(k)$$

Defines $\tilde{\Psi}_\alpha(k)$ & $\tilde{\Psi}_\alpha^\dagger(k)$

$$\left(\tilde{\Psi}_\alpha(-k)\right)^\dagger = \tilde{\Psi}_\alpha^\dagger(-k)$$ ⇝ Notation we will be using

$$\bar{\Psi}_\alpha(x) = \Psi_\beta^\dagger(\gamma^0)_{\beta\alpha} = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\bar{\Psi}}_\alpha(-k)$$

First 'bar' & then F.T.

Ex. check that

$$S_{free} = \int \frac{d^4k}{(2\pi)^4} \tilde{\bar{\Psi}}(-k) \underbrace{\left(-\gamma^\mu k_\mu - m\right)}_{(\cancel{k} - m)} \tilde{\Psi}(k)$$

Define:– $\cancel{k} = -\gamma^\mu k_\mu = \gamma^0 k^0 - \vec{\gamma}.\vec{k}$

Define :–

$$Z[\tilde{J}_\alpha(k), \tilde{\bar{J}}_\alpha(k)] = \int [D\tilde{\Psi}][D\tilde{\bar{\Psi}}] \, e^{iS + i\int \frac{d^4k}{(2\pi)^4}\left\{\tilde{\bar{J}}^\alpha(-k)\tilde{\Psi}_\alpha(k) + \tilde{\bar{\Psi}}_\alpha(-k)\tilde{J}_\alpha(k)\right\}}$$

$[D\tilde{\Psi}^+) = [D\tilde{\bar{\Psi}}]$

bcs $\text{Det}(\gamma^0) = 1$

Ex. Show that

$$\left\langle \tilde{\Psi}_{\alpha_1}(k_1) \ldots \tilde{\Psi}_{\alpha_n}(k_n)\, \tilde{\bar{\Psi}}_{\beta_1}(p_1) \ldots \tilde{\bar{\Psi}}_{\beta_n}(p_n) \right\rangle$$

$$= \frac{1}{Z[0]}\left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_1}(-k_1)}\right) \ldots \left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_n}(-k_n)}\right)$$

$$\left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)}\right) \ldots \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)}\right) Z[\tilde{J}]$$

Ex. Show that

$$Z_{free}[\tilde{J}, \tilde{\bar{J}}] = (\text{Constant}) \exp\left[-i\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}_\alpha(-k)\{(\cancel{k} - m)^{-1}\}_{\alpha\beta} \tilde{J}_\beta(k)\right]$$

Sequence doesn't matter bcs you divide by $Z[0]$ as long as we take same in num. & den.

Proof of some Results

H/W

① $I = \int \left(\prod_{i=1}^{n} d\theta_i \, dX_i \right) \exp\left(\theta_i A_{ij} X_j \right)$

We define $\tilde{X}_i = A_{ij} X_j$

$\therefore \; I = (\det A) \int \prod_{i=1}^{n} d\theta_i \, d\tilde{X}_i \exp\left(\theta_i \tilde{X}_i\right)$

The term in the exp. which will contribute is $\rightarrow$

$$\frac{1}{n!} \sum_{i_1=1}^{n} \theta_{i_1} \tilde{X}_{i_1} \cdots \sum_{i_n=1}^{n} \theta_{i_n} \tilde{X}_{i_n}$$

$$= \frac{1}{n!} \times n! \quad (\theta_1 \tilde{X}_1)(\theta_2 \tilde{X}_2) \cdots (\theta_n \tilde{X}_n)$$

[bcos $\theta_i \tilde{X}_i$ is even, we can pull through (this combn.) without picking up a '$-$' sign]

$\therefore \; I = (\det A) \int \left(\prod_{i=1}^{n} d\theta_i \, d\tilde{X}_i \right) \theta_1 \tilde{X}_1 \, \theta_2 \tilde{X}_2 \cdots \theta_n \tilde{X}_n$

$=$ (phase factor dep. on the ordering of the $d\theta_i$'s & $d\tilde{X}_i$'s)

$\times \; (\det A)$

(2) $I = \int \left(\prod_{i=1}^{n} d\theta_i \, d\theta_i^{+}\right) \exp\left(\theta_i^{+} A_{ij} \theta_j\right)$

We can treat θ_i & θ_i^{+} as indep. variables & then, as in (1),

$I = (\det A) \times$ phase factor

Aliter :- For antisymmetric A & n even

$$\theta_i = \frac{1}{\sqrt{2}}(\phi_i + i\psi_i)$$

$$\theta_i^{+} = \frac{1}{\sqrt{2}}(\phi_i - i\psi_i)$$

$$\therefore \theta_i A_{ij} \theta_i^{+} = \frac{1}{2}(\phi_i + i\psi_i) A_{ij} (\phi_j - i\psi_j)$$

$$= \frac{1}{2}\left(\phi_i A_{ij} \phi_j - i\phi_i A_{ij} \psi_j + i\psi_i A_{ij} \phi_j - i^2 \psi_i A_{ij} \psi_j\right)$$

$$= \frac{1}{2}\left[\phi_i A_{ij} \phi_j + \psi_i A_{ij} \psi_j + i \underbrace{\psi_j A_{ij} \phi_i}_{= -i\psi_j A_{ji} \phi_i} + i\psi_i A_{ij} \phi_j\right]$$

$$= \frac{1}{2}\left[\phi_i A_{ij} \phi_j + \psi_i A_{ij} \psi_j \cancel{- i\psi_i A_{ij} \phi_j + i\psi_i A_{ij} \phi_j}\right]$$

Also, note that

this is expected $\because$ both are even fns.

$$[\phi_i A_{ij} \phi_j , \psi_k A_{kl} \psi_l] \cancel{= A_{ij} A_{kl} [\phi_i \phi}$$

$$= A_{ij} A_{kl} (\phi_i \phi_j \psi_k \psi_l - \psi_k \psi_l \phi_i \phi_j)$$

$$= A_{ij} A_{kl} (\phi_i \phi_j \psi_k \psi_l - (-1)^2 (-1)^2 \phi_i \phi_j \psi_k \psi_l)$$

$$= 0$$

Even fns of Grass. variables commute

$\therefore \exp\left(\Theta_i^{+} A_{ij} \Theta_i\right)$

$= \exp\left\{\frac{1}{2}\left(\phi_i A_{ij} \phi_j + \psi_i A_{ij} \psi_j\right)\right\}$

$= \exp\left(\frac{1}{2}\left(\phi_i A_{ij} \phi_j\right)\right) \exp\left(\frac{1}{2}\left(\psi_i A_{ij} \psi_j\right)\right)$

[$\because$ their commutator is zero]

$\therefore I = i^n \int \left(\prod_{i=1}^{n} d\phi_i \, d\psi_i\right) \exp\left(\phi_i \, A_{ij}/2 \, \phi_j\right) \exp\left(\psi_i \, A_{ij}/2 \, \psi_j\right)$

Note $\rightarrow$ $d\Theta_1 \, d\Theta_1^{+} = i \, d\phi_1 \, d\psi_1$

$= i^n \times \text{phase factor} \times \frac{1}{n!} \times n! \sqrt{\frac{\det A}{2^n}} \times 2^{n/2}$

$\times \frac{1}{n!} \times n! \times \sqrt{\frac{\det A}{2^n}} \times 2^{n/2}$

$= \text{phase factor} \times (\det A)$

Better ③ can be proved by the above procedure but we we know that

$I = \int \prod_{i=1}^{n} d\Theta_i \, d\Theta_i^{+} \exp\left(\Theta_i^{+} A_{ij} \Theta_j\right)$

$= \text{phase factor} \times (\det A)$

③ $I = \int \left(\prod_{i=1}^{n} d\theta_i \right) \exp\left(\theta_i A_{ij} \theta_j \right)$

A_{ij} → antisymmetric matrix

Proof is in Cheng-Li

n even

Let us consider an antisym. matrix, or in general, an anti-Hermitian matrix →

$$A^{+} = -A$$

Let $B = iA$

$$\Rightarrow B^{+} = -iA^{+} = (-i)(-A) = iA$$

$$\Rightarrow \boxed{B^{+} = B}$$

∴ if A is anti-Herm, iA is Herm.

∴ it is always possible to diagonalise B by a unitary matrix U (to B_D), it is also possible to diagonalise A by the same unitary matrix U:—

$$UBU^{+} = B_D$$

$$\Rightarrow U(iA)U^{+} = B_D$$

$$\Rightarrow \boxed{UAU^{+} = (-i)B_D}$$

∵ B has real eigenvalues, A will always have imaginary e. values.

For real antisym. A

Jacobi's theorem

$$\det(A^T) = \det(-A)$$

$$\Rightarrow \det A = (-1)^n \det A$$

[∵ $\det M^T = \det M$ for any M]

For n odd, $\det A = 0$.

For a real anti sym. A, the imaginary e.values come in complex conjugate pairs; this is bcos

$$\sum_{i=1}^{n} \lambda_i = 0$$

For n ~~even~~ odd, there is at least one zero, bcos $\prod_i \lambda_i = \text{Det } A = 0$ for n odd

For n even, the e.values are

$i\lambda_j, -i\lambda_j$ where $j = 1, 2, \ldots, n/2$

$$\therefore \text{Det } A = \prod_{j=1}^{n/2} i(-i)\lambda_j^2 = \prod_{j=1}^{n/2} \lambda_j^2$$

$$\Rightarrow \boxed{\text{Det } A \geq 0}$$

Let $UAU^+ = A_D$

$$\therefore \theta_i A_{ij} \theta_j = \theta_i U^+_{i\alpha} (U_{\alpha\beta} A_{\beta\gamma} U^+_{\gamma\delta})(U_{\delta j} \theta_j)$$

or

$$\theta^T A \theta$$

$$= \underbrace{(\theta^+ U^+)}_{\phi}(UAU^+)\underbrace{(U\theta)}_{\phi}$$

Cannot be done in this method

For A real antisym. matrix,

$$\Theta^{+} = \Theta^{T}$$

$$\therefore \quad \Theta^{T} A \, \Theta = (\Theta^{+} U^{+}) A_D (U \Theta)$$

Define $\phi = U\Theta$

$$\therefore \quad I = (\det U) \int \left(\prod_i d\phi_i \right) \exp \Big(\underbrace{\phi_i \, (A_D)_{ij} \, \phi_j}_{\phi_i \lambda_i \delta_{ij} \phi_j \; = \; \phi_i \lambda_i \phi_i} \Big)$$

By using the fact that for a real antisym. matrix A, $\det A = (\text{Pfaffian})^2$

Every real skew-sym. matrix A can be brought into a block-diagonal A_B form by an orthogonal transfm such that

$$O A O^T = A_B = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \\ -\lambda_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \lambda_2 & \\ 0 & 0 & -\lambda_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & \lambda_{n/2} \\ -\lambda_{n/2} & 0 \end{matrix} \end{bmatrix}$$

$n \to$ even

In the odd-dim case, A_B has an additional row & column of zeros.

The e.values of this matrix are $\pm i\lambda_k$ $(\lambda_k > 0)$

$$\therefore \Theta^T A \Theta = (\Theta^T O^T)(O A O^T)\underbrace{(O \Theta)}_{\Psi}$$

$$= \Psi^T A_B \Psi$$

Let ψ_{2m-1} & ψ_{2m} be related by λ_m [where $m = 1, 2, \dots, n/2$]

$$\therefore \psi_i (A_B)_{ij} \psi_j = \psi_{2m-1} \lambda_m \psi_{2m} - \psi_{2m} \lambda_m \psi_{2m-1}$$

$i, j = 2m-1, 2m$

$$= 2\lambda_m \psi_{2m-1} \psi_{2m}$$

$$\therefore \quad I = (\det O) \int \left(\prod_{i=1}^{n} d\psi_i \right) \prod_{m=1}^{n/2} 2\lambda_m \psi_{2m-1} \psi_{2m}$$

$$= \underset{\substack{\parallel \\ \pm 1}}{(\det O)} \times \text{phase factor} \times \prod_{m=1}^{n/2} \sqrt{\lambda_m^2} \times 2^{n/2}$$

$$= \text{phase factor} \times \sqrt{\prod_{m=1}^{n/2} \lambda_m^2} \times 2^{n/2}$$

$$= \text{phase factor} \times \sqrt{\det A} \times 2^{n/2}$$

To find Z_{free} :-

Let $I = i S_{free} + i\int \frac{d^4k}{(2\pi)^4} \left[\tilde{\bar{J}}_\alpha(-k)\, \tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\alpha(-k)\, \tilde{J}_\alpha(k) \right]$

$$= i\int \frac{d^4k}{(2\pi)^4} \Big[\tilde{\bar{\psi}}(-k)\, (\not{k} - m)\, \tilde{\psi}(k) + \tilde{\bar{J}}_\alpha(-k)\, \tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\alpha(-k)\, \tilde{J}_\alpha(k) \Big]$$

[Aside :-

Expression for the propagator for fermionic fields :-

$$G(x, x') = \int e^{ik.(x-x')}\, \tilde{G}(k)\, \frac{d^4k}{(2\pi)^4}$$

$$\Rightarrow \underbrace{(i\gamma^\mu \partial_\mu - m)\, G(x, x')}_{\delta(x-x')\, \mathbb{1}_{2\times 2}} = \underbrace{\int (-\gamma^\mu k_\mu - m)\, \tilde{G}(k)\, e^{ik.(x-x')} \frac{d^4k}{(2\pi)^4}}_{= \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-x')}\, \mathbb{1}_{2\times 2}}$$

$\not{k} = -\gamma^\alpha k_\alpha$

this is in Ashoke Sen's convention :-

$\not{k} = \gamma^0 p^0 - \vec{\gamma}.\vec{p} = \gamma^\mu k_\mu$

$$\Rightarrow \tilde{G}(k) = (-\gamma^\mu k_\mu - m)^{-1} = (\not{k} - m)^{-1}$$

i.e., $\tilde{G}(k)$ is equal to the inverse of the 2×2 matrix $(\not{k} - m)$]

Claim :- $I = i\int \frac{d^4k}{(2\pi)^4} \Big[\Big(\tilde{\bar{\psi}}(-k) + \tilde{\bar{J}}_\alpha(-k) \frac{1}{\not{k} - m} \Big)$

$(\not{k} - m) \Big(\tilde{\psi}(k) + \underbrace{\frac{1}{(\not{k} - m)}}_{2\times 2}\, \underbrace{\tilde{J}(k)}_{2\times 1} \Big)$ $\quad (\not{k} - m)^{-1}$

$- \tilde{\bar{J}}(-k)\, (\not{k} - m)^{-1}\, \tilde{J}(k) \Big]$

Sign problem Dep. on your convention of F.T. & $\not{k}$

$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \rightarrow$ Ashoke Sen's convention

Note :-

$\tilde{\bar{J}}_\alpha(-k)$ is 1×2 matrix

& $(\not{k} - m)^{-1}$ is 2×2 matrix

$\therefore\ \tilde{\bar{J}}(-k)\, (\not{k} - m)^{-1}$ is again a 1×2 matrix, which is what we need

Note

$(+\not{k} - m)\, \tilde{G}(k) = \mathbb{1}$ — ①

$(+\not{k} - m)(+\not{k} + m)\, \tilde{G}(k) = +(\not{k} + m)$

$\Rightarrow (+\not{k} - m)\, \tilde{G}(k) = \not{k} + m$

$\Rightarrow (k^2 + m^2)\, \tilde{G}(k)$

$\Rightarrow \tilde{G}(k) = \frac{-(\not{k} + m)}{k^2 + m^2 - i\epsilon}$

Proof :- $I = i\int \frac{d^4k}{(2\pi)^4}\Big[\tilde{\bar{\psi}}(-k)(\not{k}-m)\tilde{\psi}(-k) + \tilde{\bar{J}}_\alpha(-k)\tilde{\psi}(-k)$
$+ \tilde{\bar{\psi}}(-k)\tilde{J}_\alpha(-k) + \tilde{\bar{J}}_\alpha(-k)(\not{k}-m)^{-1}\tilde{J}_\alpha(k)$
$- \tilde{\bar{J}}_\alpha(-k)(\not{k}-m)^{-1}\tilde{J}_\alpha(-k)\Big]$

$$= i\int \frac{d^4k}{(2\pi)^4}\Big[\tilde{\bar{\psi}}(-k)(\not{k}-m)\tilde{\psi}(-k) + \tilde{\bar{J}}_\alpha(-k)\tilde{\psi}(-k) + \tilde{\bar{\psi}}(-k)\tilde{J}_\alpha(-k)\Big]$$

(verified)

Now define :- $\tilde{\chi}(-k) = \tilde{\psi}(-k) + (\not{k}-m)^{-1}\tilde{J}(-k)$

$\tilde{\bar{\chi}}(-k) = \tilde{\bar{\psi}}(-k) + \tilde{\bar{J}}_\alpha(-k)(\not{k}-m)^{-1}$

Aliter :- After defining $\tilde{\chi}(k)$ & $\tilde{\bar{\chi}}(-k)$ as above, write I in terms of $\tilde{\chi}(k)$ & $\tilde{\bar{\chi}}(-k)$ & find the new form of $I \Rightarrow$

$\tilde{\bar{\psi}}(-k)(\not{k}-m)\tilde{\psi}(k)$

$= \Big[\tilde{\bar{\chi}}(-k) - \tilde{\bar{J}}_\alpha(-k)(\not{k}-m)^{-1}\Big](\not{k}-m)\Big[\tilde{\chi}(k) - (\not{k}-m)^{-1}\tilde{J}(k)\Big]$

$= \tilde{\bar{\chi}}(-k)(\not{k}-m)\tilde{\chi}(k) - \tilde{\bar{\chi}}(-k)\tilde{J}(k) - \tilde{\bar{J}}_\alpha(-k)\tilde{\chi}(k) + \tilde{\bar{J}}_\alpha(-k)(\not{k}-m)^{-1}\tilde{J}(k)$ — (1)

$\tilde{\bar{J}}_\alpha(-k)\tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\alpha(-k)\tilde{J}_\alpha(k)$

$= \tilde{\bar{J}}(-k)\tilde{\chi}(k) - \tilde{\bar{J}}(-k)(\not{k}-m)^{-1}\tilde{J}(k) + \tilde{\bar{\chi}}(-k)\tilde{J}(k) - \tilde{\bar{J}}(-k)(\not{k}-m)^{-1}\tilde{J}(k)$ — (2)

Adding (1) & (2) & plugging in, we get,

$$I = i\int \frac{d^4k}{(2\pi)^4}\Big[\tilde{\bar{\chi}}(-k)(\not{k}-m)\tilde{\chi}(k) - \tilde{\bar{J}}(-k)(\not{k}-m)^{-1}\tilde{J}(k)\Big]$$

Hence, $Z_{free}[\tilde{J}, \tilde{\bar{J}}]$

$$= \tilde{c} \exp\left[-i\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}(-k) (\not{k} - m)^{-1} \tilde{J}(k)\right]$$

EVEN fn. (in terms of Grassman variables)

Expression for $\langle \tilde{\Psi}_{\alpha_1}(k_1) \tilde{\Psi}_{\alpha_2}(k_2) \dots \tilde{\Psi}_{\alpha_n}(k_n) \tilde{\bar{\Psi}}_{\beta_1}(p_1) \dots \tilde{\bar{\Psi}}_{\beta_n}(p_n) \rangle$

We have,

$$\left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_1}(-k_1)}\right) \dots \left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_n}(-k_n)}\right)$$

$$\left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)}\right) \dots \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)}\right) Z[\tilde{J}, \tilde{\bar{J}}]\Big|_{\tilde{J}=0,\ \tilde{\bar{J}}=0}$$

We can apply the chain rule of diff.

bcos ~~$-i\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}(-k) (\not{k} - m)^{-1} \tilde{J}(k)$ is an even fn. of its arguments (Grassman var.)~~

$f = \tilde{\bar{J}}(-k) \tilde{\Psi}(k) + \tilde{\bar{\Psi}}(-k) \tilde{J}(k)$ is an even fn.

Here, the diff. will act on $\exp\left(i\int \frac{d^4k}{(2\pi)^4} f\right)$, as this is the part in $Z[\tilde{J}, \tilde{\bar{J}}]$ containing $\tilde{J}, \tilde{\bar{J}}$,

Now, $(2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)} \exp\left(i\int \frac{d^4k}{(2\pi)^4} f\right)$

$$= (2\pi)^4 i (i)(-1)\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{\Psi}}_{\beta_n}(-k)\, \delta(k+p_n) \times \exp(\dots)$$

$$= \cancel{(2\pi)^4} \times \frac{1}{\cancel{(2\pi)^4}} (-1)^2 \tilde{\tilde{\varphi}}_{\beta_n}(p_n) \exp(---)$$

$$= \tilde{\tilde{\varphi}}_{\beta_n}(p_n) \exp(---)$$

$$= \exp(---) \tilde{\tilde{\varphi}}_{\beta_n}(p_n) \quad [\because \exp(---) \text{ is an even } f^n]$$

$$\therefore \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)}\right) \cdots \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)}\right) \exp(---)$$

$$= \exp(---) \left[\tilde{\tilde{\varphi}}_{\beta_1}(p_1) \cdots \tilde{\tilde{\varphi}}_{\beta_n}(p_n)\right]$$

Now,
$$-(2\pi)^4 i \frac{\delta}{\delta \tilde{\tilde{J}}_{\alpha_n}(-k_n)} \exp(---)$$

$$= -\cancel{(2\pi)^4} (i) \, i \int \frac{d^4k}{\cancel{(2\pi)^4}} \tilde{\varphi}_{\alpha_n}(-k) \, \delta(k - k_n) \exp(---)$$

$$= \exp(---) \tilde{\varphi}_{\alpha_n}(k_n)$$

Hence,
$$\langle \tilde{\varphi}_{\alpha_1}(k_1) \cdots \tilde{\varphi}_{\alpha_n}(k_n) \tilde{\tilde{\varphi}}_{\beta_1}(p_1) \cdots \tilde{\tilde{\varphi}}_{\beta_n}(p_n) \rangle$$

$$= \frac{1}{Z[0,0]} \Bigg[\left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\tilde{J}}_{\alpha_1}(-k_1)}\right) \cdots \left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\tilde{J}}_{\alpha_n}(-k_n)}\right)$$
$$\left(+(2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)}\right) \cdots \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)}\right) Z[\tilde{J}, \tilde{\tilde{J}}] \Bigg]_{\substack{\tilde{J}=0 \\ \tilde{\tilde{J}}=0}}$$

2-point fn! —

$$\langle \tilde{\Psi}_\alpha(-k_1)\, \tilde{\bar{\Psi}}_\beta(k_2) \rangle$$

$$= \frac{1}{Z[0,0]} \left[\left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_\alpha(-k_1)} \right) \left((2\pi)^4 i \frac{\delta}{\delta \tilde{J}_\beta(-k_2)} \right) \tilde{e} \exp\left(-i \int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}(-k)_\delta (\not{k}-m)^{-1}_{\delta\gamma} \tilde{J}(k)_\gamma \right) \right]_{\substack{\tilde{J}=0 \\ \tilde{\bar{J}}=0}}$$

$$= \cancel{\frac{Z[0,0]}{Z[0,0]}} \left[-\cancel{(2\pi)^4} i \int \frac{d^4k}{\cancel{(2\pi)^4}} \delta(k-k_1) (\not{k}-m)^{-1}_{\alpha\gamma} \tilde{J}(k)_\gamma \right.$$

$$\left. \times \cancel{(2\pi)^4} i \int \frac{d^4k'}{\cancel{(2\pi)^4}} \delta(k+k_2)(-1) \tilde{\bar{J}}(-k)_\delta (\not{k}-m)^{-1}_{\delta\beta} \right]$$

$$\times \frac{(-i)^2}{2!} \longrightarrow \text{from the expn of } \exp(---)$$

$$= (-i)^2 \cancel{/2!} (\not{k}_1 - m)^{-1}_{\alpha\gamma} \tilde{J}(k_1)_\gamma$$

expand the exponential

$$= \cancel{\frac{\tilde{e}}{\tilde{e}}} \left[\left(-(2\pi)^4 i \frac{\delta}{\delta \tilde{\bar{J}}_\alpha(-k_1)} \right) \left(\cancel{(2\pi)^4} i \frac{\delta}{\delta \tilde{J}_\beta(-k_2)} \right) \times \frac{(-i)}{1} \int \frac{d^4k}{\cancel{(2\pi)^4}} \tilde{\bar{J}}(-k)_\delta (\not{k}-m)^{-1}_{\delta\gamma} \tilde{J}(k)_\gamma \right]_{\substack{\tilde{J}=0 \\ \tilde{\bar{J}}=0}}$$

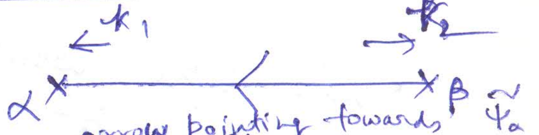
$$= -(2\pi)^4 i^2 (-i) \left[\frac{\delta}{\delta \tilde{\bar{J}}_\alpha(-k_1)} \int d^4k \underset{\substack{\uparrow \\ \text{due to passing through } \tilde{\bar{J}}(-k)}}{(-1)} \tilde{\bar{J}}(-k)_\delta (\not{k}-m)^{-1}_{\delta\beta} \delta^{(4)}(k+k_2) \right]_{\substack{\tilde{J}=0 \\ \tilde{\bar{J}}=0}}$$

$$= (-1)^2 i^2 (-i) (2\pi)^4 \int d^4k\, \delta^{(4)}(k-k_1) (\not{k}-m)^{-1}_{\alpha\beta} \delta^{(4)}(k+k_2)$$

$$\Rightarrow \langle \tilde{\Psi}_\alpha(k_1)\, \tilde{\bar{\Psi}}_\beta(k_2) \rangle$$

$$= (-1)(-i)(2\pi)^4 \, (\not{k}_1 - m)^{-1}_{\alpha\beta} \, \delta^{(4)}(k_1 + k_2)$$

$$= \underbrace{(2\pi)^4 \, i \left[(\not{k}_1 - m)^{-1} \right]_{\alpha\beta} \, \delta^{(4)}(k_1 + k_2)}_{\text{2-pt. Green's fn.}}$$



bcos arrow pointing towards $\tilde{\Psi}_\alpha$

$\tilde{\bar{\Psi}}_\beta$ is particle creator

2n-point Green's fn. :-

$$Z[\tilde{J}, \tilde{\bar{J}}] = \tilde{c} \exp\left(- \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \tilde{\bar{J}}(-k_1)_\alpha \, \Delta(k_1, k_2)_{\alpha\beta} \tilde{J}(-k_2)_\beta \right)$$

$$\cancel{\times \, \delta^{(4)}(k_1 + k_2)}$$

where $\Delta(k_1, k_2) = (2\pi)^4 \left(\frac{i}{\not{k} - m} \right)_{\alpha\beta} \delta^{(4)}(k_1 + k_2)$

$$\langle \tilde{\Psi}_{\alpha_1}(k_1) \tilde{\Psi}_{\alpha_2}(k_2) \ldots \tilde{\Psi}_{\alpha_n}(k_n) \tilde{\bar{\Psi}}_{\beta_1}(p_1) \tilde{\bar{\Psi}}_{\beta_2}(p_2) \ldots \tilde{\bar{\Psi}}_{\beta_n}(p_n) \rangle$$

$$= \underbrace{(\;-\!-\;)}_{\text{sign}} \times \underbrace{\frac{(-1)^n}{n!}}_{\text{from the expansion of the exp}(---)} \times \underbrace{(-i)^n (i)^n}_{=1} \times \cancel{\{(2\pi)^4\}^{2n}} \times \underbrace{\cancel{\frac{1}{((2\pi)^8)^n}}}_{\text{from the integral in exp}(---)}$$

$$\times \left[\Delta_{\alpha_1\beta_1}(k_1, p_1) \Delta_{\alpha_2\beta_2}(k_2, p_2) \cdots \Delta_{\alpha_n\beta_n}(k_n, p_n) + \text{all other permutations with sign} \right]$$

$$= \underbrace{(\;-\!-\;)}_{\text{sign}} \times \frac{(-1)^n}{n!} \times \underbrace{n!}_{\text{due to eqv. perm. of the pairs } (\alpha_i, \beta_i)} \left[\Delta_{\alpha_1,\beta_1}(k_1, p_1) \cdots \Delta_{\alpha_n,\beta_n}(k_n, p_n) + \text{all other inequivalent permutations with sign} \right]$$

Fixing the sign :-

(1) Analytically :-

$$(\cdots\cdots) \quad \Delta_{\alpha_1\beta_1}(k_1,p_1)\,\Delta_{\alpha_2\beta_2}(k_2,p_2)\cdots\Delta_{\alpha_n\beta_n}(k_n,p_n)$$

$\downarrow$
sign = ?

This sign would be the same as the sign obtd. to go from

$$\left[\left(\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_1}(-k_1)}\cdots\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_n}(-k_n)}\right)\left(\frac{\delta}{\delta\tilde{J}_{\beta_1}(-p_1)}\cdots\frac{\delta}{\delta\tilde{J}_{\beta_n}(-p_n)}\right)\right]$$

to

$$\left[\left(\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_1}(-k_1)}\frac{\delta}{\delta\tilde{J}_{\beta_1}(-p_1)}\right)\cdots\left(\frac{\delta}{\delta\tilde{\bar{J}}_{\alpha_n}(-k_n)}\frac{\delta}{\delta\tilde{J}_{\beta_n}(-p_n)}\right)\right]$$

This is again the same as going from

$$\cancel{\left(\tilde{\Psi}_{\alpha_1}(k_1)\,\tilde{\bar{\Psi}}_{\beta_1}(p_1)\right)\cdots}$$

$$\left[\left(\tilde{\Psi}_{\alpha_1}(k_1)\cdots\tilde{\Psi}_{\alpha_n}(k_n)\right)\left(\tilde{\bar{\Psi}}_{\beta_1}(p_1)\cdots\tilde{\bar{\Psi}}_{\beta_n}(p_n)\right)\right]$$

to

$$\left[\left(\tilde{\Psi}_{\alpha_1}(k_1)\,\tilde{\bar{\Psi}}_{\beta_1}(p_1)\right)\cdots\left(\tilde{\Psi}_{\alpha_n}(k_n)\,\tilde{\bar{\Psi}}_{\beta_n}(p_n)\right)\right]$$

$$\therefore \text{sign} = (-1)^{n-1}(-1)^{n-2}\cdots(-1)^{1}$$

$$= (-1)^{1+2+\cdots+(n-2)+(n-1)}$$

$$= \boxed{(-1)^{(n-1)n/2}}$$

(2) Pictorially :—

$$\langle \tilde{\Psi}_{\alpha_1}(k_1)\,\tilde{\Psi}_{\alpha_2}(k_2)\cdots\tilde{\Psi}_{\alpha_n}(k_n)\,\tilde{\Psi}_{\beta_1}(p_1)\,\tilde{\Psi}_{\beta_2}(p_2)\cdots\tilde{\Psi}_{\beta_n}(p_n)\rangle$$

Count only the vertical crossings

— You could also count horizontal crossings, but here they are hard to consider bcos 'n' is not specified

$$\langle \tilde{\Psi}_{\alpha_1}(k_1)\tilde{\Psi}_{\alpha_2}(k_2)\,\tilde{\Psi}_{\alpha_3}(k_3)\,\tilde{\Psi}_{\alpha_4}(k_4)\cdots\tilde{\Psi}_{\alpha_n}(k_n)\,\tilde{\Psi}_{\beta_1}(p_1)\,\tilde{\Psi}_{\beta_2}(p_2)\,\tilde{\Psi}_{\beta_3}(p_3)\,\tilde{\Psi}_{\beta_4}(p_4)\cdots\tilde{\Psi}_{\beta_n}(p_n)\rangle$$

1

2

3

$$\therefore \text{ sign} = (-1)^0(-1)^1(-1)^2\cdots(-1)^{n-1}$$

$$= (-1)^{n(n-1)/2}$$

sign before $\Delta_{\alpha_1\beta_1}(k_1,p_1)\cdots\Delta_{\alpha_n\beta_n}(k_n,p_n)$

7/9/06

Interactions involving fermions

$$e^{iS} = e^{iS_{free} + iS_{int}}$$

$$e^{iS_{int}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(iS_{int}\right)^n$$

expanded out in a Taylor series expn.

$$S_{int}\left(\tilde{\bar{\Psi}}(-k),\ \tilde{\Psi}(k),\ \dots\right)$$

$$i(2\pi)^4 \frac{\delta}{\delta \tilde{\bar{J}}(-k)} \qquad (-i)(2\pi)^4 \frac{\delta}{\delta \tilde{J}(-k)}$$

Example

$$S_{int} = \int d^4x\ \bar{\Psi}_\alpha M_{\alpha\beta} \Psi_\beta \phi$$

fermions

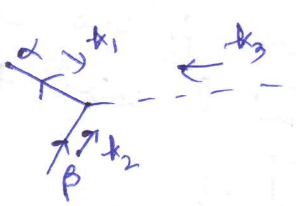
some constant matrice

scalar

Could be constructed out of γ matrices & so on

$$iS_{int} = i\int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}\ \tilde{\bar{\Psi}}_\alpha(-k_1)\, \tilde{\Psi}_\beta(-k_2)\ \tilde{\phi}(-k_3)\ (2\pi)^4\ \delta^{(4)}(k_1+k_2+k_3)\, M_{\alpha\beta}$$

(This int. vertex contains 3 fields $\rightarrow$



$$(2\pi)^4\, i\, \delta^{(4)}(k_1+k_2+k_3)\, M_{\alpha\beta}$$

(Determine the sign from the no. of crossings)

Applying path integral formulation to free quantum electrodynamics

$$S_{free} = -1/4 \int d^4x\, \eta^{\mu\mu'} \eta^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}$$

$$= 1/2 \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(-k) \underbrace{(-k^2\eta^{\mu\nu} + k^\mu k^\nu)}_{M_{\mu\nu}(-k)} \tilde{A}_\nu(k)$$

Naive propagator $\propto \left(M^{-1}(-k)\right)_{\mu\nu}$

(But it is not invertible bcos of zero eigenvalue $\rightarrow$

$(-k^2\eta^{\mu\nu} + k^\mu k^\nu) k_\nu = 0 \Rightarrow$ The 4-vector k is an eigenvector of M with zero eigenvalue

$\therefore$ $\boxed{M^{-1} \text{ is not defined.}}$

[we need to understand what is the origin of this zero e. value – there is no way to add a gauge-invariant term which will turn this into a non-zero e. value]

$(-k^2\eta^{\mu\nu} + k^\mu k^\nu) k_\nu$

$= -k^2 k^\mu + k^\mu k^2$

$= 0$

Physical origin of the zero eigenvalue:

(Original action had a gauge invariance)

Gauge invariance :—

Under $\delta A_\mu = \partial_\mu \epsilon$, $\delta S = 0$

ϵ : infinitesimal

For E.D., this holds even for finite ϵ

$\delta A_\mu = \partial_\mu \epsilon \Rightarrow \delta \tilde{A}_\mu(k) = i k_\mu \epsilon$

[The action shouldn't change under this change of $\tilde{A}_\mu(k)$ — this is possible when k_μ is directed along a zero e. value]

$\Longrightarrow$ k_μ must be an eigenvector of $M_{\mu\nu}(k)$ with zero eigenvalue.

(As a consequence of gauge invariance, this zero e. value will $\therefore$ be there)

Question :—

Why does a zero eigenvalue of $M_{\mu\nu}$ cause divergence in the 2-point function?

Consider a finite dimensional integral:

$$\left\langle \prod_i f_i(\vec{x}) \right\rangle = \frac{\int dx_1 \dots dx_n \, e^{-x^i A_{ij} x^j} \prod_i f_i(\vec{x})}{\int dx_1 \dots dx_n \, e^{-x^i A_{ij} x^j}}$$

A_{ij} is analog of $M_{\mu\nu}$

In path integral, we will have infinite no. of fields at lattice pts. instead of finite no. of x_i's

A : real symmetric

$$A = U^T A_d U \quad \text{where } U^T U = \mathbb{1}$$

U: $n \times n$ orthogonal matrix

$$\therefore \; x^i A_{ij} x^j = x^i \underbrace{U^T_{ik}}_{= U_{ki}} (A_d)_{kl} \, U_{lj} x^j$$

<u>Define</u> :- $y^l = U_{lj} x^j$

$$x^i A_{ij} x^j = y^k (A_d)_{kl} \, y^l = \sum_{\alpha=1}^{n} \lambda_\alpha y_\alpha^2$$

where $A_d = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$

Also, $x^j = (U^{-1})_{ji} y^i = (U^T)_{ji} y^i = U_{ij} y^i$

$$\Rightarrow \vec{x} = U^T \vec{y}$$

$$\prod_{i=1}^{n} dx^i = \underbrace{(\det U)^{-1}}_{\pm 1} \prod_{i=1}^{n} dy^i = \prod_{i=1}^{n} dy^i$$

$$\left\langle \prod_i f(x_i) \right\rangle = \frac{\int dy_1 \ldots dy_n \, e^{-\sum_\alpha \lambda_\alpha y_\alpha^2} \prod_{i=1}^{M} f_i(U^T \vec{y})}{\int dy_1 \ldots dy_n \, e^{-\sum_\alpha \lambda_\alpha y_\alpha^2}}$$

$\prod_{i=1}^{M} f_i(\vec{x}) \to$ polynomial in fields e.g. :- $\langle x^i x^j \rangle$

$$= \frac{\int dy_1 \ldots dy_n \, e^{-\sum_\alpha \lambda_\alpha y_\alpha^2} U^T_{ik} y^k U^T_{jl} y^l}{\int dy_1 \ldots dy_n \, e^{-\sum_\alpha \lambda_\alpha y_\alpha^2}}$$

$f_i = x_i$
$f_j = x_j$

Note :- $\int_{-\infty}^{\infty} dy \, e^{-\lambda y^2} = \sqrt{\pi/\lambda}$

defined in the full complex plane except at $\lambda = 0$ (branch-pt.)

Analytically continue it to the full complex plane for $\lambda < 0$
You might think that the above int. works only for $\lambda > 0$ — but by anal. cont. ...

The int. is ill-defined if any of the λ_i's is zero

If any of the λ_α's vanish the integral is ill-defined. Otherwise it is well defined.

For definiteness we take $\lambda_1 = 0$

Then we have :-

$$\frac{\int dy_1 \int dy_2 \ldots dy_n \, e^{-\sum_{\alpha=2}^{n} \lambda_\alpha y_\alpha^2} \prod_{i=1}^{M} f(U^T \vec{y})}{\int dy_1 \int dy_2 \ldots dy_n \, e^{-\sum_{\alpha=2}^{n} \lambda_\alpha y_\alpha^2}}$$

No λ_1 to suppress the y_1 integral

only hope is if somehow divergence cancels betw. the numerator & the denominator

The only way to avoid ~~[illegible]~~ the divergence is if $f(U^T \vec{y})$ is y^1 independent

(otherwise num. will diverge much more rapidly than den. — ..... polynomials in y_i's)

[what special prop. does y_1-indep. fn. have?]

$$S = \sum_{i,j} A_{ij}\, x^i x^j = \sum_{\alpha=2}^{n} \lambda_\alpha\, y_\alpha^2 \qquad (\because \lambda_1 = 0)$$

(Using y_α's as the basic varibles, we find $\rightarrow$

$$\begin{cases} \delta y_1 = \epsilon \\ \delta y_i = 0 \quad \text{for } i = 2, \ldots, n \end{cases}$$

leaves the action invariant

$$x^i = U^T_{ij}\, y^j = U_{ji}\, y^j$$

$$\delta x^i = U_{ji} \underbrace{\delta y^j}_{\delta_{j1}\epsilon} = \epsilon\, U_{1i}$$

In gauge th. also we saw that there was a zero e. vector & it was related to some symmetry

[What does it mean in the lang. of sym. that $f_i(U^T \vec{y})$ is indep. of y_1?

$\rightarrow$ f_i's must be inv. under the above sym. transfns.]

---

f_i being independent of y_1

means $\Longrightarrow$ f_i is invariant under the symmetry transformation $\delta y_1 = \epsilon$ & $\delta y_2 = \cdots = \delta y_n = 0$

If the action S has certain sym., the int. diverges, but the fns whose correlation fn. we are calculating also respect that sym., then the div. can be avoided.

If all operators whose cor. fn. we want to calculate are gauge-inv., div. can be avoided

Lesson :-

We expect that in a gauge theory, correlation function of gauge invariant operators will be finite.

$$\langle A_\mu(x) A_\nu(y) \rangle \to \text{infinite} \to$$ Bcos it isn't gauge inv.

$$\langle F_{\mu\nu}(x) F_{\rho\sigma}(y) \rangle \to \text{finite}$$ ⇝ but can't be calculated by taking the derivative of $A_\mu A_\nu$ cor. fn. - we would have very much liked to do that

Need to devise some tricks to do that

~~$\langle O_i(x_i) \rangle = \infty$~~

$$\langle O_i(x_i) \rangle = \frac{\int \prod_{\mu=0}^{3} [DA_\mu] e^{iS} \prod_i O_i(x_i)}{\int \prod_{\mu=0}^{3} [DA_\mu] e^{iS}}$$

gauge invariant

$$= \frac{\int \prod_{\mu=0}^{3} [D\tilde{A}_\mu(k)] e^{iS} \prod_i O_i(x_i)}{\int \prod_{\mu=0}^{3} [D\tilde{A}_\mu(k)] e^{iS}}$$

Div. is due to $\int d\tilde{A}_{\mu,\alpha}$ integrals, not from $\int d^4k$ integral

Both num. & den. are div. bcos we haven't factored out the gauge dirn. & cancelled it

Example :- $O_1(x_1) = F_{\mu_1\nu_1}(x_1) = \partial_{\mu_1} A_{\nu_1}(x_1) - \partial_{\nu_1} A_{\mu_1}(x_1)$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \left(i k_{\mu_1} \tilde{A}_{\nu_1}(k_1) - i k_{\nu_1} \tilde{A}_{\mu_1}(k_1) \right)$$

$$\delta \tilde{A}_\mu(k) = i k_\mu \tilde{\epsilon}(k)$$

This is the gauge dir.ⁿ like the y_1 dir.ⁿ

$\delta A_\mu = \partial_\mu \epsilon(x)$

$\delta \tilde{A}_\mu(k) = i k_\mu \tilde{\epsilon}(k)$ (arbit. infinitesimal fn. of k)

Choose a basis of 4-vectors such that

$e_{(1)\mu}(k) = k_\mu$, $e_{(2)\mu}(k)$, fixed for every k

fixed (chosen once & for all) $e_{(3)\mu}(k)$, $e_{(4)\mu}(k)$ are linearly independent of each other and of k_μ.

$$\tilde{A}_\mu(k) = \sum_{s=1}^{4} \tilde{B}_s(k)\, e_{(s)\mu}(k)$$

$\tilde{B}_s$: coeff. of expn.

$e_{(s)\mu}$: set of basis vectors in this 4-dim. space

Treat $\tilde{B}_s(k)$ as independent variables, & not $\tilde{A}_\mu(k)$.

[We need to understand now the gauge transfn. prop. of $\tilde{B}_s$'s]

$$\delta \tilde{B}_s(k) = ?$$

Suppose

$\tilde{B}_s(k) = \lambda_s(k)$ under gauge transfn

$\Rightarrow \delta \tilde{A}_\mu(k)$

$$= \sum_{s=1}^{4} \lambda_s(k)\, e_{(s)\mu}(k)$$

compare these

$$i\tilde{\epsilon}(k)\, k_\mu = i\tilde{\epsilon}(k)\, e_{(1)\mu}(k)$$

This will tell us

$$\lambda_1(k) = i\tilde{\epsilon}(k)$$

$$\lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0$$

(coeff. must match bcos we have a linearly indep. basis)

Example of a possible choice of basis vectors :—

$$e_{(2)} = (0, \vec{k})$$

$$e_{(3)} = (0, \epsilon^{(1)}_{(\vec{k})})$$

$$e_{(4)} = (0, \epsilon^{(2)}_{(\vec{k})})$$

$$\epsilon^{(i)}(\vec{k})\, k_i = 0$$

$$\epsilon^{(i)} \cdot \epsilon^{(j)} = \delta_{ij}$$

This choice leads to the Coulomb gauge

There are many zero e.values bcos for every k we have a zero e.value

$$\Rightarrow \delta \tilde{B}_1(k) = i\tilde{\epsilon}(k)$$

$$\delta \tilde{B}_2(k) = \cdots = \delta \tilde{B}_4(k) = 0$$

$\Rightarrow$ All gauge invariant operators, expressed in terms of $\tilde{B}_s(k)$, are independent of $\tilde{B}_1(k)$.

The action is also independent of $\tilde{B}_1(k)$.

Furthermore, $\prod_i [\mathcal{D}\tilde{A}_\mu(k)] = \prod_{s=1}^{4} [\mathcal{D}\tilde{B}_s(k)] \times$ constant

$\langle O_i(x_i) \rangle$

$$= \frac{\cancel{\int [\mathcal{D}\tilde{B}_1(k)]} \int \prod_{s=2}^{4} [\mathcal{D}\tilde{B}_s(k)] e^{iS} \prod_i O_i(x_i)}{\cancel{\int [\mathcal{D}\tilde{B}_1(k)]} \int \prod_{s=2}^{4} [\mathcal{D}\tilde{B}_s(k)] e^{iS}}$$

Zero e.value has been separated out & no divergence.

Now we have an invertible 3×3 matrix

Suppose we have charged fermions coupled to gauge fields — not only gauge fields transform, fermions also transform ($\psi \to e^{i\epsilon(x)e}\psi$)

— so $\tilde{B}_1(k)$ isn't the only variable which transforms under gauge transfs.

— soln is to take some combn. of fermionic & gauge fields so that only one of the combns transform under gauge transfs