Consistent Truncation To Three Dimensional (Super-)gravity

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For 3-dim. pure (super-)gravity,the higher derivative terms in the action can be removed by field redefinition and

The action can be reduced to the (super-)gravity action whose gravitational part contains a sum of three terms, - the Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term.

Ads/CFT correspondance suggests that even when matter fields are present, it is possible to consistently truncate the theory to standard (super-)gravity without any matter fields.

Our main goal of the work is to describe a consistent truncation procedure directly in bulk theory without any reference to Ads/CFT.

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Field Redefinition of the Bosonic Fields

We begin with a three dimensional general coordinate invt. theory of gravity coupled to an arbitrary set of matter fields.

We denote by $g_{\mu\nu}$ the metric, by ϕ the set of all the scalar fields, by Σ the set of all other tensor fields.

At the level of two derivative terms, the action takes the form:

$$S = S_0 + S_{matter} \qquad (1)$$

where

$$S_0 = \int d^3x \sqrt{-g}(R + \Lambda(\phi)) \qquad (2)$$

and S_{matter} denotes the kinetic term for the matter fields.

We shall now consider the effect of adding higher derivative terms.

We assume that the length parameter l_s that controls these higher derivative terms is small compared to the length scale l_0 over which the leading sol. varies.

With each higher derivative term in the Lagrangian density ,we associate an index n that counts how many powers of l_s accompanies this term compared to the leading term.

In order to keep track of the higher derivative terms, we introduce a derivative counting parameter λ and accompany a trem of index n by a factor of λ^n.

We will carry out our analysis in a power series expansion in λ and at the end we set $\lambda = 1$.

We introduce a new variable $P_{\mu\nu}$ defined by

$$P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}(R + \Lambda(\phi))g_{\mu\nu} \tag{3}$$

So Ricci tensor can be expressed as

$$R_{\mu\nu} = P_{\mu\nu} - (P + \Lambda(\phi))g_{\mu\nu} \tag{4}$$

where

$$P = g^{\mu\nu}P_{\mu\nu} \tag{5}$$

In this convention the most general action takes the form

$$S = S_0 + \lambda S_{cs} + \tilde{S}_{matter} + \lambda^n S_n \tag{6}$$

$\tilde{S}_{matter}$ denotes the matter terms which are quadratic and higher order in Σ, derivatives of Σ and derivatives of ϕ.

$\lambda^n S_n$ denotes all other terms i.e. manifestly general coordinate invt. up to linear order in Σ, $\partial_\mu \phi$ and their derivatives.

It is easy to see that S_n must contain at least one power of $P_{\mu\nu}$ since $P_{\mu\nu}$ independent terms either can be absorbed in $\Lambda(\phi)$ or can be included in $\tilde{S}_{matter}$.

Thus S_n has the form

$$S_n = \int d^3x\, \sqrt{-g} P^{\mu\nu} K_{\mu\nu}(\phi, \Sigma, \nabla_\rho, g_{\rho\sigma}, P_{\rho\sigma}, \lambda) \tag{7}$$

Now consider a redefinition of the metric of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda^n K_{\mu\nu} \tag{8}$$

under this

$$\begin{aligned} S_0 \to & S_0 - \lambda^n \int d^3x\, \sqrt{-g} P^{\mu\nu} K_{\mu\nu} + O(\lambda^{2n}) \\ &= S_0 - \lambda^n S_n + O(\lambda^{2n}) \end{aligned} \tag{9}$$

$$S_{cs} \to S_{cs} + O(\lambda^{n+1}) \tag{10}$$

$$\lambda^n S_n \to \lambda^n S_n + O(\lambda^{2n}) \tag{11}$$

Thus

$$S_0 + \lambda S_{cs} + \lambda^n S_n \to S_0 + \lambda S_{cs} + O(\lambda^{n+1}) \tag{12}$$

Furthermore $\tilde{S}_{matter}$ remains quadratic in Σ and $\partial_\mu \phi$ under this field redefinition.

The order λ^{n+1} terms can now be regrouped into a term of the form $\sqrt{-g}\, f(\phi)$ that can be absorbed into a redefinition of $\Lambda_0(\phi)$,

a term quadratic in Σ and $\partial\phi$ that can be absorbed into $\tilde{S}_{matter}$ and a term containing at least one power in $P_{\mu\nu}$.
Thus the resulting action may be expressed as:

$$S = S_0' + \lambda S_{cs} + \tilde{S}'_{matter} + \lambda^{n+1} S_{n+1} \tag{13}$$

where

$$S_0' = \int d^3x \sqrt{-g}(R + \Lambda'(\phi)) \tag{14}$$

$$S_{n+1} = \int d^3x \sqrt{-g} P^{\mu\nu} K'_{\mu\nu}(\phi, \Sigma, \nabla_\rho, g_{\rho\sigma}, P_{\rho\sigma}, \lambda) \tag{15}$$

Repeating this process, to any order in an expansion in λ, the action can be brought to the form:

$$S = \int d^3x\,\sqrt{-g}(R + \Lambda(\phi)) + \lambda S_{cs} + \tilde{S}_{matter} \qquad (16)$$

Now suppose $\Lambda(\phi)$ has an extremum at $\phi = \phi_0$. Introducing new fields $\xi = \phi - \phi_0$ we may express the action as

$$S = \int d^3x\,\sqrt{-g}(R + \Lambda(\phi_0)) + \lambda S_{cs} + \cdots , \qquad (17)$$

We can now carry out a consistent truncation of the theory by setting $\xi = 0$, $\Sigma = 0$.

This leaves us with a purely gravitational action with Einstein-Hilbert term, cosmological constant term and Chern-Simons term.

In supergravity theory, there are additional bosonic fields like gauge fields with Chern-Simons terms.

$$S_{gauge} = \int d^3x Tr[A \wedge dA + \frac{2}{3} A \wedge A \wedge A] \qquad (18)$$

Now under the field redefinition

$$A_\mu \to A_\mu + \delta A_\mu \qquad (19)$$

the Chern-Simons term changes by a term proportional to $\epsilon^{\mu\nu\rho} Tr\left(F_{\mu\nu}\delta A_\rho\right)$.

Thus a term of the form $\lambda^n \int \sqrt{-g}\, Tr\left(F_{\mu\nu}L^{\mu\nu}\right)$ in the action may be removed (up to order λ^{2n} terms) by a shift of A_μ proportional to $\sqrt{-g}\, \epsilon_{\mu\nu\rho}L^{\nu\rho}$.

Algorithm for Determining $\Lambda(\phi)$

Of the various parameters labelling the final theory the coefficients of the Chern-Simons terms are easy to determine since they do not get renormalized from their initial values.

On the other hand the cosmological constant term does get renormalized during the field redefinition.

we shall outline a simple procedure for finding the exact $\Lambda(\phi)$.

Suppose our initial action has the form

$$S = \int d^3x \sqrt{-g}\mathscr{L}(\phi, R_{\mu\nu}, \Sigma) + \lambda S_{CS}\,. \qquad (20)$$

Since the final truncation involves setting ϕ to constant and Σ to 0, let us consider a theory of pure gravity obtained by setting Σ to 0 and ϕ to some constant values in the above action.

We now look for a solution of the theory of the form

$$ds^2 = -l^2(1 + 1/r^2)dt^2 + l^2(1 + 1/r^2)^{-1}dr^2 + l^2r^2d\phi^2\,, \tag{21}$$

representing an AdS_3 space of size l. If we define

$$F(l, \phi) = l^3\mathscr{L}(l, \phi, \Sigma = 0) \tag{22}$$

then the metric satisfies its equation of motion if l is chosen to be at the extremum l_{ext} of F.

Now consider the form of the action obtained after a field redefinition of the metric . After setting ϕ to a constant and Σ to 0, the action takes the form:

$$S = \int d^3x \sqrt{-g}(R + \Lambda(\phi)) + \lambda S_{cs} \tag{23}$$

If we evaluate $l^3(R + \Lambda(\phi))$ for the AdS_3 background, we get a new function

$$H(l, \phi) = l^3 \left[-\frac{6}{l^2} + \Lambda(\phi) \right] \tag{24}$$

At the extremum H is

$$H(\tilde{l}_{ext},\phi) = -\sqrt{\frac{32}{\Lambda(\phi)}} \qquad (25)$$
$$= F(l_{ext},\phi)$$

Hence we get

$$\Lambda(\phi) = \frac{32}{F(l_{ext},\phi)^2} \qquad (26)$$

provided $F(l_{ext},\phi)$ is negative.

Field Redefinition of the Gravitino

We begin with an action where the purely bosonic part has already been brought into the standard form using the field redefinition.

If the theory has altogether N supersymmetries then there are N gravitino fields ψ_μ^i with $1 \leq i \leq N$. In the supergravity action the gravitino action has the form:

$$S_0^\psi = -\int d^3x \, \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^i D_\nu \psi_\rho^i \tag{27}$$

Leading gravitino equation of motion

$$D_\nu \psi_\rho^i - D_\rho \psi_\nu^i = 0 \tag{28}$$

The supersymmetry transformation law of the gravitino fields takes the form

$$\delta_S \psi_\mu^i = D_\mu \epsilon^i \tag{29}$$

Now, we consider the possibility of adding higher derivative terms in the action.

Let us denote by η the set of all the bosonic and fermionic fields coming from the matter sector with the scalars measured relative to ϕ_0.

We need to worry about terms which are at most linear in η. We shall refer to these as the dangerous terms.

Let us suppose that the first dangerous higher derivative terms in the Lagrangian density appear at order λ^k.

Terms that is proportional to the equation of motion of supergravity fields derived from the leading supergravity action can be absorbed into a redefinition of these fields at the cost of generating higher order terms.

So,we will focus on terms which do not vanish by leading equ. of motion of supergravity fields.

We consider all the order λ^k terms and organise them by their rank.

The rank of a term is defined as the total power of ψ_μ and $\bar{\psi}_\mu$. We begin with the term of lowest rank, – call it m_0.

The lowest order supersymmetry variation of the gravitino has the effect of producing a term of rank $(m_0 - 1)$.

In order for supersymmetry to be preserved, such terms need to be cancelled by some other terms.
There are two possibilities:

1) the rank $(m_0 - 1)$ terms arising from the variation of the gravitino cancel among themselves after we integrate by parts and move all the derivatives from ϵ, $\overline{\epsilon}$ to the fields.
and

2) we can try to cancel these terms against terms coming from supersymmetry variation of the bosons in a term of rank $(m_0 - 2)$.

We arrived at the following conclusion:

“It is not possible to add higher derivative dangerous terms in the action,which do not vanish by equ. of motion, in a manner consistent with supersymmetry.”

Dimensional Reduction of Five Dimensional Supergravity

we shall consider five dimensional supergravity with curvature squared term coupled to a set of vector multiplets.

We will dimensionally reduce this theory on S^2.

We will also switch on the magnetic flux through S^2.

The resultant theory is (0,4) supergravity with curved square term.

We shall concentrate our attention on the part of the action involving the bosonic fields only.

The five dimensional $N = 2$ supergravity has a Weyl multiplet, a set of vector multiplets and a compensator hypermultiplet.

After gauge fixing to Poincare supergravity, the bosonic fields of the theory include the metric g_{ab}, the two-form auxiliary field v_{ab}, a scalar auxiliary field D, a certain number (n_V) of one-form gauge fields A_a^I with $1 \leq I \leq n_V$, and an equal number of scalars M^I.

The action for bosonic fields including curvature squared terms can be written as

$$S = \frac{1}{4\pi^2} \int d^5x \sqrt{-g^{(5)}} [\mathscr{L}_0 + \mathscr{L}_1] \tag{30}$$

$$
\begin{aligned}
\mathscr{L}_0 = & -2\left(\frac{1}{4}D - \frac{3}{8}R - \frac{1}{2}v^2\right) + N\left(\frac{1}{2}D + \frac{1}{4}R + 3v^2\right) + 2N_I v^{ab} F^I_{ab} \\
& + N_{IJ}\left(\frac{1}{4}F^I_{ab}F^{Jab} + \frac{1}{2}\partial_a M^I \partial^a M^J\right) \\
& + \frac{1}{24}e^{-1} c_{IJK} A^I_a F^J_{bc} F^K_{de} \epsilon^{abcde}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\mathscr{L}_1 = & \frac{c_{2I}}{24}\Big(\frac{1}{16}e^{-1}\epsilon_{abcde}A^{Ia}C^{bcfg}C^{de}_{fg} + \frac{1}{8}M^I C^{abcd}C_{abcd} \\
& + \frac{1}{12}M^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D - \frac{1}{3}M^I C_{abcd}v^{ab}v^{cd} \\
& - \frac{1}{2}F^{Iab}C_{abcd}v^{cd} + \frac{4}{3}M^I \nabla^a v^{bc} \nabla_a v_{bc} + \frac{4}{3}M^I \nabla^a v^{bc}\nabla_b v_{ca} \\
& + \frac{8}{3}M^I\left(v_{ab}\nabla^b\nabla_c v^{ac} + \frac{2}{3}v^{ac}v_{cb}R^b_a + \frac{1}{12}v^{ab}v_{ab}R\right) \\
& - \frac{2}{3}e^{-1}M^I\epsilon_{abcde}v^{ab}v^{cd}\nabla_f v^{ef} + \frac{2}{3}e^{-1}F^{Iab}\epsilon_{abcde}v^{cf}\nabla_f v^{de} \\
& + e^{-1}F^{Iab}\epsilon_{abcde}v^c_f \nabla^d v^{ef} - \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} - \frac{1}{3}F^{Iab}v_{ab}v^2 \\
& + 4M^I v_{ab}v^{bc}v_{cd}v^{da} - M^I(v_{ab}v^{ab})^2\Big)
\end{aligned} \tag{32}
$$

where c_{IJK} and c_{2I} are parameters of the theory,

$$N = \frac{1}{6} c_{IJK} M^I M^J M^K \tag{33}$$

$$N_I = \frac{1}{2} c_{IJK} M^J M^K \tag{34}$$

$$N_{IJ} = c_{IJK} M^K \,, \tag{35}$$

We now carry out the dimensional reduction on S^2 and focus on the sector invariant under the $SO(3)$ isometry group of S^2. This can be done using the following ansatz for the five dimensional fields

$$\begin{aligned}
& ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + \chi^2(x) d\Omega^2, \qquad 0 \le \mu, \nu \le 2 \\
& v_{\theta\phi} = V(x) \sin\theta \\
& F^I_{\theta\phi} = \frac{p^I}{2} \sin\theta, \qquad F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \,,
\end{aligned} \tag{36}$$

with the mixed components of F^I_{ab} and v^I_{ab} set to zero.

we get the dimensionally reduced action to be

$$
\begin{aligned}
S \ =& -\frac{c_2 \cdot p}{96\pi}\int d^3x\Omega^{(3)}(\Gamma) \\
&+\int d^3x\sqrt{-g}\frac{\chi^2}{\pi}\Big(\frac{3}{4}+\frac{1}{4}N+\frac{c_2\cdot M}{288}\frac{1}{\chi^2} \\
&+\frac{c_2\cdot M}{72}\frac{V^2}{\chi^4}-\frac{c_2\cdot p}{288}\frac{V}{\chi^4}\Big)R^{(3)} \\
&+\int d^3x\sqrt{-g}\frac{\chi^2}{\pi}U(\chi,M^I,V,p^I,D) \\
&+\int d^3x\sqrt{-g}\frac{\chi^2}{\pi}\frac{c_2\cdot M}{192}\Big(\frac{8}{3}R^{(3)}_{\mu\nu}R^{(3)\mu\nu}-\frac{5}{6}R^{(3)2} \\
&+\frac{16}{3\chi}R^{(3)}_{\mu\nu}\nabla^\mu\nabla^\nu\chi-\frac{4}{3\chi}R^{(3)}\nabla^2\chi\Big) \\
&+\int d^3x\sqrt{-g}\tilde{\mathscr{L}}(\chi,v_{\mu\nu},M^I,F^I_{\mu\nu},R^{(3)}_{\mu\nu})
\end{aligned}
\tag{37}
$$

Here $\tilde{\mathscr{L}}(\chi, v_{\mu\nu}, M^I, F^I_{\mu\nu}, R^{(3)}_{\mu\nu})$ denotes terms which are at least quadratic in $\nabla_\mu\chi, v_{\mu\nu}, \nabla_\mu M^I$ and $F^I_{\mu\nu}$.

We first need to redefine our metric in such a manner that the coefficient of $R^{(3)}$ in the second line of the action can be absorbed into the metric. We define

$$\tilde{g}_{\mu\nu} = \psi^{-2} g_{\mu\nu} \tag{38}$$

where

$$\psi^{-1} = \frac{\chi^2}{\pi}\Big(\frac{3}{4} + \frac{1}{4}N + \frac{c_2 \cdot M}{288}\frac{1}{\chi^2} + \frac{c_2 \cdot M}{72}\frac{V^2}{\chi^4} - \frac{c_2 \cdot p}{288}\frac{V}{\chi^4}\Big) \tag{39}$$

we now define

$$
\begin{aligned}
P_{\mu\nu} &= \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}[\tilde{R} + \Lambda_0(\phi)] \\
P &= -\frac{1}{2}\tilde{R} - \frac{3}{2}\Lambda_0(\phi)
\end{aligned}
\tag{40}
$$

where $\Lambda_0(\phi)$ is a function to be determined later, and rewrite the action as

$$
\begin{aligned}
S =& -\frac{c_2 \cdot p}{96\pi}\int d^3x\Omega^{(3)}(\tilde{\Gamma}) + \int d^3x\,\sqrt{-\tilde{g}}\Big[\tilde{R} + \Lambda_0(\phi)\Big] \\
&+ \int d^3x\,\sqrt{-\tilde{g}}P_{\mu\nu}K^{\mu\nu} \\
&+ \int d^3x\,\sqrt{-\tilde{g}}\tilde{\mathscr{L}}(\chi, v_{\mu\nu}, M^I F^I_{\mu\nu}, P_{\mu\nu}, P)
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
K_{\mu\nu} =& \frac{\chi^2}{\psi\pi}\frac{c_2 \cdot M}{192}\Big[\frac{8}{3}P_{\mu\nu} - \frac{2}{3}\widetilde{g}_{\mu\nu}P + \frac{2}{3}\widetilde{g}_{\mu\nu}\Lambda_0(\phi) - \frac{16}{3\psi}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\psi \\
&+ \frac{8}{3\psi}\widetilde{g}_{\mu\nu}\widetilde{\nabla}^2\psi + \frac{16}{3\chi}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\chi - \frac{8}{3\chi}\widetilde{g}_{\mu\nu}\widetilde{\nabla}^2\chi\Big].
\end{aligned}
\tag{42}
$$

$\Lambda_0(\phi)$ is a sol. of the equ.

$$\Lambda_0(\phi) = Z(\phi) + \frac{\chi^2}{\psi\pi}\frac{c_2 \cdot M}{384}\Lambda_0(\phi)^2\,, \qquad (43)$$

and

$$Z(\chi, M^I, V, p^I, D) = \psi^3\frac{\chi^2}{\pi}U(\chi, M^I, V, p^I, D) \qquad (44)$$

In this case the required field redefinition which will remove the four derivative terms from the action is

$$\widetilde{g}_{\mu\nu} \to \widetilde{g}_{\mu\nu} + K_{\mu\nu} \qquad (45)$$

To this order the scalar field potential $-\Lambda(\phi)$ is given by

$$\Lambda(\phi) = \Lambda_0(\phi) = Z(\phi) + \frac{\chi^2}{\psi\pi}\frac{c_2 \cdot M}{384}Z^2(\phi) + O(c_2^2)\,. \qquad (46)$$

This process can now be repeated to remove the six and higher derivative terms from the action.

Our interest is in finding the exact expression for $\Lambda(\phi)$. In the AdS_3 background with constant scalar fields and vanishing tensor fields. We get

$$F(l,\phi) = -6l + l^3 Z(\phi) + 2a\frac{1}{l} \tag{47}$$

where

$$a = \frac{\chi^2}{\psi\pi}\frac{c_2 \cdot M}{192} \,. \tag{48}$$

The extremum of $F(l,\phi)$ with respect to l occurs at

$$l_{ext}^2 = \frac{1}{Z(\phi)} + \frac{1}{Z(\phi)}\sqrt{1 + \frac{2a}{3}Z(\phi)} \,. \tag{49}$$

Hence $\Lambda(\phi)$ is given by

$$\Lambda(\phi) = \frac{32}{F(l_{ext},\phi)^2} = \frac{32Z(\phi)}{W(\phi)}\left(2a\frac{Z(\phi)}{W(\phi)} + W(\phi) - 6\right)^{-2} \tag{50}$$

$$W(\phi) = 1 + \sqrt{1 + \frac{2a}{3}Z(\phi)} \tag{51}$$

Extrema of $\Lambda(\phi)$ are located at

$$\begin{aligned} \chi &= \frac{pb}{2} \\ M^I &= \frac{p^I}{pb} \\ V &= -\frac{3pb}{8} \\ D &= \frac{12}{p^2 b^2} \end{aligned} \tag{52}$$

where

$$b^3 = 1 + \frac{c_2 \cdot p}{12p^3} \tag{53}$$

The value of $\Lambda(\phi)$ at it's extremum is given by

$$\Lambda(\phi_0) = \frac{32\pi^2}{p^6}\left[1 + \frac{c_2 \cdot p}{8p^3}\right]^{-2} \tag{54}$$

Thus the final truncated theory, obtained by setting ϕ to its value at the extremum and other matter fields to zero, is given by

$$S = \int d^3x \, \sqrt{-\widetilde{g}} \, (\widetilde{R} + \Lambda(\phi_0)) - \frac{c_2 \cdot p}{96\pi} \int d^3x \Omega^{(3)}(\widetilde{\Gamma}) \, . \tag{55}$$

From this one can compute the central charges of the conformal field theory living on boundary of AdS and is given by

$$\begin{aligned} c_L &= 24\pi\left(\sqrt{\frac{2}{\Lambda(\phi_0)}} - \frac{c_2 \cdot p}{96\pi} \right) = 6p^3 + \frac{1}{2}c_2 \cdot p \\ c_R &= 24\pi\left(\sqrt{\frac{2}{\Lambda(\phi_0)}} + \frac{c_2 \cdot p}{96\pi} \right) = 6p^3 + c_2 \cdot p \end{aligned} \tag{56}$$

These results agree with the predictions.

Conclusion

1. For general three dim. theory of (super-)gravity coupled to arbitrary matter fields with arbitrary set of higher derivative terms,it is possible to consistently truncate the theory to a theory of pure (super-)gravity.

2. We have also outlined the procedure for finding the exact value of parameter of the truncated theory. And the value of $\Lambda(\phi)$ at it's extremum corresponds to cosmological constant.