Actions with local \( U(N) \) or \( SU(N) \) invariance

Gauge fields: \( B^a \mapsto \sum_i T^a_i \) of \( SU(N) \) or \( U(N) \) (vector field denoted by index \( a \))

\[ B \mu = B^a \gamma^\mu \eta_a \]

\[ G_{\mu \nu} = 2 \mu B^\nu - 2 \nu B^\mu - i [B^\mu, B^\nu] \]

\[ S_{\text{gauge}} = -\frac{1}{2 e^2} \int d^4x \text{Tr} \left( \begin{array}{c} \phi \phi^T \end{array} \right) \]

Invariant under \( B \mu \mapsto U B \mu U^{-1} i (2 \gamma^\mu) U^{-1} \)

In component form,

\[ B^a \mapsto \mathcal{R} \mathcal{B} B^a - \mathcal{L}^a \]

\[ U \gamma^a U^{-1} = \mathcal{R} \mathcal{B} \mathcal{U} \mathcal{B} \rightarrow \text{defines \( \mathcal{R} \mathcal{B} \) } \]

\[ i (2 \mu \gamma^a U^{-1} = \lambda^a \eta^a \rightarrow \text{defines \( \lambda^a \) } \]
$S_{	ext{fermion}} = \int d^4x \, \overline{\psi} (i \gamma^m D_m - m) \psi$

where $D_m = \partial_m - i B^a_m$

Invariant under

$\psi \rightarrow U \psi$ together with

gauge trs. of $B^a_m$

Suppose $R^a_\xi (U)$ represent $U$

$& \& R^a_A (T^a) \) represent $T^a$

Then $R^a_\xi (1 + i e^a \gamma^m) = 1 + i e^a R^a_A (T^a)$

$L$ defines $R^a_A (T^a)$

Suppose $\psi \rightarrow R(U) \psi$

$m_D \psi = \partial_m - i B^a_m R_A (T^a)$

$\int d^4x \, \overline{\psi} (i \gamma^m D_m - m) \psi$

is gauge invariant.

Suppose a set of scalar fields

$\phi = [\phi_1, \phi_2]$

transform as $\phi \rightarrow R^a_\xi (U) \phi$

Define $- D_m \phi = (\partial_m - i B^a_m R_A (T^a)) \phi$

Then $(D_m \phi)^+ (D^m \phi)$ is gauge invariant.

Note $- D_m \rightarrow R^a_\xi (U) D_m \psi$ shows that the action is invariant.
\[ (D\mu \phi) \rightarrow R_{\mu}^a(U)(D\mu \phi) \]

\[ S_{\text{gauge}} = -\frac{1}{2g^2} \int d^4x \text{ Tr} \left( G_{\mu
u} G^{\mu\nu} \right) \]

where \( G_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} - i [B_{\mu}, B_{\nu}] \)

To be more general, we have first the \( \frac{-1}{2g^2} \) constant factor - but what is its role?

\[ B_{\mu} = g A_{\mu} \rightarrow \text{ defines } A_{\mu} \]

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i g [A_{\mu}, A_{\nu}] \]

Check:

\[ m_n G_{\mu\nu} = g m_n F_{\mu\nu} \]

Substituting this in \( S_{\text{gauge}} \), we get,

\[ S_{\text{gauge}} = -\frac{1}{2} \int d^4x \text{ Tr} \left( F_{\mu\nu} F^{\mu\nu} \right) \]

(we will find that)

the quadratic term has no \( g \),

the cubic term has a \( g \), &

the quartic term has \( g^2 \).

\[ g \] gives a strength of the coupling

- Non-abelian gauge theory from the beginning is interesting, unlike the Maxwell theory

\[ g \] 

Cubic term gives a 3-pt. vertex

Quartic term gives a 4-pi. vertex

\[ \text{Wavely, we could have thought that} \]

\[ \text{cubic & quartic coupling could have} \]

\[ \text{dist. indep. strength} \] - but gauge

\[ \text{inv. has fixed them to be } g \leq g^2 \]
Now, for $\Phi \to R(\Phi)$

\[ m^2 \Phi = \partial_\mu - i e^a R_a (\Phi) = \partial_\mu - i g_0 a^a R_a (\Phi) \]

\[ \Phi \int d^4 x \left[ i \gamma^\mu \partial_\mu - m \right] \Phi \]

\[ \to \int d^4 x \left[ i \gamma^\mu \partial_\mu - m \right] \Phi \]

\[ + g \int d^4 x \prod_i \gamma^\mu A^{a \mu}_i \left( R_a (\Phi) \right) \]

[corr. to free Dirac action for the multicomponent fermion field]

[This has a cubic interaction vertex & the coefficient of this term is given by $g^i$]

[the same $g^i$ that controlled the couplings of the gauge fields, also controls that for the fermion fields]

[\( i, j \to \) gauge indices]

[Dirac indices are suppressed]

\[(R^a (\Phi))_{ij}\]

Let discrete choice depending on which ref \( R \) the fermion belongs to

For Maxwell field, suppose you couple to $A^\mu$ & $\Phi$, with strengths $e$ & $e'$ which can vary continuously in principle.

Here, the couplings can't be varied continuously, it must be some discrete choice.

- in $SU(2)$ it is spin $1/2, 3/2$, etc.

nothing in the theory tells us that \( e, e' \) should be discrete.

which ref. we use, or gauge group, we use, will be fixed by exp.]
Brief review of continuous groups & their unitary representations

Group $G$: Collection of $N \times N$ unitary matrices $U$ labelled by one or more continuous parameters such that if $U_1 \in G$, $U_2 \in G$, then $U_1 U_2 \in G$.

If we choose all possible $N \times N$ unitary matrices, we are back to $U(N)$.

$SU(2)$ $\rightarrow$ all possible $2 \times 2$ matrices with det 1

But, we can also refer $SU(2)$ by a subset of the $3 \times 3$ matrices by containing the no. of independent parameters - set of all $3 \times 3$ orthogonal matrices $3 \times 3$ real unitary matrices (spin 1 rep.) or spin $\frac{3}{2}$ rep. or $4 \times 4$ matrices.

Defining representation

(For exceptional groups, you can't have a defining rep.) - a rep. where you can put simple group elements close to identity.

$U = 1 - \frac{1}{k} \sum_{\alpha=1}^{k} T^\alpha$ collection of $N \times N$ hermitian matrices

where $k$: dimension of the group/algebra

$\{T^\alpha\}$: Basis of a $k$-dimensional real vector space.

A generic element of the vector space $\sum_{\alpha=1}^{k} \beta^\alpha T^\alpha$ where $\beta^\alpha$: real numbers.
For you can think of the \( \beta \)-parameters \( \beta_a \) as forming a \( k \)-dim. vector space which give a generic generator \( \sum \beta_a T_a \).

If \( T_a \in G \), \( T_b \in G \), we can show that
\[
[T_a, T_b] \in G.
\]

\[
[T_a, T_b] = i \sum_{c \in \text{ Real constants}} f_{abc} T_c
\]

It follows from the above fact that
If \( U_1 \in G \), \( U_2 \in G \),
then \( U_1 U_2 U_1^{-1} U_2^{-1} \in G \).

\[
U_1 = 11 - i \alpha_a T_a + O(\alpha^2)
\]
\[
U_2 = 11 - i \beta_a T_a + O(\beta^2)
\]

Calculate \( U_1 U_2 U_1^{-1} U_2^{-1} \rightarrow 11 + (\text{const})\alpha_a \beta_b + \text{higher order terms}

\[
[T_a, T_b] = i \text{ if } a \neq c.
\]

A generic element of \( G \) can be written as
\[
U = \exp (i \alpha_a T_a) \rightarrow \text{ real numbers}
\]
Claim: \( \forall U \in G \& \forall a \in A, U^a U^{-1} \in A \) 

This algebra doesn't contain all possible Herm. matrices.

Hence 
\[ U^a U^{-1} = \sum_{l=1}^{K} \rho_{ba} \] 
\( \Rightarrow \) Real numbers.

We proceed an analogous result for \( SU(n) \), where we had Herm. matrices.

If we take a \( G \)-valued function \( U(x) \), then \( i \not\in UU^{-1} \in A \)

\[ \Rightarrow i \not\in UU^{-1} = \sum a \alpha^a \tau_a \]

real numbers.

Proof of (i):

\[ U \left( \frac{1}{2} - i \epsilon a \right) U^{-1} \in G \]

\[ = \frac{1}{2} - \epsilon a \] if \( \tau_a \in A \)

\& \( U \in G \).

This implies 
\( U^a U^{-1} \in A \)

Proof of (ii):

Let \( \epsilon \) be an infinitesimal parameter

\& \( n^u \) be an arbitrary vector.

\( (U(x) + \epsilon n^u) U(x) + O(\epsilon^2) \not\in G \)

\( (U(x) + \epsilon n^u \frac{\partial}{\partial x} U(x) + O(\epsilon^2) U(x)^{-1} \not\in G \)

\( \Rightarrow U(x) \in G \)

\( U(x + \epsilon n^u) \in G \)

\( U(x + \epsilon n^u) U(x)^{-1} \in G \)
\begin{align*}
\text{Pair:} & \quad (U(x) + \epsilon n^\mu \partial_\mu U(x) + O(\epsilon^2)) U(x) \\
& = (\Phi(x) + \epsilon n^\mu \partial_\mu U(x) + O(\epsilon^2)) U(x) \\
& = i \epsilon x^\mu \gamma^\mu a + O(\epsilon^2) \tag{the whole thing is in the group and is infinitesimally close to identity} \\
\text{(This shows that)} & \quad i \epsilon x^\mu \gamma^\mu a = \chi^\mu \gamma^\mu a \\
& \text{for some real } \chi^\mu \gamma^\mu (x) \tag{real} 
\end{align*}

Introduce \( k \) vector fields \( B^a_\mu \) (where \( a = 1, 2, \ldots, k \)).

Gauge transformation of \( B^a_\mu \) under \( \psi \) is 
\( B^a_\mu (x) \rightarrow \text{R}_{\text{ab}} (\psi) B^b_\mu (x) - \chi^a_\mu (x) \)

\( \text{Gauge invariant action} \) (is constructed as follows) 

Define: 
\( B_\mu = \sum_{a=1}^k B^a_\mu \gamma^a \)

\( G_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i [B_\mu, B_\nu] \)

\( S_{\text{gauge}} = -\frac{1}{2g^2} \int dy^2 \text{Tr} (G_{\mu \nu} G^{\mu \nu}) \)

(Prove that \( S_{\text{gauge}} \) is gauge-invariant)

Proof: 
1. Show that \( U \) matrix:
   \( B_\mu \rightarrow U B_\mu U^{-1} + i \epsilon U B_\mu U^{-1} \)

(Multiply \( B^a_\mu \) by \( \gamma^a \) in \( B^a_\mu \rightarrow \text{R}_{\text{ab}} B^b_\mu - \chi^a_\mu \).)


\[ G_{\mu \nu} \rightarrow U \ G_{\mu \nu} \ U^{-1} \]

\[ \nabla \left( G_{\mu \nu} \right) \rightarrow \nabla \left( U \ G_{\mu \nu} \ U^{-1} \right) = \nabla \left( U \ G_{\mu \nu} \ U^{-1} \right) \]

Note:

\[ \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = \left[ U \frac{\partial}{\partial x^i} U^{-1}, U \frac{\partial}{\partial x^j} U^{-1} \right] - i \left[ \frac{\partial}{\partial x^i} U^{-1}, U \frac{\partial}{\partial x^j} U^{-1} \right] - \left[ U \frac{\partial}{\partial x^i} U^{-1}, \frac{\partial}{\partial x^j} \right] - i \left[ \frac{\partial}{\partial x^i} U^{-1}, \frac{\partial}{\partial x^j} U^{-1} \right] \]
Matter fields (fermions, scalars) belong to certain representation.

Under a gauge transformation by $U(2)$:

$$
\begin{pmatrix}
\Psi \\
\psi_m
\end{pmatrix} \rightarrow \mathcal{R}_g(U) \begin{pmatrix}
\Psi \\
\psi_m
\end{pmatrix}
$$

We need to take a covariant derivative - ordinary derivative won't work here.

Review of group representation

M-dimensional unitary representation.

For every $U \in G$, there is an $M \times M$ unitary matrix $\mathcal{R}_g(U)$ such that

$$\mathcal{R}_g(U_1) \mathcal{R}_g(U_2) = \mathcal{R}_g(U_1 U_2)$$

for $U_1 \in G$, $U_2 \in G$.

$$\mathcal{R}_g(1) = \begin{bmatrix} 1 \end{bmatrix}$$

For $U = 1 + i \epsilon \mathcal{T}^1$, a small parameter:

$$\mathcal{R}_g(1 + i \epsilon \mathcal{T}^1) = \begin{bmatrix} 1 \end{bmatrix} - \epsilon \mathcal{R}_g(T^1) \mathcal{R}_g(T^2)$$

This expression follows.
Similarly, define $R_A(T^2) = R_A(T)^2$.

Now, let us consider

$$R_{\mathcal{E}} \left( 1 - i \sum_{\alpha=1}^{k} n_{\alpha} T_{\alpha} \right)$$

if we neglect $O(e^2)$ term.

$$= R_{\mathcal{E}} \left( 1 - i e_{1} T^{1} \right) R_{\mathcal{E}} \left( 1 - i e_{2} T^{2} \right) \ldots$$

$$= (1 - i e_{1} R_{\mathcal{E}}(T^{1}) \left( 1 - i e_{2} R_{\mathcal{E}}(T^{2}) \right) \ldots$$

$$= (1 - i \sum_{\alpha=1}^{k} n_{\alpha} R_{\mathcal{E}}(T_{\alpha}))$$

But, $R_{\mathcal{E}}(1 - i \sum_{\alpha=1}^{k} n_{\alpha} T_{\alpha}) \rightarrow 1 - i e R_{\mathcal{E}}(\sum_{\alpha=1}^{k} n_{\alpha} T_{\alpha})$

Comparing these two, we get

$$R_{\mathcal{E}}(\sum_{\alpha} n_{\alpha} T_{\alpha}) = \sum_{\alpha} n_{\alpha} R_{\mathcal{E}}(T_{\alpha})$$

This is the analog of $R_{\mathcal{E}}(U_{1}) R_{\mathcal{E}}(U_{2}) = R_{\mathcal{E}}(U_{1} U_{2})$ (this is for the group).

Recall $[T_{\alpha}, T_{\beta}] = i f_{\alpha \beta \gamma} P_{\gamma}$ (sum over $(a, b, c)$)

$$[R_{\mathcal{E}}(T_{\alpha}), R_{\mathcal{E}}(T_{\beta})] = i f_{\alpha \beta \gamma} R_{\mathcal{E}}(T_{\gamma})$$
Proof:

\[ R_a(UVW) = R_a(U) R_a(V) R_a(W) \]

Take

\[ U = (1 + i \eta \alpha_a T^a) \]

\[ V = (1 + i \gamma \beta_a T^a) \]

\[ W = (1 - i \eta \alpha_a T^a) \]

\( \xi, \eta \) infinitesimally close to identity

We will get,

\[ UVW = 1 - i \gamma \alpha_a \beta_a [T^a, T^b] \]

\[ = (1 + i \gamma \alpha_a \beta_a) (1 - i \gamma \alpha_a \beta_a + i \gamma \alpha_a \beta_a + i \gamma \alpha_a \beta_a) \]

\[ = 1 - i \gamma \alpha_a \beta_a \left[ \alpha_a \beta_a [T^a, T^b] + \gamma \alpha_a \beta_a [T^a, T^b] \right] \]

\[ = 1 - i \gamma \alpha_a \beta_a \left[ R_a(T^a), R_a(T^b) \right] + i \gamma \beta_a R_a(T^b) \]

\( \xi, \eta \) terms can be compared on both sides - but not \( O(\xi^2) \) \( O(\eta^2) \)

Terms as we are neglecting them

Think of \( \xi \) \( \eta \) as independent parameters & \( \xi, \eta \) as if we have carried out Taylor series expansion retaining terms up to \( O(\xi) \) \( O(\eta) \) for the first step.

Then compare each term of the Taylor series expansion on both sides.

Note: \( U = 1 + i \gamma \alpha_a T^a \)

Without \( \eta \) dependence

Also,

\[ UVW = 1 + i \gamma \beta_a T^a - i \gamma \alpha_a \beta_a f_{abc} T^c \]

\[ \therefore R_a(UVW) \]

\[ = 1 + i \gamma \beta_a R_a(T^a) \]

\[ = 1 + i \gamma \beta_a R_a(T^a) \]

Compare two sides

\[ \Rightarrow \left[ R_a(T^a), R_a(T^b) \right] = i f_{abc} R_a(T^c) \]

(Proved)
If $UT^aU^{-1} = T^aR^a$, then

$$Ra(U)Ra(T^a)(Ra(U))^{-1} = Ra(T^a)Ra$$

(To prove this, follow the same strategy which we established it in the first place)

$$U(1 - i \epsilon \Delta T^a) U^{-1} = 1 - i \epsilon \Delta T^a R R^a$$

Use

$$Ra(U(1 - i \epsilon \Delta T^a) U^{-1}) = Ra(U)Ra(1 - i \epsilon \Delta T^a)(Ra(U))^{-1}$$

(Evaluate both sides & we will get the desired result)

$$Ra(U) (1 - i \epsilon \Delta R^a(T^a) (Ra(U))^{-1} = 1 - i \epsilon \Delta R^a(T^a) R R^a$$

Suppose $U(x)$ is a group valued function of $x$.

$$i \Theta U U^{-1} = \lambda^m(x) T^a$$

(Given this, we can show)

Then, $i \Theta A(Ra(U))(Ra(U))^{-1}$

We defined $\lambda^m(x)$ by this relation

$U(x)$ is a group valued function.

$Ra(U)$ is defined.

(Here comes the proof which we choose).

(To prove this, follow the steps we took)

Proof:

Let $n$ be a 4-vector.

$$U(x + \epsilon n^m) (U(x))^{-1}$$

($\lambda^m(x)$ is our starting point) $k = (U(x) + \epsilon n^m U(x)) U(x)$

$$= 11 + \epsilon n^m U(x) (U(x))^{-1}$$
Apply $R_{a}$ on both sides

\[ \begin{align*}
R_{a} (U_{a}e^{+en_{a}}) (U_{c}x^{-1}) & = R_{a} (U_{a} + e n_{a} \partial_{a} U_{c}x^{-1}) \\
& = R_{a} (I + e n_{a} \partial_{a} U_{c}x^{-1}) (U_{c}x^{-1}) \\
& = R_{a} (I - i e n_{a} \partial_{a} U_{c}x^{-1}) (T_{a})
\end{align*} \]

Now, \( R_{a} (I - i e n_{a} \partial_{a} U_{c}x^{-1}) (T_{a}) \)

\[ = I + i e n_{a} \partial_{a} U_{c}x^{-1} R_{a} (T_{a}) \]

Again, \( R_{a} (U_{c}e^{+en_{a}}) (U_{c}x^{-1}) \)

\[ \begin{align*}
& = R_{a} (U_{c}x + e n_{a}) R_{a} (U_{c}x^{-1})^{-1} \\
& = R_{a} (U_{c}x) R_{a} (U_{c}x^{-1})^{-1} \\
& = R_{a} (U_{c}) R_{a} (U_{c}x^{-1})^{-1} \\
& = I + e n_{a} \partial_{a} R_{a} (U) R_{a} (U)^{-1}
\end{align*} \]

Comparing, we get

\[ \partial_{a} R_{a} (U) R_{a} (U)^{-1} = -i \chi_{a}^{\mu}(x) R_{a} (T_{a}) \]

Note:

\[ \begin{align*}
\partial_{a} R_{a} (U) R_{a} (U)^{-1} & = R_{a} (U) \partial_{a} \Psi + 2 \partial_{a} R_{a} (U) R_{a} (U)^{-1} R_{a} (U) \Psi \\
& - i \partial_{a} R_{a} (U) \frac{B_{a}^{\mu}}{\gamma_{\mu}} R_{a} (U) \Psi \\
& = R_{a} (U) \partial_{a} \Psi
\end{align*} \]
Suppose $\psi$ is an $M$-component fermion such that
\[ \psi \rightarrow R_g(U) \psi \quad \text{under gauge transformation by } U(z) \]

Define
\[ D^a \psi = \partial^a \psi - i B^a(x) T^a \psi \]

in normal $SU(n)$ gauge theory.

**Hint:** $B^a_{\mu} T^a \rightarrow U B^a_{\mu} T^a U^{-1} - i \bar{\psi} \partial_{\mu} \psi \quad$ (the transformed field)

Show that
\[ B^a_{\mu} R^a_A (T^a) = R_g(U) B^a_{\mu} R^a_A (T^a) R_g(U) - i \bar{\psi} \partial_{\mu} \psi R_g(U) R_g(U) -1 \]

Ex. Prove this

(Note: This OPEF doesn't refer to the space-time spin, i.e., the spin of the particles)
Action (for the fermions)\[ \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi \] is gauge invariant.

Use \[ R_a (U) + R_a (U) = \mathbb{1} \]

(Ra's canceled by the unitarity of the reps.)

Suppose we have M complex scalars which transform as

\[ (\phi_1, \ldots, \phi_M) \rightarrow R_a (U) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_M \end{pmatrix} \]

Define:\[ m \phi = \partial_\mu \phi - i \bar{\psi} \Gamma_\mu \psi R_a (T^a) \phi \]

\[ -(D_\mu \phi)^+ D^\mu \phi - \frac{1}{2} m^2 \phi^+ \phi \]

is gauge invariant.

In a theory with a given gauge group, one can have many different fermions & scalars in different representations.

Because we saw that the transverse law for Bם was fixed once & for all - By appropriate coupling to Bם, we added fermions & scalars.
Example

Fermions $\Psi$ in $M$ dimensional representation $R$

Fermions $\Phi$ in $M$ dimensional rep $R'$

Scalar $\Phi$ in $M''$ " " $R''$

# $D_m \Psi = \partial_m \Psi - i B^a m_a R^a (T^a) \Psi$

# $D_m \phi = \partial_m \phi - i B^a m_a R^a (T^a) \phi$

Save $B^a$ couples to all of these

These are all gauge-invariant terms in the action

But there are not the only gauge inv. terms we can add to the action - we can also add $\chi (\phi^+ \phi)^2$ + $\alpha \sum_{a=1}^{12} \sum_{b=1}^{12} \sum_{c=1}^{12} C_{abc} \phi_a \phi_b \phi_c$

($\phi^+ \phi$ in gauge inv.)

Support, the gauge grp. $G = SU(2)$
Suppose \( Y \) is in spin \( 2j_1 + 1 \) rep
\( X \) is in spin \( 2j_2 + 1 \) rep
\( \phi \) is in spin \( 2j_3 + 1 \) rep

\( c_{abc} \) should be chosen such that the total spin is zero

\[
2j_1 + 1 \quad 2j_2 + 1 \quad 2j_3 + 1
\]

\[
|s_1 - s_2|, |s_1 - s_2| + 1, \ldots, |s_1 + s_2|
\]

We must have

\[
|s_1 - s_2| \leq j_3 \leq |s_1 + s_2|
\]

(C \( j_3 \) must be equal to one of these to get a spin-zero rep.)

\( c_{abc} \rightarrow \) Clebsh-Gordon coefficients

We must be able to combine any of these with \( j_3 \) to get a singlet or spin-zero rep.

Given any 2 reps, in the past, we know what rep. we get — take a prod., & see if we can get a singlet rep., which is gauge invariant. Can be done for any gauge group.
Product groups

Suppose $G_1$ is a group.
$G_2$ is a group.

Product group $G_1 \times G_2$ is defined as

\[ \{(U, V)\} \text{ such that } U \in G_1, V \in G_2. \]

(Product rule)

\[ (U_1, V_1)(U_2, V_2) = (U_1U_2, V_1V_2). \]

Gauge theories based on product group

Introduce gauge fields $B^a_\mu$ (a=1,...,K)

Dimension of the first group, i.e., $G_1$

\[ C^a_\mu \text{ } (a = 1, 2, ..., K_1) \]

Dimension of the second group, i.e., $G_2$

\[ L^a_\mu \text{ } (a = 1, 2, ..., K_2) \]

Suppose $T^a_1, ..., T^a_{K_1}$ are the generators of $G_1$

& $L^a_1, ..., L^a_{K_2}$ " " " " of $G_2$

Define $B^a_\mu = B^a_\mu T^a, C^a_\mu = C^a_\mu L^a$
Gauge field laws:

Under \((U(x), G(x)) \in G_1 \times G_2\):

\[ B^m_\mu \rightarrow U B^m_\mu U^{-1} - i \partial^\mu U U^{-1} \]
\[ C^m_\mu \rightarrow V C^m_\mu V^{-1} - i \partial^\mu V V^{-1} \]

Gauge invariant action:

\[ \text{Define: } \quad G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu - i [B^\mu, B^\nu] \]
\[ H^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu - i [C^\mu, C^\nu] \]

The gauge invariant action:

\[ -\frac{1}{2g_1^2} \int d^4x \text{ Tr } \left( G^{\mu\nu} G_{\mu\nu} \right) \]
\[ -\frac{1}{2g_2^2} \int d^4x \text{ Tr } \left( H^{\mu\nu} H_{\mu\nu} \right) \]

Where

\( g_1, g_2 \) are arbitrary constants

Fields in one part don't couple with fields in another. It is a sum of 2 indep. actions. But once we introduce matter fields, they can couple to both. Pure gauge fields don't have int.

- just put the results together.
Representation of product groups

(We must have a single matrix, not a pair of matrices)

Suppose \( R \) is an \( M \)-dimensional rep. of \( G_1 \) & \( S \) is an \( N \)-dimensional rep. of \( G_2 \).

\[ \forall \ U \in G_1, \text{ we have an } M \times M \text{ matrix } R_{G_1}(U) \]

\[ (R_{G_1}(U))_{mn} \quad m,n = 1,2,\ldots,M \]

Similarly, \( \forall \ V \in G_2, \text{ we have an } N \times N \text{ matrix } S_{G_2}(V) \)

\[ (S_{G_2}(V))_{\sigma\tau} \quad \sigma,\tau = 1,2,\ldots,N \]

Construct an \( MN \times MN \) matrix:

\[ (R_{G_1 \times G_2}(U,V))_{m\sigma,\ n\tau} \]

this pair can take \( MN \) values

\[ (R_{G_1}(U))_{mn} (S_{G_2}(V))_{\sigma\tau} \]

Need to prove:

\[ (R_{G_1 \times G_2}(U_1,V_1)(U_2,V_2))_{m\sigma,\ n\tau} = R_{G_1 \times G_2}(U_1V_1) R_{G_1 \times G_2}(U_2,V_2)_{m\sigma,\ n\tau} \]
Prove this (using the defns).

\[ R((u, v)) = R(u) \circ S(v) \quad \rightarrow \text{this is (2.6) irreducible} \]

\[ R(u, v) = R_{\alpha_1}(u) \]

\[ R_{\alpha_1}(u, v) = S_{\alpha_2}(v) \quad \text{when all the } S's \text{ have been mapped to identity} \]

Suppose we have a fermion

\[ \psi_{m\sigma} \quad m = 1, \ldots, M \]

\[ \sigma = 1, \ldots, N \]

\[ (\mathcal{D}_\mu \psi)_{m\sigma} \quad \text{(3.24)} \]

\[ \mathcal{D}_\mu \psi = \partial_\mu \psi_{m\sigma} - i \epsilon_B^{a\alpha} (R_{\alpha_1}(T^a))_{mn} \psi_{n\sigma} \]

\[ - i \epsilon^{\alpha \beta} S_{\alpha_2}(L^\beta) \psi_{m\sigma} \]

We can show that

\[ (\mathcal{D}_\mu \psi)_{m\sigma} \rightarrow (R_{\alpha_1 \times \alpha_2}(u, v))_{m_0, n} \psi_{n_2} \]

Then \( F(i \gamma^\mu \mathcal{D}_\mu - m) \psi \) will be gauge-invariant.

So, \( B_{\mu} \) and \( C \) are not decoupled any more.
Pure gauge theory based on gauge group \( G \)

Fields: \( B^a \quad a = 1, \ldots, K \)

\( \gamma \) runs over the generators of the gauge group

Define: \( B^a_{\mu} = B^a_{\mu} \Phi^a \rightarrow \) generators of \( G \)

\[
\tilde{g}^{a\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - i [ B^a_\mu, B^a_\nu ] = \tilde{g}^{a\nu} \Phi^a
\]

\[
\tilde{g}^{a\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc} B^{b\mu} B^{c\nu}
\]

Gauge field action

\[
S_{\text{gauge}} = -\frac{1}{2g^2} \int \text{Tr} \left( \tilde{g}^{a\nu} \tilde{g}^{a\mu} \right)
\]

\[
= -\frac{1}{2g^2} \text{Tr} \left( \Phi^a \Phi^b \right) \int d^4x \, \tilde{g}^{a\nu} \tilde{g}^{a\mu}
\]

Convention: \( \text{Tr} \left( \Phi^a \Phi^b \right) = \frac{1}{2} \delta^{ab} \rightarrow \text{Achieved by choice of } \Phi^a \)

\[
S_{\text{gauge}} = \frac{1}{4g^2} \int G^{a\nu} G^{a\mu} d^4x
\]

\( B^a_{\mu} = g A^a_{\mu} \rightarrow \) defines \( A^a_{\mu} \)

\[
S_{\text{gauge}} = -\frac{k}{2g}
\]

\[
F^{a\mu\nu} = \partial_{(\mu} G^{a\nu)} - \partial_{\nu} G^{a\mu}
\]

\( [\hat{T}^a, \hat{T}^b] = \delta^{ab} \delta \rightarrow \) defines \( \hat{T}^a \)

\( \delta \) depends on the particular choice of basis - on the particular \( \hat{T}^a \)'s we have chosen
\[ F_{\mu \nu} = \frac{1}{8} \varepsilon_{\mu \nu \alpha \beta} F^{\alpha \beta} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A_\mu A_a A_b \]

\[ S_{\text{gauge}} = -\frac{1}{4} \varepsilon_{\mu \nu \alpha \beta} F^{\mu \nu} F^{\alpha \beta} \]

\[ = S_{\text{free}} + S_{\text{int}} \]

(contains quadratic piece)

(contains cubic & higher order piece)

where,

\[ S_{\text{free}} = -\frac{1}{4g} \int dx \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha \right) \]

\[ S_{\text{int}} = -\frac{1}{4g} \int dx \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) f^{abc} \left( A_\alpha A_\beta - \varepsilon_{\alpha \beta \gamma} A_\gamma A^{\alpha \beta} \right) \]

\[ -\frac{1}{4g} \int dx f^{abc} f^{de} \left( A_\alpha A_\beta - \varepsilon_{\alpha \beta \gamma} A_\gamma A^{\alpha \beta} \right) A_\alpha A_\beta A_\gamma A_\delta \]

\[ \text{Gauge transformation} \]

\[ B_\mu \rightarrow B_\mu + \frac{i}{g} \left[ \pi_\mu B^a \right] (x) - \frac{1}{2} \chi^{a}_{\mu} (x) \]

where \( B_\mu (x) \) is defined through

\[ U(x) \pi^a U^{-1}(x) = R^a_{\mu}(x) \]

\( \chi^{a}_{\mu} (x) \) is defined through

\[ \chi^{a}_{\mu} (x) = \partial_\mu \omega^a (x) \]

\[ U = 1 - i \frac{g}{2} \int \frac{d^4x}{(2\pi)^4} \pi^a \left( x \right) \]

\[ \rightarrow \text{infinitesimal gauge transformation} \]
(We have explicitly taken \( g' \) out — just a choice of normalization for convenience)

**Ex.** Show that

\[
\mathcal{L}(\phi) = \delta \phi + g \int \text{d}^4x \, \epsilon^{abc} \partial_a \phi \partial_b \phi \partial_c \phi
\]

\[
\phi(x) = \int \frac{d^4k}{(2\pi)^4} \phi(k) e^{ikx}
\]

**Ex.**

\[
B^a \rightarrow B^a + 8 B^a
\]

where

\[
8 B^a = g \int \text{d}^4x \, \epsilon^{abc} \partial_b B^c \partial_c \phi
\]

Consequently,

\[
A^a \rightarrow A^a + 8 A^a
\]

where

\[
8 A^a = \frac{1}{g} \int \text{d}^4x \, \epsilon^{abc} \partial_b \phi \partial_c \phi
\]

- **Gauge invariance of \( S \)**
  
  \( S S = 0 \) under this transformation

**Naive path integral quantization**

\[
A^a(\vec{x}) = \int \frac{d^4k}{(2\pi)^4} \tilde{A}^a(x) e^{ikx}
\]

\[
S_{\text{free}} = \int \frac{d^4k}{(2\pi)^4} \tilde{A}^a(\vec{k}) \left( \hat{M}^a_{\mu\nu} + \frac{i}{\hbar} \partial^a \partial_\mu \right) \tilde{A}^a(\vec{k})
\]

\[
\langle \tilde{A}^a(\vec{k}_1), \tilde{A}^a(\vec{k}_2) \rangle = \delta_{ab} \frac{1}{(2\pi)^4} \delta^{(4)}(\vec{k}_1 + \vec{k}_2)
\]

\[
= \frac{1}{(2\pi)^4} \delta^{(4)}(\vec{k}_1 + \vec{k}_2)
\]

So we will face the same problem as we faced in quantizing Maxwell action.
[We should think \( M_{\mu\nu}(k) \) as a \( 4k \times 4k \) matrix — we have \( 4k \) fields]

But \( (M(k^2))^{-1} \) does not exist because \( M_{\mu\nu}(k) \) has a zero eigenvalue.

\[
-k^2 \eta_{\mu\nu} + \kappa^2 \eta_{\mu\nu} k_\nu = 0
\]

(This \( 4 \times 4 \) matrix has an e.value 0 with e.vector \( k \nu \))

**Physical origin of the zero eigenvalue**

In the \( g \to 0 \) limit, gauge transformation is

\[
\delta A_\mu(x) = \alpha_\mu \rightleftharpoons \frac{\delta A_\mu(x)}{2\pi^2} \rightleftharpoons \int \frac{dk}{2\pi^2} e^{ikx} \delta A_\mu(k)
\]

\[
\delta S_{free} = 0 \text{ under this}
\]

(The origin of the zero e.value is related to the gauge invariance of action)
Why does zero eigenvalue cause divergence?

Consider a finite dimensional integral

\[
\langle f(x) \rangle = \frac{\sum_{i=1}^{n} \delta i \cdot e^{-\lambda_i x_i^2}}{\sum_{i=1}^{n} \delta i \cdot e^{-\lambda_i x_i^2}}
\]

Some polynomial in \( x_1, \ldots, x_n \)

\( A = U^T A_2 U \) where \( A_2 = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \)

& \( U^T U = I \)

Orthogonal matrix

Define \( \tilde{y} = U x \)

\[
\langle f(y) \rangle = \frac{\sum_{i=1}^{n} \delta i \cdot e^{-\sum_{i=1}^{n} \lambda_i y_i^2}}{\sum_{i=1}^{n} \delta i \cdot e^{-\sum_{i=1}^{n} \lambda_i y_i^2}}
\]

(After change of variables)

\[
[e^{-x^T A x} = e^{-y^T A_2 U x} = e^{-y^T A_2 x} = e^{-\sum \lambda_i y_i^2}]
\]

On \( \mathbb{R}^n \) also, we have to do this kind of integral except that you must do an infinite no. of integrals instead of a finite no.

If \( \lambda_i \neq 0 \) & \( \lambda_i > 0 \), integral is well-defined.

If \( \lambda_i < 0 \) for some \( i \), we can still define it by analytic continuation.
\[
\frac{\int e^{-\lambda y^2} y^2 \, dy}{\int e^{-\lambda y^2} \, dy} = \text{(constant)} \times \frac{1}{\sqrt{\lambda}} \quad \text{for } \lambda > 0
\]

For $\lambda < 0$, we can have
\[
\int e^{-\lambda y^2} \, dy = (\sqrt{\pi}) \times \frac{1}{\sqrt{-\lambda}} \quad \text{for } \lambda < 0
\]

But we can't make it work with $\lambda = 0$ --> divergence

Num. will be more div. than den. if
if $f$ is $y^1$-dep. & is some polynomial of $y^1$ -- no hope to get rid of div.

\[
\text{If } \lambda = 0, \text{ all } x^i \text{ has the same value for all } y^i \text{ at fixed } y^2 \Rightarrow y^1 \text{ integral diverges}
\]

\[
\text{If the action has some symmetry, action remaining constant along a dirn. in the configuration space, ind. of action along that dirn. isn't damped}
\]

\[
\text{If the action has some symmetry, the operator whose correlators for } f \text{ are calculated, is unchanged in the sym. dirn., the div. cancels out -- i.e., the operator must be gauge-inv. & should have the symmetries of the action.}
\]

On this case, the sym. is gauge inv., & the op. should then also be ""'

Conclusion \rightarrow In order to get sensible answer for correlation functions in a gauge theory, we must consider correlation functions of gauge invariant operators. [gauge inv. op. are the only observables in gauge theories] these are the only observables in the theory.
(This resolves the conceptual problem — what is the cause & we must do to resolve the problem (insider only gauge inv. obs.) Practical diff. has to develop our pert. theory to calculate correct form of gauge inv. objects like \( G^a_{\mu
u}(x) \).

2-pt. fn. like \[ \langle G^a_{\mu}(x) G^b_{\lambda}(y) \rangle \]
can be calculated Lorentz indices should be same.

---

**Basic Strategy**

- Gauge orbit (value of action remains the same as this orbit)
- Gauge slice

**Configuration Space**

Coordinate space

Some complicated curve — all pts. on this curve are related by gauge transformation.

Problem comes from the integral along a orbit. Locally action has the same value on each orbit only once. Curve should cut each orbit same no. of times — even if it cuts each orbit twice, it isn't a problem.

Choose one pt. from each orbit — can take curve which intersects each orbit only once.

If equiv. do not perform the \( y^2 \)-integral. Take integral over the gauge slice.
Instead of a 2-D int. in this case, we are restricting to a 1-D int. along gauge slice — avoiding the extra dim.

\[ \int I = \frac{\int dx \, dy \, e^{-2y^2} y^2}{\int dx \, dy \, e^{-2y^2}} \]

(Both num. & den. are well-defined)

Symmetry \( x \to x + a \)

\[ y = f(x) \]

\[ f'(x) > 0 \]

(Any monotonically \( f \) will do)

\( y = f(x) \)

\[ f'(x) \]

\( (y - f(x)) \)

(We will force \( y \) to be equal to \( f(x) \) & restrict to a 1-D integral)

\[ \text{Insert } \delta(y - f(x)) \]

\[ I = \frac{\int dx \, dy \, e^{-2y^2} y^2 \delta(y - f(x))}{\int dx \, dy \, e^{-2y^2} \delta(y - f(x))} \]

\[ = \frac{\int dx \, e^{-\lambda(f(x))^2} (f(x))^2}{\int dx \, e^{-\lambda(f(x))^2}} \]
(This has problem b/c the final ans. depends on the choice of \( f(x) \) — although, \( f(x) \) is \( \text{m.n.} \), the final ans. is finite) \( \Rightarrow \) so for each gauge choice, \( f(x) \) only once.

\[
\text{Put } u = f(x), \quad x = g(u) \\
\therefore \quad dx = g'(u) \, du
\]

\[
\int_{-\infty}^{\infty} du \, g'(u) \, e^{-u^2} = \int_{-\infty}^{\infty} du \, g'(u) \, e^{-u^2}
\]

So the procedure is almost right, but not completely.

The whole meaning of gauge inv. is that the ans. should not depend on which gauge slice we choose — of course we take a gauge fix & calculate, but the ans. should be gauge choice indep.

(By gauge choice, we break manifest gauge inv. — I is no longer inv. under \( x \rightarrow x + a \))

(Take \( f(x) = x \) & \( f(x) = x^3 \) for another & see what happens)

We will show that the original integral can be manipulated to give

\[
\int_{-\infty}^{\infty} dy \, e^{-y^2} \, f_1(y) \, \delta(y - f(x)) = \int_{-\infty}^{\infty} dy \, e^{-y^2} \, f_1(y) \, \delta(y - f(x))
\]
\[
\begin{align*}
&= \frac{\int dx \ e^{-\lambda (f(x))^2} \ f'(x)}{\int dx \ e^{-\lambda f(x)^2} \ f'(x)} \\
&= \frac{\int du \ e^{-\lambda u^2} \ u^2}{\int du \ e^{-\lambda u^2}} \\
&= \frac{\int dx \ dy \ e^{-\lambda (x^2+y^2)} \ (x^2+y^2)}{\int dx \ dy \ e^{-\lambda (x^2+y^2)}} \\
&= \frac{\int dx \ e^{-\lambda x^2} \ x^2}{\int dx \ e^{-\lambda x^2}} \\
\text{Original integral} &= \frac{\int S^2 dr \ dr \ e^{-\lambda r^2} \ r^2}{\int S^2 \ e^{-\lambda r^2} \ dr} \\
&= \frac{\int S^2 \ e^{-\lambda r^2} \ dr}{\int S^2 \ e^{-\lambda r^2} \ dr}
\end{align*}
\]
We haven't taken into account that diff. gauge or bit has diff. vol. -- have to account for it to get the correct answer.