\[ S_{\text{tot}} = S + S_{g.f.} + S_{\text{ghost}} \]

gauge fixing term

**Example:** Take the gauge fixing \( \lambda \rightarrow \alpha \frac{\lambda}{\sqrt{2}} \)

\[ S_{\text{tot}} = -\frac{1}{4} \int d^4x \, F_{\mu\nu}^a(x) \cdot F^{\mu\nu}_{\alpha} \]

\[ -\frac{1}{2\alpha} \int d^4x \, \partial^\mu A^\alpha_m (x) \cdot \partial^\mu A^\alpha_k (x) \]

\[ + \int d^4x \, b_a(x) \times \partial^\mu c_a(x) \]

\[ + \int f_{abc} \lambda d^4x \]

\[ + S_{\text{fermion + scalar}} + \text{coupling between fermions \& scalars} \]

\[ F_{\mu\nu}^a = \partial^\mu A^a_v - \partial^\nu A^a_v + g f_{abc} A^b \wedge A^c \]

**Step 1:** Count dimension of each term:

\[ S_{\text{free}} = -\frac{1}{4} \int d^4x \, \left( \partial^\mu A^\alpha_v - \partial^\mu A^\alpha_m \right) \cdot \left( \partial^\mu A^\alpha_v - \partial^\mu A^\alpha_m \right) \]

\[ \text{dimension } 4 \]

\[ \Rightarrow A^\alpha_v \text{ has dimension } 1. \]

(The term \( \int d^4x \, b_a(x) \partial^\mu c_a(x) \) tells us)

\[ b_a \text{ has dimension } 1+\lambda \]

\[ c_a \text{ has dimension } 1-\lambda \]

We'll choose \( \lambda = 0 \)

(for convenience)

This is allowed because everywhere we have the scale of \( b_a \) & \( c_a \).
\[ F_{\mu \nu} = 2 \partial_{\mu} A_{\nu} - 2 \partial_{\nu} A_{\mu} + g f_{abc} (\partial_{\mu} A^a - \partial_{\nu} A^a) A^b A^c \]

\[ + g^2 f_{abc} f^{d'bc} A^a A^b A^c A^d' \]

(Each term has dimension 4)

All terms have dimension 4 operators

Since all coupling constants have dim. \( \geq 0 \)

\( \Rightarrow \) so all ops. have dim. \( \leq 4 \)

\( \Rightarrow \) First condition of power counting
renormalizability is satisfied.

But terms like \( \int dx A^a A^a \), \( \int dx b^a c^b A^a A^c \) are not there.

(2 Gauge field prop.)

This has an intrinsic div. \( \nabla \cdot \) 

Not subdivergences which could be cancelled by other counterterms.
The 3-pt. coupling (e.g., \(g\)) of gauge fields is related to the 4-pt. coupling (\(g^2\)) of gauge fields is made finite by adjusting \(2g \tilde{z}_A^{3/2}\).

\[ \text{requires adjusting } \tilde{z}_A \]

(This diagram is quadratically divergent — not only the leading term, but also the term with first derivative w.r.t. \(k\) is div. in the Payton series expr.

\[ \tilde{z}^{1/2} \tilde{z}^{1/2} \tilde{z}_A \]

requires adjusting \(\tilde{z}_A \tilde{z}_g \tilde{z}^{2/2} \tilde{z}_A\) (because we had \(g^2\)).

(But this has already been dealt with making the lower order diag. finite).

\[ \text{requires adjusting } \tilde{z}^{1/2} \tilde{z}^{1/2} \tilde{z}_A \tilde{z}_g \tilde{z}^{2/2} \tilde{z}_A \]

\[ \text{not adjustable} \]
The problem becomes more complicated if we include fermions, scalars, etc.

Consider fermions $\Psi, \bar{\Psi}$ with mass $m$.

Require adjusting $\bar{\Psi} \gamma_4, \bar{m} \bar{\Psi}$

$\bar{\Psi} \gamma_4, \bar{m} \bar{\Psi}$

$\bar{\Psi} \gamma_4 \bar{\Psi} \approx 1/2 \bar{\Psi} \gamma_4$

$(\bar{\Psi} \gamma_4 \bar{\Psi})$ can no longer be independently adjusted

So no need to consider them separately.

But our original gauge in.

Lagrangian had all of

with dim. $< 4$ — the problem is that we are not quantising that action directly — we have broken manifest gauge in. by gauge fixing, adding ghost fields, etc.

Our Feyn. rules don't have the original gauge in., we began with — the fact that original 3-ft. for. was related 4-ft. for., etc. is no longer manifest.

Want identities that we derive from sym. various

Green's ft. by certain eqns — so if LHS finite

RHS must be.
We will show that the gauge fixed action is invariant under certain symmetry transformations, known as **BRST symmetry**.

Symmetry parameter $\xi$ is Grassmann number.

So it will anti-commute with $i, \gamma, \partial, \psi, \xi$ etc.

(not allowed to be function of space-time coord., it's a single no.)

Feynman classes of gauge fields, scalars & fermions are identified to infinitesimal transformations of these fields under a gauge transformation with parameter $\xi \phi^a(x)$.

\[ S_\xi \phi^a(x) = -\frac{1}{2} \int d^4x \, \xi \phi^a(x) \phi^b(x) \]

\[ = \frac{i}{2} \int d^4x \, \xi \phi^a(x) \phi^b(x) \]

\[ \text{if we choose} \]

\[ S_{\text{g.f.}} = -\frac{1}{2} \int d^4x \, F^a(x) \cdot F_a^b(x) \]

\[ \text{for fields of various fields} \]

\[ e.g. \rightarrow D^a \phi^a(x) \]

\[ S_{\text{g.f.}}(x) = -\frac{1}{2} \int d^4x \, F_a(x) \phi^a(x) \]

Claim: \[- S \left( S + S_{\text{g.f.}} + S_{\text{ghost}} \right) = 0 \]

(invariant part)
Problem Set 2: Date Due: October 20, 2006

1. Consider an action for a scalar field $\phi$ coupled to a fermion field $\psi$:

$$S = \int d^4x \left[ -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 + \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \lambda \bar{\psi} \gamma^\mu \psi \partial_\mu \phi \right]$$

(a) Derive the Feynman rules for this theory.
(b) Using these Feynman rules calculate

$$\langle \bar{\psi}_{\alpha_1}(k_1) \bar{\psi}_{\alpha_2}(k_2) \psi_{\beta_1}(p_1) \psi_{\beta_2}(p_2) \rangle_c$$

to order $\lambda^2$. Here $\langle \rangle_c$ denotes connected Green’s function.

2. In a non-abelian gauge theory based on the gauge group $G$, consider a gauge transformation of the gauge field $B^a_\mu$ by the group valued function $U_2(x)$, followed by another gauge transformation by the group valued function $U_1(x)$. Show that this is equivalent to transforming the original gauge field by the group valued function $U_1(x)U_2(x)$. 

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Note: All external lines are connected for each of them.
Problem Set 3: Date Due: November 25, 2006

1. Derive the Feynman rules of quantum electrodynamics if we choose the
   gauge fixing term in the action to be
   \[-\frac{1}{2\alpha} \int d^4x \ H(x, A) H(x, A)\]
   where
   \[H(x, A) = \partial ^\mu A_\mu + A_\mu A_\mu.\]

2. Consider a parametrization of the SU(2) group element as
   \[U = \exp(i\alpha \sigma_3) \exp(i\beta \sigma_1) \exp(i\gamma \sigma_3)\]
   where \(\sigma_1, \sigma_2\) and \(\sigma_3\) are Pauli matrices. Find an expression for the
   Haar measure of the group in terms of the parameters \(\alpha, \beta, \gamma\).

3. Consider a field theory of a scalar field \(\phi\) and a fermion field \(\psi\) with action:
   \[S = \int d^4x \left[ -\frac{1}{2} \eta ^{\mu \nu} \partial _\mu \phi \partial _\nu \phi - \frac{1}{2} M^2 \phi^2 + \bar{\psi} (i\gamma ^\mu \partial _\mu - m) \psi \right]\]
   (a) Now introduce a new fermion field \(\chi\) through the relation:
       \[\psi = e^{i\lambda \phi} \chi\]
       and express the action in terms of the fields \(\phi\) and \(\chi\). Here \(\lambda\) is a
       constant.
   (b) Now forget about the original action and work with this new action regarding \(\phi\) and \(\chi\) as independent fields, and \(\lambda\) as a small parameter. To order \(\lambda\), calculate the S-matrix element relevant
       for computing the decay of a \(\phi\) particle into two \(\chi\) particles. Assume that \(M > 2m\) so that this decay is energetically possible.

4. Consider a field theory of a fermion field \(\psi\) and a scalar field \(\phi\) with action:
   \[S = \int d^4x \left[ -\frac{1}{2} \eta ^{\mu \nu} \partial _\mu \phi \partial _\nu \phi - \frac{1}{2} M^2 \phi^2 + i\bar{\psi} \gamma ^\mu \partial _\mu \psi + g \bar{\psi} \psi \phi \right]\]
   where \(g\) is a coupling constant. Determine the minimal set of other terms you will need to add to the action to make the theory renormalizable.
12/11/06

\[ S_{\text{total}} = S + S_{g.f.} + S_{\text{ghost}} \]

Consider a gauge theory with gauge group \( G \) & fermions \( \psi \) in representation \( R_f \), and scalars \( \phi_k \) in representation \( R_s \).

**Infinitesimal gauge transformation:**

\[
\delta A^a_{\mu}(x) = -\partial_\mu \Theta^a(x) - g f^{abc} \phi_b(x) A^c_\mu(x) \]

\[
\delta \psi(x) = -ig \gamma^\mu \Theta^a(x) (R_f(\Gamma))_a^b \psi^b(x) + \psi^b(x)
\]

\[
\delta \phi_k(x) = -ig \Theta^a(x) (R_s(\Gamma))_a^k \phi_k(x)
\]

Infinitesimal gauge transformations together give the symmetry of \( S \).

Choose

\[
S_{g.f.} = \frac{1}{2g} \int d^4x \, F^a(x) \cdot F^a(x)
\]

(Repeated indices are summed over)

Some fields, e.g., \( \partial_\mu A^a(x) \) (for Lorentz gauge)

Claim: \( S_{\text{total}} \) has a BRST symmetry with Grassmann valued symmetry transformations \( S \).
\[ S_{Y}^{\alpha}(x) = -ig \mathcal{F}^{\alpha \beta}(x) \Phi^{\beta}(x) \]
\[ S_{\Phi}(x) = -ig \mathcal{F}^{\alpha \beta}(x) \mathcal{F}^{\beta \gamma}(x) \Phi^{\alpha}(x) \]
\[ S_{\Phi_{g}}(x) = -1/2 \alpha \mathcal{F}^{\alpha \beta}(x) \Phi^{\alpha}(x) \]

The term \( S_{\Phi_{g}}(x) \) is a fixed term, i.e., it's not gauge algebra.

Both LHS & RHS are Grassman even-odd.

Grassman prop. are okay.

\[ \delta S = 0 \]
\[ \delta S_{gf} = -1/2 \alpha \int d^4x \Phi^{\alpha}(x) \delta \mathcal{F}^{\alpha \beta}(x) \mathcal{F}_{\alpha \beta}(x) \]
\[ = -1/2 \alpha \int d^4x \Phi^{\alpha}(x) \sum \frac{S_{\Phi_{g}}^{\alpha}(x)}{S_{\Phi_{g}}^{\alpha}(y)} \delta \mathcal{F}^{\alpha \beta}(x) \]
where $F_a^\phi(x)$: Transform of $F_a(x)$ under a gauge transformation $\phi$. 

$S_{\text{ghost}} = - \int d^4x \, d^4y \, \delta^a(x) \frac{\delta F_a^\phi(x)}{\delta \delta^a(y)} \left. c^\phi(y) \right|_{\phi = 0}$

$S_{S_{\text{ghost}}} = - \int d^4x \, d^4y \, \delta^a(x) \frac{\delta F_a^\phi(x)}{\delta \delta^a(y)} \left( \frac{\delta F_a^\phi(x)}{\delta \delta^a(y)} \right) c^\phi(y)$

Hence, $S_{\text{total}} = - \int d^4x \, d^4y \, \delta^a(x) \left( \delta \left. \frac{\delta F_a^\phi(x)}{\delta \delta^a(y)} \right|_{\phi = 0} \right) c^\phi(y)$

It's a functional derivative, so we have an integral matrix in the $x$-$y$ space as well as in the $y$-$\mu$ space.

$F_a^\phi(x)$ is actually an infinite no. of $\phi$.
\[ \delta S_{\text{total}} = - \int d^4x \, d^4y \, b^0(x) \int \frac{S}{\delta \phi^2} \left( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \right) \bigg|_{\phi=0} \]

\[ + \frac{8 F_a^0(x)}{8 \phi^2(y)} \bigg|_{\phi=0} \left( - \frac{1}{2} g f_{abc} g_c^0(y) c^d(y) c^d(y) \right) \]

where \( \left( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \right) \bigg|_{\phi=0} \) is the transform of \( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \) under a gauge transformation, \( \phi \).

In general, single and double derivatives are not related.

But commutator of gauge transformations = a single gauge transformation.

\[ \delta \left( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \right) \text{ involves 2 successive gauge transformations.} \]

\[ \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \text{ involves a single derivative.} \]

Both \( \delta \left( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \right) \) and \( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \) are single gauge transformations.

\[ \delta \left( \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \right) \text{ and } \frac{\delta F_a^0(x)}{\delta \phi^2(y)} \bigg|_{\phi=0} \text{ are not related.} \]
(The commutator of the \( \mathfrak{g} \) double deriv. can be regarded as a single deriv.)

Given any quantity \( \Omega \),

\[
\int \left[ \frac{\delta}{\delta \phi(x)} \left( \frac{\delta \phi(y)}{\delta \phi(x)} \right) \right] \phi = 0
\]

localized at \( x \)

\[
\int \left[ \frac{\delta}{\delta \phi(y)} \left( \frac{\delta \delta \phi(x)}{\delta \phi(x)} \right) \right] \phi = 0
\]

localised at \( y \)

\[
= \int d^4y \left[ \frac{\delta \phi(y) \delta \phi(y)}{\delta \phi(x)} \right] \phi = 0
\]

for some

\[
\int \delta \phi(y) \delta \phi(x) \text{ independent of } \phi
\]

but depends

\[
\phi(x) \neq 0, \; \gamma, \; \tau, \; b, \; d
\]

We are trying to prove these things in general irrespective of the choice of \( F_a(x) \) — otherwise we could, for e.g., directly put \( F_a = \mu A a \) & show that the terms cancel.

The simplest choice of \( \phi \) is fermion fields because it doesn't involve deriv. — for gauge fields, it is.

\( \phi(x) \) involves derivatives.
Take \( \Omega = \gamma^\mu (v) \)

\[
-\frac{\delta}{\delta \phi_i (x)} \delta \phi_i (x) = \gamma (R_\phi (T \phi))_{\text{int}} \psi^\mu (v) \] 

\[
\frac{\delta}{\delta \phi_i (x)} \bigg|_{\phi = 0} \left[ \frac{\delta \phi_i (x)}{\delta \phi_j (y)} \right]_{\phi = 0} = -g^2 \left[ R_\phi (T \phi), R_\phi (T \phi) \right]_{\text{int}} \psi^\mu (v) \delta^{(n)} (v-z) \delta^{(n)} (v-y) 
\] 

\[
\frac{\delta}{\delta \phi_i (x)} \left( \frac{\delta \phi_i (x)}{\delta \phi_j (y)} \right)_{\phi = 0} = -g^2 \left[ R_\phi (T \phi), R_\phi (T \phi) \right]_{\text{int}} \psi^\mu (v) \delta^{(n)} (v-z) \delta^{(n)} (v-y) 
\]
\[ = \int f(a) e^{-ig \mathcal{R}_\alpha(T^\alpha_{st'}(u))} \delta^{(w)}(u-z) \]

\[ \delta^{(v)}(u-w) \]

\[ \text{it is a gauge transformation with} \]

\[ \epsilon^a(u) = \int f(a) \delta^{(w)}(u-z) \delta^{(v)}(u-y) \]

\[ \epsilon^a(u) = \int f(a) \delta^{(w)}(u-z) \delta^{(v)}(u-y) \delta^d(x) \delta^d(y) \]

\[ \text{(as per requirement)} \]

\[ \frac{\delta^4 \epsilon^a(u)}{\delta \theta^a(w)} = -i \frac{\mathcal{R}_\alpha(T^\alpha_{st'}(u))}{\delta \theta^a(w)} \delta^{(v)}(u-w) \]

\[ \text{with } \epsilon^a(u) = \int f(a) \delta^{(w)}(u-z) \delta^{(v)}(u-y) \delta^d(x) \delta^d(y) \]

\[ \epsilon^a(u) = \int f(a) \delta^{(w)}(u-z) \delta^{(v)}(w-y) \]

\[ (\text{we can choose } \epsilon^a(u), \eta^a(y) \text{ equal to Dirac delta and Kronecker delta}) \]

\[ \int \left( \frac{\delta}{\delta \theta^a(0)} \right) \left( \frac{\delta}{\delta \theta^a(y)} \right) \phi = \int \left[ \frac{\delta}{\delta \theta^a(0)} \left( \frac{\delta}{\delta \theta^a(0)} \right) \phi \right] = \int \left( \frac{\delta}{\delta \theta^a(0)} \right) \phi = \int \epsilon^a(u) \]

\[ = \int d^4w \left( \frac{\delta^4}{\delta \theta^a(w)} \right) \phi \]

\[ \text{satisfies} \]

\[ \int \left( \frac{\delta}{\delta \theta^a(0)} \right) \left( \frac{\delta}{\delta \theta^a(y)} \right) \phi = \int \left[ \frac{\delta}{\delta \theta^a(0)} \left( \frac{\delta}{\delta \theta^a(0)} \right) \phi \right] = \int \left( \frac{\delta}{\delta \theta^a(0)} \right) \phi = \int \epsilon^a(u) \]

\[ = \int d^4w \left( \frac{\delta^4}{\delta \theta^a(w)} \right) \phi \]
\[ e^a \rightarrow \text{prof. of the group } 4 \text{ it doesn't depend on what quantity you are applying on} \]

Hence,

\[ S_{\text{total}} = - \int d^3x \int d^3y d^3z \frac{\delta \phi \phi}{\delta \phi} \theta(x) \theta(y) \theta(z) \]

\[ = \frac{1}{2} \left[ \int d^4w \frac{\delta F_a \phi}{\delta \phi} \right] \left. \frac{\partial F_a \phi}{\partial \phi} \right|_{\theta = 0} \]

\[ + \int d^3x d^3y \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ + \int d^3x d^3y \frac{\delta F_a \phi}{\delta \phi \theta(x)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ = \frac{1}{2} \left[ \int d^4w \frac{\delta F_a \phi}{\delta \phi} \right] \left. \frac{\partial F_a \phi}{\partial \phi} \right|_{\theta = 0} \]

\[ + \frac{1}{2} \int d^3x d^3y \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ + \frac{1}{2} \int d^3x d^3y \frac{\delta F_a \phi}{\delta \phi \theta(x)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ = 0 \]

\[ g \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ + \frac{1}{2} \int d^3x d^3y d^3z \frac{\delta F_a \phi}{\delta \phi \theta(x)} \left. \delta F_a \phi \right|_{\theta = 0} \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ = g \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} + \frac{1}{2} \int d^3x d^3y d^3z \frac{\delta F_a \phi}{\delta \phi \theta(x)} \left. \delta F_a \phi \right|_{\theta = 0} \frac{\delta F_a \phi}{\delta \phi \theta(y)} \left. \delta F_a \phi \right|_{\theta = 0} \]

\[ (\text{anti-sym.}) \]