

5/4/06

Renormalization group

One-dimensional Ising model

$$Z = \sum_{\{s_i\}} e^{-\beta \mathcal{H}}$$

$$\mathcal{H} = -J \sum_{i=1}^N s_i s_{i+1} - K \sum_{i=1}^N s_i$$

$$Z = \sum_{\{s_i\}} e^{J \sum_{i=1}^N s_i s_{i+1} + \hat{K} \sum_{i=1}^N s_i}$$

where $\hat{J} = \beta J$
 $\hat{K} = \beta K$

We can add to \mathcal{H} a constant term :-

$$\mathcal{H} = -J \sum_{i=1}^N s_i s_{i+1} - K \sum_{i=1}^N s_i - AN$$

\downarrow
constant term

~~...~~

$\therefore Z(\hat{J}, \hat{K}, \hat{A}, N) \rightarrow$ fn. of these 4

Take $N = 2M$ (even) (In the thermodyn. limit it doesn't matter whether we take even or odd)

$$Z(\hat{J}, \hat{K}, \hat{A}, 2M)$$

$$= \sum_{s_1, s_2, \dots, s_{2M}} T_{s_1, s_2} T_{s_2, s_3} \dots T_{s_{2M-1}, s_{2M}} T_{s_{2M}, s_1}$$

Note :- $Z = \sum_{\{s_i\}} e^{J \sum_{i=1}^N s_i s_{i+1} + \hat{K} \sum_{i=1}^N s_i + \hat{A}N}$

$$T = e^{\hat{A}} \begin{pmatrix} e^{\hat{J} + \hat{K}} & e^{-\hat{J}} \\ e^{-\hat{J}} & e^{\hat{J} - \hat{K}} \end{pmatrix}$$

$$\hat{A} + \hat{J} s s' + \frac{1}{2} \hat{K} (s + s')$$

Transfer matrix $\leftarrow T_{ss'} = e^{\dots}$

In the thermodyn. limit we want \hat{A} to be const. - an additive energy per site - can't change anything physically

Previously, we didn't have the factor $e^{\hat{A}}$

Sum over s_2, s_4, s_6, \dots (even lattice sites)

$$= \sum_{s_1, s_3, s_5, \dots, s_{2M-1}} (T^2)_{s_1, s_3} (T^2)_{s_3, s_5} (T^2)_{s_5, s_7} \dots (T^2)_{s_{2M-1}, s_1}$$

Now, $T^2 = e^{2\hat{A}} \begin{pmatrix} e^{2(\hat{J}+\hat{K})} + e^{-2\hat{J}} & e^{\hat{K}} + e^{-\hat{K}} \\ e^{\hat{K}} + e^{-\hat{K}} & e^{-2\hat{J}} + e^{2(\hat{J}-\hat{K})} \end{pmatrix}$

Define :- $\tilde{s}_k = s_{2k-1}$ where $k=1, 2, \dots, M$

$$\tilde{T} = T^2$$

$$\therefore Z(\hat{J}, \hat{K}, \hat{A}, 2M) = \sum_{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M} (\tilde{T})_{\tilde{s}_1, \tilde{s}_2} (\tilde{T})_{\tilde{s}_2, \tilde{s}_3} \dots (\tilde{T})_{\tilde{s}_M, \tilde{s}_1}$$

only diff. is T has been replaced by \tilde{T} & instead of $2M$ lattice sites, we have M sites to sum over

Define new parameters $\tilde{A}, \tilde{J}, \tilde{K}$ such that

$$\tilde{T} = e^{\tilde{A}} \begin{pmatrix} e^{\tilde{J}+\tilde{K}} & e^{-\tilde{J}} \\ e^{-\tilde{J}} & e^{\tilde{J}-\tilde{K}} \end{pmatrix}$$

$$\therefore \textcircled{1} e^{\tilde{A} + \tilde{J} + \tilde{K}} = e^{2\hat{A}} (e^{2(\hat{J}+\hat{K})} + e^{-2\hat{J}})$$

$$\textcircled{2} e^{\tilde{A} + \tilde{J} - \tilde{K}} = e^{2\hat{A}} (e^{-2\hat{J}} + e^{2(\hat{J}-\hat{K})})$$

$$\textcircled{3} e^{\tilde{A} - \tilde{J}} = e^{2\hat{A}} (e^{\hat{K}} + e^{-\hat{K}})$$

3 eqns in 3 unknowns

$\textcircled{1} / \textcircled{2}$ gives us \tilde{K}

$\textcircled{1} \times \textcircled{2} \times \textcircled{3}^2$ gives us \tilde{A}

$\frac{\textcircled{1} \times \textcircled{2}}{\textcircled{3}^2}$ gives us \tilde{J}

