"Rubric" lattice in any dimension.

D = 1:

D = 2:
Square

Dynamical variables

'Spin' on each lattice site

\[ E(p_1, \ldots, p_N) = -J \sum_{i<j} S_i \cdot S_j - K \sum S_i \]

implies nearest neighbour pairs

J, K constants

Example

1. \[ D = 1 \]

\[ E = -J \sum_{i=1}^{N} S_i \cdot S_{i+1} - K \sum_{i=1}^{N} S_i \]

2. \[ D = 2 \]

\[ E = -J \sum_{i,j=1}^{N} S_{ij} \]

To make it more symmetric

\[ -\frac{J}{2} \sum_{i,j=1}^{N} (S_i S_j + S_{i+1} S_{i+2}) \]

What do we mean by \( S_{N+1} \)?

\[ \Rightarrow \text{Periodic boundary condition:} \]

\[ S_{N+1} = S_1 \]

Periodic boundary makes calculations simpler

Could have written:

\[ \sum_{i} S_i, S_{i-1} \]

\[ + \sum_{i} S_i, S_{i+1} \]

But under a shift of \( i \), the first sum will also be of the form \( \sum_i S_i, S_{i+1} \)
\[ E = -\frac{J}{2} \sum_{i=1}^{N} \sum_{\beta=1}^{N} \left( S_{i}^{x} S_{i+1}^{x} + S_{i}^{y} S_{i+1}^{y} + S_{i}^{z} S_{i+1}^{z} \right) \]

- \[ K \sum_{i=1}^{M} \sum_{\beta=1}^{N} \delta_{i,\beta} \]

**Periodic boundary conditions:**
\[ S_{N+i}^{\beta} = S_{i}^{\beta} \]
\[ S_{i}^{z}, S_{i+1}^{z} = S_{i}^{z} \]

**Motivation:**

> A Model for ferromagnetism.

Take a lattice & at each lattice site we have a magnetic dipole.

At \( i^{th} \) site, \( \vec{\mu}_{i} = \mu_{0} (\sin \Theta_{i} \cos \Phi_{i}, \sin \Theta_{i} \sin \Phi_{i}, \cos \Theta_{i}) \)

\[ \vec{H} = \sum_{i} \left( \frac{\vec{B} \cdot \vec{\mu}_{i}}{2I} + \frac{\mu_{0}^{2}}{2I \sin^{2} \Theta_{i}} \right) - \alpha \sum_{i} \frac{\vec{\mu}_{i}}{\langle i | i \rangle} \]

- \[ B . \sum \vec{\mu}_{i} \]

**External magnetic field**
\( (0, 0, B) \)

\[ Z = \sum_{i=1}^{N} (d_{0} \cdot d_{i} \cdot d_{0}, \Phi_{i} \cdot \Phi_{i}) e^{-\beta H} \]

\[ = C (kT)^{N} \sum_{i=1}^{N} (d_{0} \cdot d_{i} \cdot \sin \Theta_{i}) e^{-\beta \mu_{0}^{2} / 2I} \]

\[ + \text{constant} \]
Approximation:
Replace the angular integration by a sum over two values: \( \theta = 0 \) and \( \theta = \pi \).

\[
\mathcal{Z} \to C(\mathbf{k}r)^N \sum_{s_i = \pm \frac{1}{2}, s = \pm \frac{1}{2}, \ldots, s = \pm 1} \exp(-i \mathbf{r} \cdot \mathbf{p})
\]

\[
H' = -a \mu_0^2 \sum s_i^2 + \sum B_i \sum x_i
\]

\[
\sum_{i=1}^N \frac{s_i + 1}{2}
\]

\[
E = -J \sum_{i=1}^N s_i A_{i+1} - K \sum_{i=1}^N A_i
\]

\[
Z = \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_N = \pm 1} \exp(i \mathbf{S} \cdot \mathbf{A} + \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{C})
\]

\[
\sum_{i=1}^N \frac{s_i + 1}{2}
\]

Analysis of one dimensional Ising model using model:

This may capture qualitative features of what is happening, but features like ferromagnetic should show up.
Define:
\[ T_{s_{i+1}} = e^{\beta J s_i s_{i+1} + \beta K} (s_i + s_{i+1}) \]

We may write:
\[ Z = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} \prod_{i=1}^{N} e^{\beta J s_i s_{i+1} + \beta K} (s_i + s_{i+1}) \]
\[ = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} T_{s_1, s_2} T_{s_2, s_3} T_{s_3, s_4} \cdots T_{s_{N-1}, s_N} T_{s_N, s_1} \]

\( T_{s_{i+1}} \) can be thought of as a 2 \( \times \) 2 matrix:
\[ \begin{pmatrix} 1 & s \pm 1 \pm s' \pm 1 \end{pmatrix} \]

\[ \Rightarrow Z = T^N \quad \left( T^N \right)_{s=1, s'=1} \]

Here:
\[ T = \begin{pmatrix} e^{\beta J + \beta K} & e^{-\beta F} \\ e^{-\beta F} & e^{\beta J - \beta K} \end{pmatrix} \]

It is called the transfer matrix.

\[ \Rightarrow Z = \lambda_1^N + \lambda_2^N \quad \lambda_1, \lambda_2 \text{ are eigenvalues of } T. \]

Eigenvalue eqn:
\[ (\lambda - e^{\beta J + \beta K}) (\lambda - e^{\beta J - \beta K}) - e^{-2\beta J} = 0 \]

Solve:
\[ \lambda = e^{\beta J} \cosh \beta K \pm \sqrt{e^{2\beta J} \cosh^2 \beta K - 2 \sinh (2\beta J)} \]
\[ = e^{\beta J} \sinh \beta K + e^{-2\beta J} \quad e^{2\beta J} \cosh^2 \beta K - e^{2\beta J} + e^{-2\beta J} \]

\[ \geq 0 \]
→ Both eigenvalues are real.

\[ Z = (\lambda_1^N + \lambda_2^N) \]

\[ F = -kt \ln Z = -kt \ln (\lambda_1^N + \lambda_2^N) = -kt \ln \left( \lambda_1^N + (\rho \lambda_2)^N \right) \]

\[ \lim_{n \to \infty} -kt \ln \lambda_1^N \]

\[ \Rightarrow F \text{ is never singular.} \]

\[ \Rightarrow \text{no phase transition} \]

Total magnetic moment

\[ M = \mu_0 \sum_i x_i \]

\[ \langle M \rangle = \frac{\sum \sum _{s_i = \pm 1} \sum _{s_j = \pm 1} - e^{-\beta E} \mu_0 \sum_i s_i}{\sum \sum _{s_i = \pm 1} \sum _{s_j = \pm 1} e^{-\beta E}} \]

\[ Z = \sum \sum _{s_i = \pm 1} \sum _{s_j = \pm 1} e^{\beta \sum s_i + \rho \sum s_i} \]

Take \( \lambda_1 \) to be the bigger eigenvalue:

\[ e^{\beta \gamma} \cosh \gamma \]

\[ + \frac{1}{2} e^{2 \beta \gamma} \cosh \gamma \sinh 2\beta \gamma \]

\[ > 0 \]

Do no branch pt. from sheet.

Also argument of \( \log \) is always \( n \to \infty \) for total.
\[ \frac{1}{2} \frac{\partial^2}{\partial K} = \beta \left( \sum_{i=1}^{N} s_i \right) = \frac{P}{\mu_0} \langle M \rangle \]

\[ \langle M \rangle = \frac{\mu_0}{q} \frac{\partial}{\partial K} \ln Z = \frac{\partial F}{\partial K} = -\mu_0 \frac{\partial F}{\partial K} \]

\[ F = -kT \ln Z \]
\[ = -\frac{1}{\beta} \ln Z \]

**Exercise:** Show that

\[ \langle M \rangle = \mu_0 N \frac{\sinh (\beta \mu_0)}{\sqrt{\cosh^2 \beta \mu_0 - 2 e^{-2\beta \mu_0} \sinh 2\beta \mu_0}} \]

**Properties:**

1. \( \langle M \rangle \rightarrow -\langle M \rangle \) under \( \beta \rightarrow -\beta \)
   
   \( k = \mu_0 \beta B \)

   (Magnetization flips sign on switching the mag. field in the opp. dirn.)

2. \( \langle M \rangle = 0 \) for \( \beta = 0 \) for any \( B \).
   
   There is no spontaneous magnetization

\[ \lim_{K \rightarrow 0^+} \langle M \rangle = 0 \]

**Suppose**

\[ \langle M \rangle = C \frac{\sinh \beta K}{\sinh \beta K} \]

\[ \lim_{K \rightarrow 0^+} \langle M \rangle = C \]

\[ \lim_{K \rightarrow 0^-} \langle M \rangle = -C \]

Exactly at \( k = 0 \), \( \langle M \rangle \) is ill-defined.

(This is the way you see spontaneous magnetization)
Equivalence of Ising model with other systems

Consider a cubic lattice.

\( s_i \) : variable associated with \( i \)th lattice site which can take two values \( s_i^{(1)}, s_i^{(2)} \).

Examples:
- Random alloy: a cubic lattice where the \( i \)th site may have an atom of type A or type B.
- Lattice gas: a cubic lattice where the \( i \)th site is either occupied or vacant.

\( \{ s_i \} \rightarrow \) labels a configuration

\[
E(\{ s_i \}) = \sum_{\langle ij \rangle} E^{(u)}(s_i, s_j) + \sum_{i} E^{(v)}(s_i)
\]

\( \rightarrow \) sum over nearest neighbours

We'll now map this to the Ising model.

Introduce a new lattice variable \( \pi_i \) for each site:

\[
\pi_i = \begin{cases} 1 & \text{for } s_i = s_i^{(1)} \\ -1 & \text{for } s_i = s_i^{(2)} \end{cases}
\]

\( \{ \pi_i \} \leftrightarrow \{ s_i \} \)
Consider a new function:

\[ E(\delta_{i\gamma}) = -J \sum_{\langle i,j \rangle} \delta_{i,j} \delta_{\gamma} + \sum_i \delta_i + \text{constant} \]

\[ = \sum_{\langle i,j \rangle} \left( -J \delta_i \delta_{\gamma} - \frac{k}{\hbar} (\delta_i + \delta_{\gamma}) \right) + \text{constant} \]

\[ n \text{ number of nearest neighbours of a given atom} \]

\[ \rho_k \sum_{\langle i,j \rangle} = \frac{K}{N} \sum_i \delta_i + \frac{k}{\hbar} \sum_{\langle i,j \rangle} \delta_i \delta_{\gamma} \]

\[ \sum_{\langle i,j \rangle} E^{(0)}(\delta_{i\gamma}) = \frac{1}{N} \sum_{i,j} E^{(0)}(\delta_{i\gamma}) = \frac{1}{N} \sum_{i,j} \delta_i \delta_{\gamma} \]

\[ = \frac{1}{N} \sum_{\langle i,j \rangle} \left( E^{(0)}(\delta_i) + E^{(0)}(\delta_{\gamma}) \right) \]

\[ \sum_{\langle i,j \rangle} E^{(0)}(\delta_{i\gamma}) = \sum_{\langle i,j \rangle} \left( E^{(0)}(\delta_i) + E^{(0)}(\delta_{\gamma}) \right) \]

\[ \text{magnetically it is spin under if each} \]

\[ \text{we can choose } E^{(0)}(\delta, \delta) \]
Consider the Ising model energy:

$$E(\{\sigma_i\}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \sum_i h_i \sigma_i$$

$$= \sum_{\langle i,j \rangle} \left\{ -J \sigma_i \sigma_j - k_B \left( \beta_i + \beta_j \right) \right\}$$

$$\Rightarrow \sum_{\langle i,j \rangle} \left\{ -J \beta_i \beta_j - k_B (\beta_i + \beta_j) + A \right\}$$

Goal: Choose J, K & A such that the pair energy in the original model matches the pair energy in the Ising model.

1. \(\sigma_i = \sigma^{(1)}, \sigma_j = \sigma^{(1)} \Rightarrow s_i = 1, s_j = 1\)
   
   \(E^{(1)}(\sigma^{(1)}, \sigma^{(1)}) + \frac{k_B}{n} E^{(1)}(\sigma^{(1)}) = J - 2k_B h + A\)

2. \(\sigma_i = \sigma^{(1)}, \sigma_j = \sigma^{(2)} \Rightarrow s_i = 1, s_j = -1\)
   
   \(E^{(1)}(\sigma^{(1)}, \sigma^{(2)}) + \frac{1}{n} \sqrt{E^{(1)}(\sigma^{(1)}) + E^{(1)}(\sigma^{(2)})} = J + A\)

3. \(\sigma_i = \sigma^{(2)}, \sigma_j = \sigma^{(2)} \Rightarrow s_i = -1, s_j = -1\)
   
   \(E^{(1)}(\sigma^{(2)}, \sigma^{(2)}) + 2k_B E^{(1)}(\sigma^{(2)}) = -J + 2k_B h + A\)

\(\Rightarrow\) Solve for J, K & A.

This gives an exact map between the original model & Ising model.
A model in which the system is in a 2D lattice with periodic boundary conditions. The model is described by the following equations:

- $S_i$: spin at site $i$
- $J$: coupling constant
- $K$: external field

The model can be described in terms of a grand canonical ensemble with the partition function $Z$.

The total number of spins is fixed, and the system is described by the partition function $Z$.

Each site is fixed, and the system is described by the partition function $Z$.

Mathematical expressions are also included, such as:

- $Z = \sum_i e^{\beta S_i}$
- $\beta = \frac{1}{kT}$
- $k$: Boltzmann constant
- $T$: temperature

The model is further analyzed with different values of $J$ and $K$, leading to different phases and behaviors.
\[ Z = \sum_{\{A_{ij} = \pm 1\}} e^{-\beta E} \]
\[ = \sum_{\{A_{ij} = \pm 1\}} \prod_{i=1}^{M} \exp \left\{ \beta J \sum_{\delta_{ij} \in \{\pm 1\}} \delta_{ij} \cdot \delta_{i+1,j} \right\} \]
\[ + \frac{1}{2} \beta J \sum_{\delta_{ij} \in \{\pm 1\}} \delta_{ij} \cdot \delta_{ij+1} \]
\[ + \frac{1}{2} \beta K \sum_{\delta_{ij} \in \{\pm 1\}} \delta_{ij} \cdot \delta_{j} \]

Suppose \( \vec{\sigma}, \vec{\sigma}' \) are two \( N \)-dimensional vectors.

\[ \vec{\sigma} = (\sigma_1, \ldots, \sigma_N), \quad \vec{\sigma}' = (\sigma'_1, \ldots, \sigma'_N) \]

Define: \( T(\vec{\sigma}, \vec{\sigma}') = \exp \left\{ \beta J \sum_{\delta_{ij} \in \{\pm 1\}} \delta_{ij} \cdot \delta_{ij}' \right\} \)
\[ + \frac{1}{2} \beta J \sum_{\delta_{ij} \in \{\pm 1\}} (\sigma_i \sigma_i' + \sigma_j \sigma_j') \]
\[ + \frac{1}{2} \beta K \sum_{\delta_{ij} \in \{\pm 1\}} (\sigma_i' \sigma_i + \sigma_j' \sigma_j) \]

\[ T(\vec{\sigma}, \vec{\sigma}') = T(\vec{\sigma}', \vec{\sigma}) \]

Define:

\[ \vec{\sigma}^{(i)}: \text{\( N \)-dimensional vector for each} \]
\[ \{\delta_{ij}^{(i)} = \pm 1\} \quad i=1, \ldots, M \]
\[ \vec{\sigma}^{(i)} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_N^{(i)}) \]

Each \( \vec{\sigma}^{(i)} \) takes \( 2^N \) values.
\[ Z = \sum_{\{\sigma^{(\alpha)}\}} \prod_{i=1}^{M} T(\sigma^{(1)}, \sigma^{(i+1)}) \]

\[ T(\sigma^{(1)}, \sigma^{(2)}) \cdot T(\sigma^{(2)}, \sigma^{(3)}) \cdot \cdots \cdot T(\sigma^{(M-1)}, \sigma^{(M)}) \cdot T(\sigma^{(M)}, \sigma^{(1)}) \]

(Each \( T \) can be thought of as a \( 2^N \times 2^N \) matrix, hence each of \( \sigma^{(\alpha)} \) is as can take \( 2^N \) values)

\[ Z = \text{Tr}_T \left( T^M \right) \]

\( T \) a \( 2^N \times 2^N \) matrix

\[ Z = \sum_{\alpha} 2^M \]

\( \sum \) over eigenvalues

\[ = \left( \lambda_{\text{max}} \right)^{\alpha} \left[ 1 + \sum_{\alpha} \left( \frac{\lambda_{\alpha}}{\lambda_{\text{max}}} \right)^M \right] \]

\( \sum \) over all \( \alpha \) except the maximum e.v.

\[ F = -k T \ln Z \]

\[ = -k T \ln \lambda_{\text{max}} + \ln \left( 1 + \sum_{\alpha} \left( \frac{\lambda_{\alpha}}{\lambda_{\text{max}}} \right)^M \right) \]

\[ \lim_{N \to \infty} -N k T \ln \lambda_{\text{max}} \]

Problem reduces to calculating the maximum eigenvalue of the transfer matrix.
The problem reduces to diagonalizing a matrix. Upon over $f$, the Jordan form is transformed into a matrix form of the problem.
Result for $T_{\text{max}}$ (For $K = 0$)

Define $\phi = \beta \varphi$, $\Theta = \tan^{-1} e^{-2\phi}$

\[ Y_k = \cosh^{-1} \left[ \frac{\cosh 2\phi \cosh 2\Theta}{\cosh \tanh 2\phi \sinh 2\phi} \right] \]

$k = 0, 1, 2, \ldots, 2N - 1$

The positive value

\[ T_{\text{max}} = \left( 2 \sinh 2\beta \varphi \right)^{N/2} e^{\frac{1}{2} \left( \eta - \eta_1 + \eta_2 - \eta_3 \right) + \ldots + (-1)^{2N-1} \eta_N} \]

This is also $N$-dependent, there are $N$ terms in the sum

\[ Z = \text{Tr}(T^N) \]

$T$ is a $2^N \times 2^N$ matrix

\[ T(\sigma, \sigma') = \exp \left[ \beta \sum_{j=1}^{N} \sigma_j \sigma_j' + \frac{1}{2} \beta J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} + \alpha \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \sigma_j' \right] \]

$\sigma = (\sigma_1, \ldots, \sigma_N)$, $\sigma' = (\sigma_1', \ldots, \sigma_N')$

$\sigma_i = \pm 1$, $\sigma_i' = \pm 1$
\[ Z = \sum_{\mathcal{M}} \chi_{\mathcal{M}} \]

Eigenvalues of \( T \)

\[ \chi_{\mathcal{M}} = -kT \chi_{\mathcal{M}} \]

\[ -kT \ln \chi_{\mathcal{M}} = -M \frac{\chi_{\mathcal{M}}}{M+\alpha} \]

We need to find \( \chi_{\text{max}} \)

**Result for \( \chi_{\text{max}} \) (This is for zero magnetic field):**

Define \( \phi = \beta J \) and \( \alpha = e^{-2\phi} \)

\[ \cosh V_k = \cosh (2\phi) \cosh (2\theta) - \cos \frac{2\pi k}{N} \frac{\sinh (2\phi)}{\sin (2\theta)} \]

\[ \chi_k > 0 \]

\[ \chi_{\text{max}} = \left[ \frac{2 \sinh 2\phi J}{\sinh 2\phi J} \right]^{N/1} \cdot \frac{1}{N} \left( X_1 + X_2 + \ldots + X_N \right) \]

\[ F = -MNkT \left[ \frac{N}{2} \ln \left( \frac{2 \sinh 2\phi J}{\sinh 2\phi J} \right) + \frac{1}{2} \sum_{i=1}^{N} Y_{2i-1} \right] \]

\[ = -\frac{1}{2} MNkT \left[ \ln \left( \frac{2 \sinh 2\phi J}{\sinh 2\phi J} \right) + \frac{1}{N} \sum_{i=1}^{N} Y_{2i-1} \right] \]

For \( N \) large,

Cosh \( V_k \) will be a smooth function of \( k \) as \( \frac{k}{N} \) changes by unity. It changes very little.
Define \( U = \frac{c}{N} \)
\( i = NU \)

\[ Y_{2i-1} = Y_{2(Nu-1)} - Y_{2Nu} \]

\[ f = \frac{1}{N} \lim_{N \to \infty} \left[ \ln \left( \frac{2 \sinh(2\beta J)}{2} \right) + \int_0^1 du Y_{2nu} \right] \]

\[ \cosh Y_{2nu} = \cosh(2\phi) \cosh(2\phi) - \cos(2\pi N) \sinh(2\phi) \sinh(2\phi) \]

Once we write \( Y_{2nu} = \cosh^{-1}(\ldots) \), there is no \( N \)-dependence & we get a smooth \( f_N \).

Using model is independent under the interchange \( M \leftrightarrow N \); nothing for \( i \) depends on the ratio \( M/N \).

Ex. Show that

\[ \int_0^1 du Y_{2nu} = \ln \left( \frac{2 \cosh^2(2\beta J)}{\sinh 2\beta J} \right) + \frac{1}{4\pi} \int_0^\infty dx \ln \left( \frac{4x^2}{(14 - x^2)^2} \right) \]

where

\[ K = \frac{2}{\cosh(2\phi) \coth(2\phi)} \]
Average total energy \( \bar{E} = -\frac{\partial \beta}{\partial \beta} \ln 2 \)
\[ = -\frac{\partial}{\partial \beta} (-\beta F) \]
\[ = \frac{\partial}{\partial \beta} (\beta F) \times \text{MN} \]

\[ e = \frac{\bar{E}}{\text{MN}} = \frac{\partial}{\partial \beta} (\beta F) \]

\( \triangleright \) Average energy per site (Av. energy/site)
\[ f = -\beta T \ln (2 \cosh 2\beta J) + \frac{1}{2\pi} \int_0^T d\omega \frac{\partial}{\partial \beta} \frac{e^{i\omega\beta}}{\sqrt{1 - k^2 \sin^2 \omega}} \]

\( k \) depends on \( \beta \)

\[ e = -J \operatorname{coth} (2\beta J) \left( 1 + \frac{2}{\pi} K' K_1 (K) \right) \]

where \( K' = 2 + \tanh^2 (2\beta J) - 1 \), \( K^2 + (K')^2 = 1 \)

and \( K_1 (K) = \int_0^{\frac{T_1}{k}} \frac{d\omega}{\sqrt{1 - k^2 \sin^2 \omega}} \)

\[ C = \frac{\partial e}{\partial T} \]

Specific heat per site
\[ C = k \frac{2}{\pi} (\beta J \operatorname{coth} 2\beta J)^2 \left\{ 2K_1 (K) - 2E_1 (K) - (1 - K) (1/2 + K' K_1 (K)) \right\} \]

where \( E_1 (K) = \int_0^{T_1} d\omega \sqrt{1 - k^2 \sin^2 \omega} \)

let us analyze & see if there is any source of singularity at some finite temp. — we’ll analyze free energy & its derivatives
In $2 \cosh 2\beta_3$ doesn't have any singularity at exactly 0 & $\infty$.

If $|k| < 1$, $\sqrt{1-k^2}$ is never zero & we never get any singularity for the 2nd term of its derivatives.

There are no singularities in f or any of its derivatives if $|k| < 1$.

(We need to analyze functional dependence of $k$ on $T$ & see if $|k|$ can be 1 or more.)

$$K = \frac{2}{\cosh 2\phi \cosh 2\beta_3} = \frac{2 \sinh 2\phi}{\cosh^2 2\phi}$$

$$= \frac{4 \left( e^{2\beta_3} - e^{-2\beta_3} \right)}{(e^{2\beta_3} + e^{-2\beta_3})^2}$$

$$= \frac{4 \left( y - y^{-1} \right)}{(y + y^{-1})^2}$$

$\rightarrow 0$ as $y \rightarrow 1$

$\rightarrow 0$ as $y \rightarrow \infty$

Put $y = e^{2\beta_3}$

$1 \leq y < \infty$

Find the maximum of this function.

Answer: $y = \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$

At the maximum, the function = 1

Critical point: $T_c$ is given by $e^{2\beta_3} = \sqrt{\frac{2}{2} + 1}$
$K=1$ at $T=T_c$

Q. How does $K$ behave near $T=T_c$?

i.e. near $y=y_c$?

Near $y=y_c$,

$$K \approx 1 - A(y-y_c)^2$$

$A > 0$ (since $K_n > 0$)

Also, $K$ has a delta-like form, as $y \to y_c$; it'll also behave as a max, as a function of $T$.

1. $f$ is non-singular.
2. $E_j(k)$ is non-singular.
3. $K_l = \frac{1}{\sqrt{1-k^2}}$ where $k = 1-\eta$ and $\eta$ small

\[
\eta = \sqrt{1-k^2} = \sqrt{1-(1-2\eta^2) \sin^2 \varphi}
\]

4. $K_l(k) = \int_0^{\kappa \sqrt{2 \eta}} d\varphi$

We note that $K_l$ diverges at $\varphi = \kappa$ but is small for $\kappa < \kappa_c$.

This divergent piece suggests a modification.
(Divergence comes from large t region in terms of the t variable) large contribution to the integral

\[ \int \frac{e^{\sqrt{t} \eta}}{\sqrt{t}} \, dt \sim \ln \frac{1}{\xi t'} \frac{1}{\sqrt{\eta}} \]

\[ k^1 k_1(k) \sim \ln \frac{1}{\sqrt{\eta}} + \text{finite as } \eta \to 0. \]

\[ k^1 k_1(k) \sim \sqrt{\eta} \ln \frac{1}{\sqrt{\eta}} \to 0 \text{ as } \eta \to 0. \]

\[ \therefore \text{ it has no divergence as } k \to 1. \]

\[ C \sim \frac{2k}{\pi} \left\{ \frac{k_1 \cosh (2k_c J)}{2} \right\}^2 \frac{1}{\sqrt{\eta}} \]

\[ + \text{ finite \ as } \eta \to 0 \]

\[ \sim C \sim \frac{2k}{\pi} \left\{ \frac{k_1 \cosh (2k_c J)'}{2} \right\}^2 \frac{1}{\sqrt{\eta}} \]

\[ + \text{ finite.} \]

This clearly is the sign that there is a transition cut-off.

There is one more thermal and the magnetic part of the system that gives an idea of the system fitting in place of spins, then.