

## Statistical Mechanics Exam

Marks distribution: 10 + 9 + 9 + 12

**Instructions:** Some of the problems could involve tedious algebra. Please ensure that you take utmost care in carrying out the algebra. Getting the correct final result is important. Also for your own benefit, in some cases you may want to check the consistency of the final answer by analyzing the problem in more than one way.

1. Consider a cylinder of radius  $a$ , kept vertically, containing  $N$  molecules of a monatomic gas.  $m$  is the mass of each atom,  $g$  is the acceleration due to gravity, and  $T$  is the temperature of the gas. The upper end of the cylinder is closed with a piston of weight  $w$ . In the equilibrium situation the piston is at a height  $h$  above the lower end of the cylinder. The height  $h$  is large so that the effect of the gravitational pull on the atoms cannot be ignored. The temperature is sufficiently large so that we can use classical statistical mechanics.
  - a) Beginning with the definition of the canonical partition function, find an expression for the free energy  $F$  of the system as a function of  $a$ ,  $N$ ,  $m$ ,  $g$ ,  $h$ .
  - b) Using the result of part a), and the equation  $dF = -dW - SdT$  where  $dW$  is the work done by the system, and  $S$  is the entropy of the system, calculate the net force exerted by the gas on the piston, i.e. calculate the weight  $w$  in terms of  $a$ ,  $N$ ,  $m$ ,  $g$ ,  $h$ .
  - c) Calculate the equilibrium density of atoms at a height  $x$  above the bottom of the cylinder.
2. Consider a one dimensional ideal monatomic Bose gas containing  $N$  atoms each of mass  $m$ , confined in a periodic box of length  $L$ . Find the temperature  $T$  at which the fraction  $f$  of the total number of particles will be in the ground state, assuming that  $f \ll 1$  but  $fN \gg 1$ .  $T$  should be determined in terms of  $N$ ,  $L$ ,  $f$  and  $m$ . (Note that you need to do this calculation for large but finite  $L$  and  $N$ .)

3. Consider a two dimensional classical gas containing  $N$  atoms each of mass  $m$ , which interact with each other via a potential  $v(\vec{r}_i, \vec{r}_j)$  of the form

$$\begin{aligned} v(\vec{r}_i, \vec{r}_j) &= v_0 \quad \text{for } |\vec{r}_i - \vec{r}_j| \leq a \\ &= v_1 \quad \text{for } a < |\vec{r}_i - \vec{r}_j| \leq b \\ &= 0 \quad \text{for } |\vec{r}_i - \vec{r}_j| > b. \end{aligned}$$

The gas is confined to a box of area  $A$ . Calculate the correction to the equation of state of the system, i.e. to the ratio  $PA/(NkT)$ , to order  $(N/A)$ . Note that for a two dimensional gas,  $P$  is the force per unit length exerted by the gas on the one dimensional boundary of the two dimensional box.

4. Consider a one dimensional Ising model with  $2N$  sites, with variable  $s_i$  at the  $i$ -th site taking values  $\pm 1$  and satisfying periodic boundary condition  $s_{i+2N} \equiv s_i$ . The energy associated with a given configuration is given by

$$E = -J \sum_{i=1}^{2N} s_i s_{i+1} - A \sum_{i=1}^{2N} s_i s_{i+2}.$$

Calculate the free energy per site of this model in the thermodynamic limit to first order in  $A$ .

Hint: You may find it useful to work with  $N$  new variables  $(s_{2i-1}, s_{2i})$  with  $1 \leq i \leq N$ . The new variable takes four possible values.

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Electron in a magnetic field B along z

$$E = \underbrace{\frac{eB\hbar}{mc} \left( j + \frac{1}{2} \right)}_{E_{Lz} = E'} + \frac{p_z^2}{2m} = E_z$$

These are the Landau levels

For each  $j$ , there are  $\infty$  no. of states.  
 $\rightarrow \infty$  degeneracy (infinite volume limit)

Q) If we put the system in a finite box what happens to the degeneracy?

Take a box of length  $L_z$  in the z-direction &  $L_1, L_2$  in x, y direction.

Suppose  $N(j)$  is the number of states for energy level  $j$ .

(Implicitly assuming the volume is large)

No. of states between  $E'$  and  $E' + dE'$

$$= N(j) \times \frac{dE'}{\frac{eB\hbar}{mc}}$$

$$= \frac{mc}{eB\hbar} N(j) dE'$$

For  $B=0$ , we have free particles

No. of states between  $E'$  &  $E' + dE'$

$$\# \text{ in the range } dp_x dp_y = \frac{L_1 L_2}{h^2} dp_x dp_y$$

$$= \frac{L_1}{2\pi\hbar} dp_x \cdot \frac{L_2}{2\pi\hbar} dp_y$$

final result will not depend on cubic or cyl. box in the large  $V$  limit  $\checkmark$

H.o. waveps fall off exponentially & most of them won't be affected in the large  $V$  limit even if we confine them in a box. The ones with very large excitations will be affected in a box - but not in  $V \rightarrow \infty$  limit. So there will still be degeneracy on confinement within a box.

