Free Fermions and Thermal AdS/CFT - II

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Outline

Exact Solution at Finite N

Gauge theory Partition function S^3 [B.Sundborg and Aharony *et.al*]

- Start with the matrix model,

$$Z = \int [dU] \ \exp\left[\sum_{n=1}^{\infty} \frac{a_n(T)}{n} \ TrU^n \ TrU^{\dagger n}\right]$$

- Expand the exponential to obtain for the integrand

$$\exp\left[\sum_{n=1}^{\infty} \frac{a_n(T)}{n} \ TrU^n \ TrU^{\dagger n}\right] = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \prod_j a_j^{k_j} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{k}}(U^\dagger)$$

$$z_{\vec{k}} = \prod_j k_j! j^{k_j} \quad and \quad \Upsilon_{\vec{k}}(U) = \prod_{j=1}^{\infty} (TrU^j)^{k_j}.$$

Frobenius formula

- Use Frobenius formula,

$$\Upsilon_{\vec{k}}(U) = \sum_{R \in U(N)} \chi_R(C(\vec{k}))\, Tr_R U.$$

- $\chi_R(C(\vec{k}))$: character of the conjugacy class $C(\vec{k})$ of the permutation group S_l, ($l = \sum j k_j$) in the representation R of $U(N)$. $\vec{k} = \{k_1, k_2, ...\}$.

Orthonormality

- Orthogonality relation between the characters of $U(N)$,

$$\int [dU]\; Tr_R(U)\; Tr_{R'}(U^\dagger) = \delta_{RR'}.$$

- Carry out the integral over holonomy U.

The Exact N partition function : $\lambda = 0$

- Finally we obtain

$$Z(\beta) = \sum_{\vec{k}} \frac{\prod_j a_j^{k_j}}{z_{\vec{k}}} \sum_R \left[\chi_R(C(\vec{k}))\right]^2 .$$

The Special Case

- In a special case where $a_n = 0$ for $n > 1$ the partition function becomes

$$Z(\beta) = \sum_{k=0}^{\infty} \sum_{R} \frac{1}{k!} \left[d_R(S_k)\right]^2 a_1^k .$$

- $\chi_R(C(\vec{k})) \to d_R(S_k) \to$ dimension of the representation.

The (a,b) Model: Effective Action

- This method can easily be generalized to the (a,b) model.
- The effective action for (a,b) model is given by,

$$S_{eff} = a_1(\lambda, T) TrU TrU^\dagger + \frac{b_1(\lambda, T)}{N^2} \left(TrU TrU^\dagger \right)^2 .$$

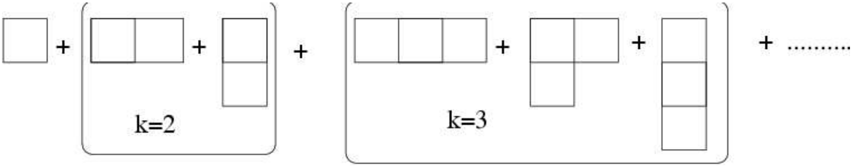
Exact N partition function for (a,b) Model

- The exact N partition function for (a,b) model is given by,

$$Z(a_1, b_1) = \sum_{k=0}^{\infty} \sum_{l=0}^{k/2} \frac{a_1^{k-2l} b_1^l k!}{N^{2l} l! (k-2l)!} \sum_R \frac{d_R^2(S_k)}{k!} .$$

Rearrange the sum over representation

- One can rearrange the sum over representations of $U(N)$ in terms of the number of boxes of the corresponding Young tableaux.



- $n_i \rightarrow$ Number of boxex in the ith row of the Young tableaux.

Partition function at $\lambda = 0$

- Therefore the partition function reads as

$$Z(\beta) = \sum_{k=0}^{\infty} \sum_{\{n_i\}=0}^{\infty} \frac{1}{k!} [d_R(S_k)]^2 \, a_1^k \, \delta(\Sigma_{i=1}^{N} n_i - k).$$

Dimension Formula

- The dimension $d_R(S_k)$ is given by the formula

$$d_R(S_k) = \frac{k!}{h_1!h_2!...h_N!} \prod_{i<j} (h_i - h_j),$$

where,

$$h_i = n_i + N - i,$$

with

$$h_1 > h_2 > ... > h_N \geq 0.$$

Taking the large N Limit

Partition function as stat. mech system

- In $N \to \infty$ the limit, the partition function can be viewed as a statistical mechanical system.
  - The group characters $\to$ entropy contribution .
  - a_1 (b_1) $\to$ Boltzmann suppression factors.

Phase Transition

- Interplay between these two factors.
- The balance between them leads to a dominant representation at any particular value of the temperature.
- At large N, as we vary temperature the dominance of saddle points changes => **phase transition.**

Continuum Limit and Saddle Point Equations

Definitions

- In $N \to \infty$ limit, let us define

$$n(x) = \frac{n_i}{N}, \quad h(x) = \frac{h_i}{N}, \quad x = \frac{i}{N} \quad x \in [0,1] \; .$$

- In this limit h(x) and n(x) are related by,

$$h(x) = n(x) + 1 - x.$$

- The function $n(x)$ or $h(x)$ captures the profile of the large N Young tableaux.

Constraint on h

$$h(x) > h(y) \quad for \quad y \; > \; x.$$

Introduce Young tableaux density

- Introduce the density of boxes in the Young tableaux $u(h)$ defined by

$$u(h) = -\frac{\partial x(h)}{\partial h}.$$

Normalization

- By definition, it obeys the normalization

$$\int_{h_L}^{h_U} dh\, u(h) = 1, \quad h_L = h(1) \;\; and \;\; h_U = h(0).$$

Constraint

- From the monotonicity of $h(x)$, it follows that $u(h)$ obeys the constraint

$$u(h) \leq 1.$$

The effective action: $\lambda = 0$

- In terms of the saddle point density the partition function becomes : $Z(\beta) = \int [dh] \exp[-N^2 S_{eff}]$.

$$
\begin{aligned}
-S_{eff} &= \int_{h_L}^{h_U} dh \fint_{h_L}^{h_U} dh' u(h) u(h') \ln|h - h'| \\
&- 2 \int_{h_L}^{h_U} dh\, u(h)\, h \ln(h) + k' + 1 + k' \ln(a_1 k').
\end{aligned}
$$

Total No. of boxes

- Where k' is related to total number of boxes 'k' in a Young tableaux

$$
k = N^2 \int_{h_L}^{h_U} dh\, h\, u(h) = N^2 k'.
$$

The Saddle point Equation: $\lambda = 0$

- Varying S_{eff} with respect to $u(h)$, we obtain the saddle point equation,

$$\int_{h_L}^{h_U} dh' \frac{u(h')}{h-h'} = \ln h - \frac{1}{2}\ln\left[a_1 k'\right] = \ln\left[\frac{h}{\xi}\right]$$

- where $\xi^2 \equiv a_1 k'.$

One important note

- ξ involves k'.
- Which in turns depends on the density $u(h)$.
- We will therefore have to solve the equation self-consistently.

The Solutions : Saddle point densities

Two kinds of saddle points

- In presence of constraint on $u(h)$ there are two different possible solutions $\rightarrow$ depending on the value of the parameter ξ.
- We will sketch the procedure and show the final result in this talk.

The Saddle points

Solution Class 1

$$0 \leq u(h) < 1; \quad h \in [q, p].$$

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The Saddle points

Solution Class 1

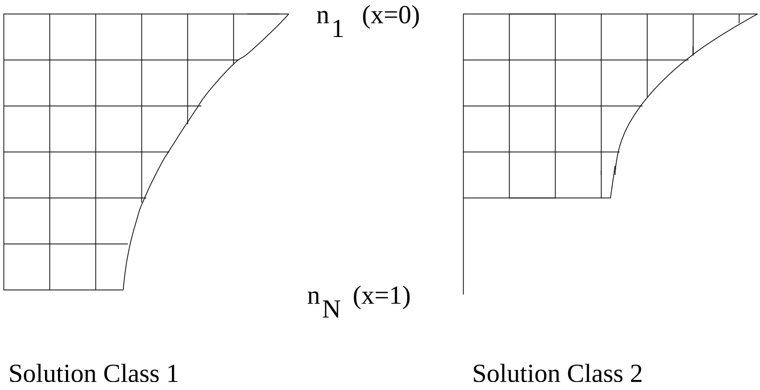
$$0 \leq u(h) < 1; \qquad h \in [q, p].$$

Solution Class 2

$$\begin{aligned} u(h) &= 1 & h \in [0, q] \\ &= \tilde{u}(h) & h \in [q, p] \end{aligned}$$

$$\text{with} \quad 0 \leq \tilde{u}(h) < 1.$$

- As we vary ξ, one will have to switch from one of the branches to the other.



Generic plots of Young tableaux

The sketch of the analysis

Resolvent

- Introducing the resolvent $H(h)$ defined by

$$H(h) = \int_{h_L}^{h_U} dh' \frac{u(h')}{h - h'}.$$

- Find the expression for $H(h)$ for two different solution classes.
- Young tableaux density is given by,
  $u(h) = -\frac{1}{2\pi i}\left[H(h + i\epsilon) - H(h - i\epsilon)\right]$ for $h \in [q, p]$.
- The support of $u(h)$ as well as k' is determined by expanding $H(h)$ for large h and matching with
  $H(h) \sim \frac{1}{h} + (k' + \frac{1}{2})\frac{1}{h^2} \quad (\text{as } h \to \infty)$.

The Saddle densities: Solution Class 1 : $\lambda = 0$

- The Young tableaux density in this solution class is given by,

$$
\begin{aligned}
u(h) &= \frac{1}{\pi}\cos^{-1}\left[\frac{h-1}{2\,\xi} + \frac{\left(\xi - \frac{1}{2}\right)^2}{2\,\xi\,h}\right] \\
&= \frac{2}{\pi}\cos^{-1}\left[\frac{h+\xi-1/2}{2\sqrt{\xi h}}\right]. \quad h \in [q,p]
\end{aligned}
$$

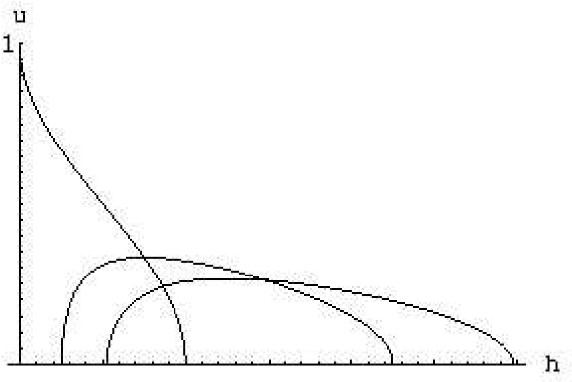
The supports : p and q

- The support of $u(h)$ as well as k' is given by,

$$\begin{aligned}\sqrt{q} &= \sqrt{\xi} - \frac{1}{\sqrt{2}}\,, \\ \sqrt{p} &= \sqrt{\xi} + \frac{1}{\sqrt{2}}\end{aligned}$$

- $k' = \sqrt{q\,p} + \frac{1}{4} \;=>\; a_1 = \frac{4\,\xi^2}{4\,\xi - 1}$

- q is a real and positive quantity => this branch of solution exists for $\xi \geq \frac{1}{2}$.
- **Conclusion :** this class of solutions only exist for $a_1 \geq 1$ or $T > T_H$ where $a_1(T_H) = 1$



Plot of u(h) Vs. h : Solution class 1

The Saddle densities: Solution Class 2: $\lambda = 0$

- The Young tableaux density in this solution class is given by,

$$\tilde{u}(h) = \frac{1}{\pi} \cos^{-1}\left[\frac{h-1}{2\xi}\right] , \quad h \in [q, p]$$

The supports: p and q

- The support of $u(h)$ is given by,

$$\begin{aligned} q &= 1 - 2\xi \, , \\ p &= 1 + 2\xi \end{aligned}$$

- $k' = \xi^2$

- $\xi \leq \frac{1}{2}$.

Two possible solutions

- From the definition $a_1 k' = \xi^2$ we obtain

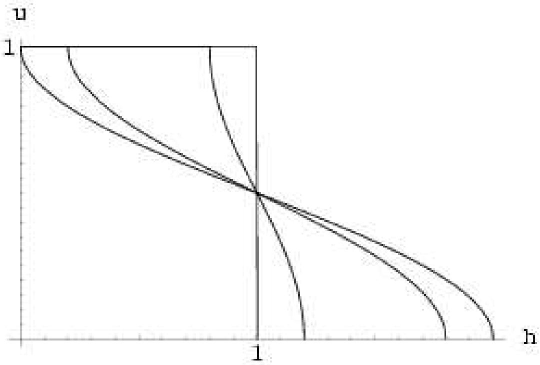
$$\begin{aligned} \textit{Either } \xi &= 0 \quad (2A) \\ \textit{or}, \ a_1 &= 1 \quad (2B). \end{aligned}$$

Solution 2A : $\xi = 0$

- Uniform distribution: $u(h) = 1 \quad h \in [0, 1].$
- This is a saddle point for any value of a_1.
- This saddle point corresponds to the trivial representation *i.e.* $n_i = 0$.

Solution 2B : $a_1 = 1$

- A family of saddle points labeled by ξ .
- Exists only at $a_1 = 1$ *i.e* at $T = T_H$.



Plot of $u(h)$ vs. h: Solution Class 2

The Free Energies

- In large N limit the free energy is given by,

$$F = -T \ln Z \ = N^2 T S^0_{eff} \ .$$

- S^0_{eff} is the value of effective action at the (dominant) saddle point.

Zero coupling Free energy: Solution Class 1

$$F = -N^2 T \left[\xi - \frac{1}{2} \ln(2\xi) - \frac{1}{2} \right] \leq 0.$$

Zero coupling Free energy: Solution Class 2

$$F = 0 + \mathcal{O}(\frac{1}{N^2})$$

Zero coupling phase diagram

$T < T_H$

- For low enough temperature ($T < T_H$) when $a_1 < 1$ there exists only one saddle point *i.e.* $\xi = 0$.
- This saddle point corresponds to trivial representation.
- Zero free energy.

$T = T_H$

- $a_1 = 1$.
- A family of solutions parameterized by $0 < \xi \leq \frac{1}{2}$.
- In fact $\xi = 0$ configuration is a limiting member of this family.
- A finite fraction of rows of Young tableaux are empty.
- Zero free energy.

$T > T_H$

- As we further increase the temperature $T > T_H$, then $a_1 > 1$.
- ξ has a solution greater than $\frac{1}{2}$.
- An exchange of dominance of the saddle point at $T = T_H$.
- All the rows of Young tableaux are filled.
- $a_1 > 1$ phase has negative free energy of order $\mathcal{O}(N^2)$.

Confinement-deconfinement phase transition

Extension to non-zero coupling

- To see the complete phase structure we have to turn on small 't Hooft coupling.

(a,b) Model

- We will consider the (a,b) model to explore the weak coupling phase diagram.

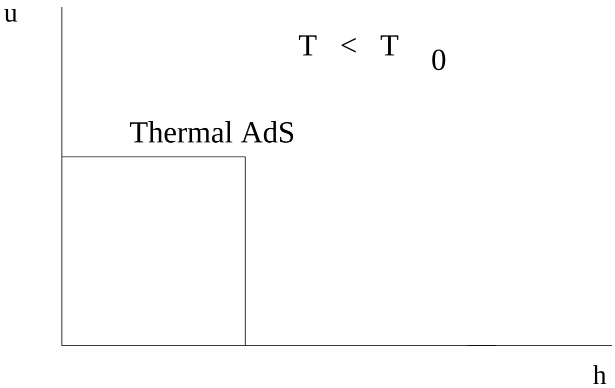
$$S_{eff} = a_1(\lambda, T) TrUTrU^\dagger + \frac{b_1(\lambda, T)}{N^2} \left(TrUTrU^\dagger\right)^2 .$$

- All the expressions for $u(h)$ and the supports p and q for both the solution classes are same as that of zero coupling.
- We will only discuss the phase diagram and the corresponding pattern of Young tableaux for different saddle points.

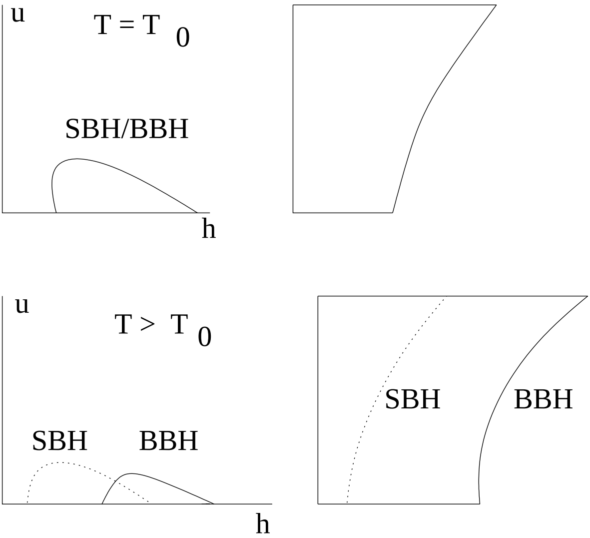
Weak coupling phase diagram

$T < T_0$

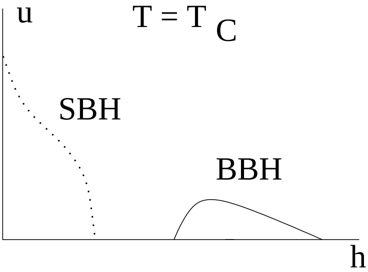
- For temperature $T < T_0$ there exists only one saddle point, $\xi = 0$. This saddle point corresponds to trivial representation.

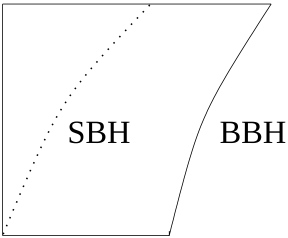


$T = T_0$ and $T > T_0$



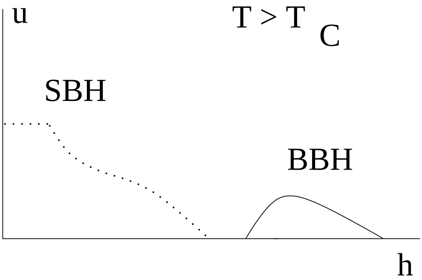
$T = T_c$

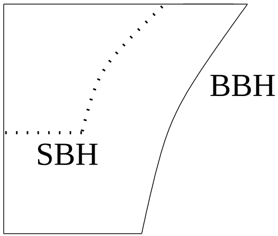




- BH-String transition : identified by Alverez Gaume-C. Gomez-H. Liu-Spenta Wadia

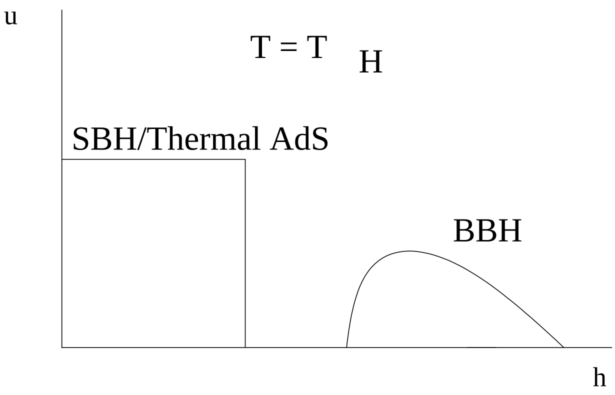
$T > T_c$





$T = T_H$

- At $T = T_H$ saddle point corresponds to SBH and saddle point corresponds to global AdS merge.
- $T > T_H$ BBH is the only saddle point.



Free Fermionic Phase space Description

Young tableauxVs. eigenvalue distribution

- There exists a simple relation between $(h, u(h))$ and $(\theta, \sigma(\theta))$.
- The saddle point for which $u(h)$ and h relation is one to one, we have a mapping,

$$u = \frac{\theta}{\pi}\,, \quad \frac{h}{2\pi} = \sigma(\theta)$$

- Where as, for the other saddle point, the mapping is non trivial, but simple.

$$\begin{aligned} u &= \frac{\theta}{\pi}\,, \quad \frac{h_+ - h_-}{2\pi} = \sigma(\theta) \\ \sin^2\frac{\theta_0}{2} &= \frac{1}{2\xi} \end{aligned}$$

Interpretation

- The above relations have a natural interpretation in terms of free fermionic picture.
- The eigenvalues θ's of the holonomy matrix U behave like coordinates of fermions.
- On the other hand the representation of $U(N)$ also have an interpretation in the language of free fermions with n_i's are like momenta. [Douglas]
- This suggests that eigen value density is like a position distribution.
- Young tableauxdensity is like momentum distribution.

- Define a phase-space distribution which gives rise to these individual distributions.

Phase space density

- At saddle point define a phase space density

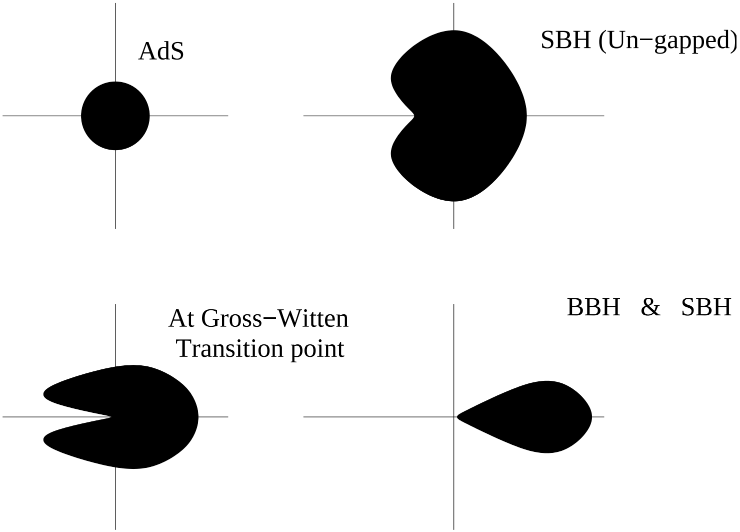
$$\begin{aligned}\rho(h,\theta) &= \frac{1}{2\pi} \quad (h,\theta)\in R \\ &= 0; \quad \textit{otherwise}\end{aligned}$$

- Then the position distribution and momentum distributions are given by,

$$\begin{aligned}\sigma(\theta) &= \int_0^\infty \rho(h,\theta)dh \\ u(h) &= \int_{-\pi}^{\pi} \rho(h,\theta)d\theta\end{aligned}$$

- We therefore see that the large N saddle points of the gauge theory effective action, which correspond to the Thermal AdS, the small black hole and the big black hole can all be thought of in terms of a particular configuration in a free fermionic phase space.

- There is a particular shape associated to each of them.


AdS
SBH (Un−gapped)
At Gross−Witten
Transition point
BBH & SBH

Summary

- We studied thermal gauge theory by evaluating its partition function at finite N.
- Taking large N limit of the full partition function gives us a new perspective on some already known facts about the phase diagram of gauge theory.
- We saw there is a close relation between Young tableaux density and eigen value density.
- This identification has a natural meaning in terms of Free fermion phase space distribution.
- We found different phase space distributions for different saddle points of the partition function.
- This formulation may be helpful to reconstruct the local theory in bulk with all its redundancies.