String Beta-function in the Doubled Formalism

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Plan of Talk

- Some brief introduction and motivation.
- The doubled formalism
- Some quantum aspects of the doubled formalism.
- Quantum equivalence from vanishing of the doubled beta-function

T-folds

- Just as in one can define a string background using diffeomorphism or gauge transformation transition functions, one can use T-dualities as transition functions.
- This leads to non-geometric string backgrounds
- T-duality on a geometric background can lead to a non-geometric background.
- Lifts of D=4 supergravity backgrounds include non-geometric backgrounds.

Quantum Equivalence

- We should be sure that important quantum aspects of the ordinary string go over to the doubled formalism.
- Confirm necessity of parts of the doubled formalism for quantum consistency.
- Are there any differences that appear in the doubled approach?
- What happens when we consider non-geometric backgrounds?

Simple T-duality

- The simplest form of T-duality relates strings on a circle of radius R to those on a circle of radius $1/R$.
- It identifies winding modes in one theory with momentum modes in the other. The winding modes are a stringy effect.
- X is interchanged with $\tilde{X}$, where $\tilde{X} = X_L - X_R$.
- Would like to put X and $\tilde{X}$ on an equal footing.

Double the Co-ordinates

- We consider a sigma model whose target space is locally a T^d bundle over a base space N.
- The base space has co-ordinates Y^m on a given patch.
- Normally we would consider the d momenta $\mathcal{P}^i = \mathcal{P}^i_\alpha d\sigma^\alpha$ on the fibres, locally $\mathcal{P}^i_\alpha = \partial_\alpha X^i$. The T-duality group would be $O(d, d; \mathbb{Z})$.
- Instead we consider the $2d$ momenta $\mathcal{P}^I = \mathcal{P}^I_\alpha d\sigma^\alpha$.
- $\mathcal{P}^I_\alpha = \partial_\alpha \mathbb{X}^I$, where $\mathbb{X}^I$ are the co-ordinates on the doubled torus.

The Doubled Lagrangian

- The doubled Lagrangian is then

$$\mathcal{L}_d = \frac{\pi}{2}\mathcal{H}_{IJ}\mathcal{P}^I \wedge *\mathcal{P}^J + \mathcal{L}(Y).$$

  where we have set to zero a possible 1-form connection on the base $\mathcal{A}^I = \mathcal{A}^I_m dY^m$.
- We have introduced $\mathcal{H}$, the doubled metric on the fibres.
- We can also introduce L, an $O(d,d)$ invariant metric such that

$$L^{-1}\mathcal{H}L^{-1}\mathcal{H} = 1\!\!1,$$

  i.e we can raise and lower indices with L.

The Constraint

- Clearly we have doubled the degrees of freedom. We introduce a constraint to halve their number again.
- The constraint is $S^I{}_J \mathcal{P}^J - *\mathcal{P}^I = 0$, where $S^I{}_K = L^{IJ}\mathcal{H}_{JK}$.
- $S^2 = \mathbb{1}$ which ensures the consistency of the constraint (S also defines an almost real structure).
- By looking in the correct basis we will see that the constraint forces the co-ordinates to be chiral scalars.

Manifest $O(d,d;\mathbb{Z})$ Invariance

- The Lagrangian is invariant under rigid $GL(2d,\mathbb{R})$ transformations acting as

$$\mathcal{H} \to h^t \mathcal{H} h\,, \qquad \mathcal{P} \to h^{-1}\mathcal{P}\,,$$

(which act on the co-ordinates as $\mathbb{X} \to h^{-1}\mathbb{X}$

- However, we must also preserve the constraint, which requires us to restrict to an element of $O(d,d) \subset GL(2d,\mathbb{R})$.
- Further, we must preserve the periodicities of the co-ordinates so the symmetry group must be broken again to $O(d,d;\mathbb{Z})$.

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Introducing the Vielbein

- We can introduce a vielbein $\mathcal{V}$ for the metric $\mathcal{H}$ which is an element of $O(d,d)$ invariant under the left action of $O(d) \times O(d)$: $\mathcal{H}$ is a coset metric on $O(d,d)/O(d) \times O(d)$.
- $$\mathcal{H}_{IJ} = \left(\mathcal{V}^t\right)_I{}^A \delta_{AB} \mathcal{V}^B{}_J \,.$$
- $\mathcal{V}$ is chosen so that

$$L^{AB} = \begin{pmatrix} \mathbb{1}^{ab} & 0 \\ 0 & -\mathbb{1}^{a'b'} \end{pmatrix} .$$

Chirality

- In this $O(d) \times O(d)$ frame $A = (a, a')$ etc, and the constraint becomes

$$\begin{aligned} \mathcal{P}^a &= *\mathcal{P}^a \\ \mathcal{P}^{a'} &= -*\mathcal{P}^{a'} . \end{aligned}$$

- Equivalently (for flat worldsheet)

$$\begin{aligned} \mathcal{P}^a_- &= 0 , \\ \mathcal{P}^{a'}_+ &= 0 . \end{aligned}$$

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In terms of Co-ordinates

- For a flat fibre this is locally

$$\begin{aligned} \partial_- X^a &= 0\,, \\ \partial_+ X^{a'} &= 0\,. \end{aligned}$$

- For a toroidal worldsheet where $z = \sigma_1 + \tau\sigma_0$,

$$\begin{aligned} \bar{\partial} X^a &= 0\,, \\ \partial X^{a'} &= 0\,. \end{aligned}$$

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Polarisation

- To make contact with the undoubled theory we must choose a polarisation, this is a projector that picks which T^d subtorus within the T^{2d} is considered physical.
- We are picking a $GL(d, \mathbb{R})$ subgroup of $O(d, d)$ such that the fundamental **2d** of $O(d, d)$ splits into the fundamental **d** and its dual **d**′ of $GL(d, \mathbb{R})$.
- T-dual theories have the same doubled theory, and are obtained from it by different choices of polarisation.
- The earlier $O(d, d; \mathbb{Z})$ transformations of the doubled geometry can instead be thought of as transformations of the polarisation, relating the theory to different T-dual undoubled theories.

The polarisation projector

is given by

$$\Phi = \begin{pmatrix} \Pi \\ \tilde{\Pi} \end{pmatrix} .$$

This allows recovery of the physical and dual co-ordinates

$$\begin{aligned} X^i &= \Pi \mathbb{X}^I , \\ \tilde{X}^i &= \tilde{\Pi} \mathbb{X}^I . \end{aligned}$$

We also let $P = \Pi \mathcal{P}$ and $Q = \tilde{\Pi} \mathcal{P}$. In this basis

$$L = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} .$$

T-folds

- Recall T-folds can have T-duality transition functions and are in general not manifolds.
- However, in the doubled formalism they are described by a doubled torus which is a manifold.
- For non-geometric backgrounds there is no global choice of polarisation.

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Equivalence with the standard string sigma-model

- Classical equivalence with the standard sigma-model can be demonstrated by solving the constraint for the P^i in terms of the Q_i and using the classical equations of motion.
- The current $J_I = \mathcal{H}_{IJ}\mathcal{P}^J - L_{IJ} * \mathcal{P}^J$ is conserved and its vanishing implies the constraint in fact only $J^i = 0$ is necessary).
- Quantum equivalence can be shown by gauging this current.
- Adding the necessary terms only leaves the action gauge invariant if we include a topological term.

The Topological Term

- The topological term is given by

$$\mathcal{L}_t = \pi \tilde{\Pi}_{i[I} \Pi^i{}_{J]} d\mathbb{X}^I \wedge d\mathbb{X}^J = \pi dX^i \wedge d\tilde{X}_i \, .$$

- A non-standard normalisation is also required to show equivalence.
- Quantum equivalence was also shown in a specific case using Dirac brackets.

Holomorphic Factorisation

- A key issue in using the doubled formalism for quantum calculations is how to apply the constraint.
- Our method was to work in the basis where the constraint can be thought of as a chirality constraint and use techniques for writing the partition function of chiral bosons.
- Even for simple geometries this is difficult as you must separate the contribtions of the left and right bosons before you factorise them.
- The basis where the constraint is a chirality constraint is not the one in which the co-ordinates have integral periods.

The results

- We were able to reproduce the one-loop partition function that is well known for the standard string sigma-model.
- The topological term and unusual normalisation were needed, as in other methods.
- From consideration of higher loops it seems a T-duality invariant dilaton is appropriate for the formalism.

What we want to do.

- Calculating the beta-function provides an additional test of the quantum consistency of the doubled formalism.
- We wonder whether there are any additional conditions on allowed backgrounds.

The Usual Method.

- The action is expanded in fluctuations using the Background Field Expansion.
- We use the fluctuation propagator to perform contractions and find we are left with a term proportional to

$$R_{ij}\partial_\alpha X^i \partial^\alpha X^j \frac{1}{\epsilon}$$

- Renormalisation to remove the Weyl pole requires the vanishing of the background Ricci tensor.

Applying the Constraint

- We need a new method to apply the constraint; we want an action for chiral scalars.
- The PST procedure allows us to impose the self-duality constraints on the scalars.

For chiral scalar $P = RX + R^{-1}\tilde{X}$ and anti-chiral scalar $Q = RX - R^{-1}\tilde{X}$

$$
\begin{aligned}
S &= \int -\frac{1}{2}dP \wedge *dP - \frac{1}{2}dQ \wedge *dQ + \frac{(\mathcal{P}_m u^m)^2}{2u^2} + \frac{(\mathcal{Q}_m v^m)^2}{2v^2} \\
&= \frac{1}{2}\int -(R\partial_1 X)^2 - (R^{-1}\partial_1\tilde{X})^2 + 2\partial_0 X \partial_1 \tilde{X}\,.
\end{aligned}
$$

More generally $\mathcal{L} = -\frac{1}{2}\mathcal{H}\partial_1 X \partial_1 X + \frac{1}{2}L\partial_0 X \partial_1 X$.

A Fibred Set-up

We will consider a a doubled torus fibred over a base with co-ordinates Y. This has action

$$\mathcal{L} = -\mathcal{G}_{\alpha\beta}\partial_1 X^\alpha \partial_1 X^\beta + \mathcal{L}_{\alpha\beta}\partial_0 X^\alpha \partial_1 X^\beta + \mathcal{K}_{\alpha\beta}\partial_0 X^\alpha \partial_0 X^\beta \, ,$$

where

$$\mathcal{G} = \begin{pmatrix} \mathcal{H}(Y) & 0 \\ 0 & -G(Y) \end{pmatrix}, \; \mathcal{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}, \; \mathcal{K} = \begin{pmatrix} 0 & 0 \\ 0 & G(Y) \end{pmatrix}.$$

The Background Field Expansion

- Now we want to write X as a classical background plus a quantum fluctuation, there is a procedure for doing this in nice co-ordinates.
- We write the fluctuation in normal co-ordinates, ξ, which are tangent to geodesics of the background.
- These are contravariant, so expanding in ξ involves only tensors.
- The first order term in the expansion in ξ vanishes by the equations of motion of the background.

Background Field Expanding

- In the standard case at second order we get a kinetic term for ξ plus $R_{kijl}\xi^i\xi^j\partial_\alpha X^k\partial^\alpha X^l$.
- The expansion for a general tensor involves terms like $D_\alpha T$ and $D_\alpha D_\beta T$: $\mathcal{L}$ and $\mathcal{K}$ lead to many terms.
- Using equations of motion there are many simplifications to

$$\begin{aligned}\mathcal{L}_{(2)} \;=\;& -\mathcal{G}_{\alpha\beta}\partial_1\xi^\alpha\partial_1\xi^\beta + \mathcal{L}_{\alpha\beta}\partial_0\xi^\alpha\partial_1\xi^\beta + \mathcal{K}_{\alpha\beta}\partial_0\xi^\alpha\partial_0\xi^\beta \\ & -2\partial_\alpha\mathcal{G}_{\gamma\beta}\partial_1X^\gamma\xi^\alpha\partial_1\xi^\beta - \frac{1}{2}\partial_\alpha\partial_\beta\mathcal{G}_{\gamma\delta}\partial_1X^\gamma\partial_1X^\delta\xi^\alpha\xi^\beta \\ & +2\partial_a G_{bg}\xi^a\partial_0\xi^b\partial_0X^g + \frac{1}{2}\partial_a\partial_b G_{bg}\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_0X^\delta\end{aligned}$$

Vielbeins and Propagators

- We introduce a vielbein which allows us to move to the chiral basis where we know how to find the fluctuation propagators, this will introduce extra terms with vielbein derivatives.
- The divergence of the propagator for a chiral scalars with a flat FJ style action was known.
- More complex contractions can be related to these (e.g. $<\xi\partial\xi\xi\partial\xi>$).
- The pole behaviour of propagators of different chirality obey $\frac{1}{2}(\Delta_+ + \Delta_-) \sim \Delta$

Result

- In the case of flat base metric and zero B-field we find exactly the usual Ricci tensor after applying the background equations of motion.
- We also find agreement with the standard string picture when we allow non-trivial base metric and B-field
- There is no Lorentz anomaly.
- There was no need to use any information about isometry.
- Setting the duality invariant doubled dilaton to zero rather than the standard one actually greatly simplifies comparison to the standard case.

Summary

- The doubled formalism is a manifestly T-duality invariant formulation of string theory.
- The partition function on a torus is the same as that for the standard sigma-model
- We can use a FJ style action and background field expansion to reproduce the requirement of vanishing Ricci tensor, in more general cases we reproduce the beta-function equations.

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