

① For 2 identical Bosons, out of  $j=2, 1, 0$ , we can consider only  $j=2, 0$ ;  $j=2$  must be present to get  $m=2, -2$  — Now we have 5 states & need one more — this comes from  $j=0$ . So  $j=1$  is excluded. Rotational invariance tells us that whole subspaces should be present or absent. || Also,  $j=2, 0$  gives sym. wavefn.

**Problem set 2**

**Date due: April 7, 2006**

1. Consider a system with  $N$  independent sites and  $2N$  identical spin 1 bosonic particles distributed equally among the sites so that each site contains two particles. The whole system is kept in a uniform magnetic field so that the Hamiltonian of the system is given by  $-\mu\vec{B} \cdot \sum_i \vec{S}_i$ , with the sum over  $i$  running over all the  $2N$  bosons.

a) Calculate the canonical partition function and specific heat of this system.

b) Repeat the calculation if each site contained two spin 3/2 fermionic particles instead of two spin 1 bosonic particles.

Suppose  $j=0, 1, 2, 3$   
one is always sym.

$\rightarrow j=0, 1, 2, 3$   
 $j=0, 2$  gives anti-sym. wave function

2. Consider a system of  $N$  spin half electrons. A fraction 1/2 of them are polarized along the  $z$  direction, a fraction 1/4 are polarized along the  $x$  direction and another fraction 1/4 are polarized along the  $-z$  direction. Calculate the density matrix of the system in a basis of  $S_z$  eigenstates.

3. Consider a set of non-interacting particles moving in a harmonic oscillator potential. The parameters of the potential are such that the angular frequency is  $\omega$ .

a) Find the expression for the grand partition function of the system in quantum statistical mechanics. Consider both cases, when the particles are fermionic and the particles are bosonic.

b) Consider the high temperature limit at a fixed value of the total number of particles, and show that the result agrees with that of classical statistical mechanics. (For this comparison you need to calculate the grand partition function of a system of classical particles moving in a harmonic oscillator potential.) Find the specific heat of the system in this limit.

4. a) Find an expression for the grand partition function  $Q$  of two dimensional ideal Bose gas, and express  $V^{-1} \ln Q$  as a function of the fugacity  $z$  and temperature  $T$  in the thermodynamic limit.

③  
check

b) Find the average number of particles per unit area in the thermodynamic limit as a function of  $z$  and  $T$ .

c) Show that there is no Bose-Einstein condensation for a two dimensional ideal Bose gas.

- d) Calculate various other thermodynamic quantities like entropy per unit volume, total energy per unit volume and specific heat at constant volume, as a function of  $T$  and  $N$  at low temperature for fixed density.
5. Consider an ideal gas of spin 1 bosons in three dimensions. The system is placed in a uniform magnetic field  $\vec{B}$ , and in this field the total Hamiltonian of the system acquires an additional term (besides the kinetic energy) of the form  $-\mu\vec{B} \cdot \sum_i \vec{S}_i$  where  $\vec{S}_i$  denotes the spin operator of the  $i$ -th particle. This corresponds to bosons carrying magnetic moment  $\mu$ .
- a) For a given temperature find an expression for the critical density of particles at which Bose-Einstein condensation takes place.
- b) Find an expression for the total magnetization (which is defined as the ensemble average of  $\mu \sum_i \vec{S}_i$ ) of the system both below and above the critical density.
- c) Calculate the magnetization in the  $B \rightarrow 0$  limit both below and above the critical density.
6. Consider an ideal gas of spin 1/2 massless fermions. In this case we need to use relativistic relation between energy and momentum i.e.  $e(\vec{p}) = c|\vec{p}|$ .
- a) Find the expression for the grand partition function and the number of particles in the system in terms of  $T, V, z$ .
- b) Show that in the high temperature limit the equation of state reduces to the classical result. (For this comparison you first need to find the equation of state for an ideal gas of relativistic massless particles obeying Boltzmann statistics.) Also find the lowest order correction to the equation of state.
- c) More realistically, if we have massless fermions, then fermion - antifermion pairs may be produced without violating any conservation laws. (Antifermions have similar properties as the fermions but are distinguishable from the fermions.) Thus the conserved quantity for a given system is the difference between the total number of fermions and antifermions but not the number of fermions and antifermions individually. Repeat parts a) and b) for such a system.

$$\frac{N_1 - N_2}{N_1 + N_2} = \frac{1}{R} \frac{\partial \ln Z}{\partial \mu}$$

$$N_1 + N_2 = ?$$

