BLACK HOLE ENTROPY FUNCTIONS AND ATTRACTOR EQUATIONS

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(G. L. Cardoso, B. de Wit and SM)
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PLAN OF THE TALK

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INTRODUCTION

- An important feature of extremal black hole solutions is that scalar fields tend to fixed values at the horizon which are determined by the black hole charges.
- This fixed point behaviour is encoded in the so-called attractor equations which are a crucial ingredient in comparing the macroscopic and microscopic black hole entropy.
- This aspect has been studied extensively in the context of $N = 2$ supergravity in four space-time dimensions. The inclusion of higher-derivative interactions into the effective actions often played a crucial role.

- In the presence of higher derivative interactions, it is very difficult to explicitly construct black hole solutions and exhibit the attractor phenomenon. However, by concentrating on the near horizon geometry, one can usually determine the fixed point values and the attractor phenomenon can be described in terms of a variational principle for a so called "entropy function".
- For BPS black holes the attractor equations can be understood entirely from supersymmetry enhancement at the horizon. Can they correspond to special solutions of these more general attractor equations based on such variational principle?

THE REDUCED ACTION AND ENTROPY FUNCTION

Consider the action for a general system of abelian vector gauge fields, scalar and matter fields coupled to gravity. The geometry is a sphere S^2 times a $1+1$ dimensional space-time.
The general field configuration is given by,

$$ds^2{}_{(4)} = v_1\Big(-r^2dt^2 + \frac{dr^2}{r^2}\Big) + v_2\Big(d\theta^2 + \sin^2\theta\, d\varphi^2\Big)\,,$$

$$F_{rt}{}^I = e^I\,, \qquad F_{\theta\varphi}{}^I = \frac{p^I}{4\pi}\sin\theta$$

p^I denote the magnetic charges and the fields e^I are dual to the electric charges. In addition to the fields e^I, v_1 and v_2 there may be a number of other fields which we collectively denote by u_α.

In the above background, the dual field strengths are given by,

$$\begin{aligned} G_{\theta\varphi\, I} &= -v_1 v_2 \,\sin\theta\, \frac{\partial\mathcal{L}}{\partial F_{rt}{}^I} = -v_1 v_2 \,\sin\theta\, \frac{\partial\mathcal{L}}{\partial e^I}\,, \\ G_{rt\, I} &= -v_1 v_2 \,\sin\theta\, \frac{\partial\mathcal{L}}{\partial F_{\theta\varphi}{}^I} = -4\pi\, v_1 v_2\, \frac{\partial\mathcal{L}}{\partial p^I} \end{aligned}$$

These two tensors can be written as $q_I\, \sin\theta/(4\pi)$ and f_I. Hence one obtains,

$$\begin{aligned} q_I(e,p,v,u) &= -4\pi\, v_1 v_2\, \frac{\partial\mathcal{L}}{\partial e^I}; \\ f_I(e,p,v,u) &= -4\pi\, v_1 v_2 \frac{\partial\mathcal{L}}{\partial p^I} \end{aligned}$$

- Our aim is to obtain a description in terms of the charges p^I and q_I, rather than in terms of p^I and e^I.

We define the reduced Lagrangian by the integral of the full Lagrangian over S^2,

$$\mathcal{F}(e,p,v,u) = \int \mathrm{d}\theta\, \mathrm{d}\varphi\, \sqrt{|g|}\, \mathcal{L}$$

The conjugate quantities q_I and f_I take the form,

$$q_I = -\frac{\partial \mathcal{F}}{\partial e^I}\,, \qquad f_I = -\frac{\partial \mathcal{F}}{\partial p^I}$$

Under electric/magnetic duality, the Lagrangian does not transform as a symplectic function, but the combination $\mathcal{F}(e,p,v,u) + \frac{1}{2}[e^I q_I + f_I p^I]$ transforms as a function. The combination $e^I q_I - f_I p^I$ also transforms as a symplectic function under $q_I \longrightarrow \tilde{q}_I$ and $p^I \longrightarrow \tilde{p}^I$, where, $(\tilde{p}^I, \tilde{q}_I)$ are related to (p, q) by $\mathrm{Sp}(2n+2;\mathbb{R})$ rotation:

$$\tilde{p}^I = U^I{}_J p^J + Z^{IJ} q_J \,; \quad \tilde{q}_I = V_I{}^J q_J + W_{IJ} p^J$$

and $U^I{}_J$, $V_I{}^J$, W_{IJ}, Z^{IJ} are constant real $(n+1)\times(n+1)$ submatrices subject to

$$U^{\mathrm{T}}V - W^{\mathrm{T}}Z = V^{\mathrm{T}}U - Z^{\mathrm{T}}W = 1$$
$$U^{\mathrm{T}}W = W^{\mathrm{T}}U \,, \quad Z^{\mathrm{T}}V = V^{\mathrm{T}}Z$$

One can show that the quantity

$$\mathcal{E}(q,p,v,u) = -\mathcal{F}(e,p,v,u) - e^I q_I$$

transforms as a function under electric/magnetic duality. The above definition takes the form of a Legendre tansform. $\mathcal{E}(q,p,v,u)$ is in fact minus the Hamiltonian density associated with the reduced Lagrangian density.

- $v_{1,2}$ and u_α are determined by demanding $\mathcal{E}$ to be stationary under variations of v and u:

$$\frac{\partial \mathcal{E}}{\partial v} = \frac{\partial \mathcal{E}}{\partial u} = 0; \quad q_I = \text{constant}$$

- The first two equations are interpreted as the attractor equations and the function $2\pi\mathcal{E}$ coincides with the entropy function proposed by Sen (0506177).
- The Wald entropy is directly proportional to the value of $\boxed{\mathcal{E}}$ at the stationary point:

$$\mathcal{S}_{\text{macro}}(p,q) = 2\pi\mathcal{E}\Big|_{\text{attractor}}$$

- This derivation of the entropy function applies also to Lagrangians containing higher-derivative terms.

N=2 SUPERGRAVITY AND BPS ENTROPY FUNCTION

- $D = 4$, $N = 2$ supergravity theories with n abelian vector multiplets are based on a holomorphic prepotential function $F(X)$ which depends on complex fields X^I with $I = 0, 1, \ldots, n$.
- The resulting "Special geometry" can be defined in terms of a symplectic section V given by $(X^I, F_I(X))$, where $F_I = \frac{\partial F(X)}{\partial X^I}$ $(I = 0, 1, \ldots, n)$.
- One can incorporate higher-order derivative interactions involving the square of the Weyl tensor and the prepotential function is expressed as $F(X, A)$.

- The BPS attractor equations are given by,

$$\mathcal{P}^I = 0\,, \qquad \mathcal{Q}_I = 0\,, \qquad \Upsilon = -64\,,$$

$$\mathcal{P}^I \equiv p^I + \mathrm{i}(Y^I - \bar{Y}^I); \quad \mathcal{Q}_I \equiv q_I + \mathrm{i}(F_I - \bar{F}_I)$$

Υ is a complex scalar field equal to the square of the $N = 2$ auxiliary field $T_{ab}{}^{ij}$ of the Weyl multiplet. Y^I and Υ are related to X^I and A respectively by a uniform rescaling.

- The BPS attractor equations can also be described by a variational principle based on an entropy function.

$$\Sigma(Y,\bar{Y},p,q) \;=\; \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - q_I(Y^I+\bar{Y}^I) + p^I(F_I+\bar{F}_I)$$

$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$ denoted as free energy, is defined by

$$\begin{aligned}\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) \;&=\; -i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right)\\ &\quad -2\mathrm{i}\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon\right)\end{aligned}$$

where $F_\Upsilon = \partial F/\partial\Upsilon$.

Varying the BPS entropy function Σ with respect to the Y^I and using the homogeneity of the function $F(Y,\Upsilon)$, one obtains,

$$\delta\Sigma = \mathcal{P}^I\delta(F_I+\bar F_I) - \mathcal{Q}_I\,\delta(Y^I+\bar Y^I)$$

$\Rightarrow$ stationary points of Σ satisfy the attractor equations. Explicit calculation gives,

$$\mathcal{S}_{\rm macro}(p,q) = \pi\,\Sigma\Big|_{\rm attr} = \pi\Big[|Z|^2 - 256\,\mathrm{Im}\,F_\Upsilon\Big]_{\Upsilon=-64}$$

- The entropy functions $\mathcal{E}$ and Σ for $N=2$ supergravity systems are in fact related and they lead to identical results for BPS black holes.

The relevant $N = 2$ supergravity Lagrangian involving the vector multiplets, possible effects from the Weyl multiplet and including a hypermultiplet compensator, is given by (Cardoso, de Wit, Käppeli, Mohaupt, hep-th/0009234),

$$
\begin{aligned}
&8\pi e^{-1}\mathcal{L} = \\
&\quad i\mathcal{D}^{\mu}F_I\mathcal{D}_{\mu}\bar{X}^I - iF_I\bar{X}^I(\frac{R}{6} - D) - \frac{i}{8}F_{IJ}Y_{ij}{}^{I}Y^{Jij} - \frac{i}{4}\hat{B}_{ij}F_{AI}Y^{Iij} \\
&\quad +\frac{i}{4}F_{IJ}(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}{}^{ij}\varepsilon_{ij})(F^{-Jab} - \frac{1}{4}\bar{X}^J T^{ijab}\varepsilon_{ij}) \\
&\quad -\frac{i}{8}F_I(F_{ab}^{+I} - \frac{1}{4}X^I T_{abij}\varepsilon^{ij})T^{ab}{}_{ij}\varepsilon^{ij} \\
&\quad +\frac{i}{2}\hat{F}^{-ab}F_{AI}(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}{}^{ij}\varepsilon_{ij}) \\
&\quad +\frac{i}{2}F_A\hat{C} - \frac{i}{8}F_{AA}(\varepsilon^{ik}\varepsilon^{jl}\hat{B}_{ij}\hat{B}_{kl} - 2\hat{F}_{ab}^{-}\hat{F}^{-ab}) - \frac{i}{32}F(T_{abij}\varepsilon^{ij})^2 + \text{h.c.} \\
&\quad -\frac{1}{2}\varepsilon^{ij}\bar{\Omega}_{\alpha\beta}\mathcal{D}_{\mu}A_i{}^{\alpha}\mathcal{D}^{\mu}A_j{}^{\beta} + \chi(\frac{1}{6}R + \frac{1}{2}D)
\end{aligned}
$$

We consider a restricted class of solutions and assume a set of constraints :

$$
\begin{aligned}
\hat{B}_{ij} &= Y_{ij}{}^{I} = 0 \\
R(\mathcal{V})_{\mu\nu}{}^{i}{}_{j} &= R(A)_{\mu\nu} = \mathcal{D}_{\mu} X^{I} = \mathcal{D}_{\mu} A_{i}{}^{\alpha} = 0
\end{aligned}
$$

And we have,

$$
\begin{aligned}
\hat{A} &= (T_{ab}{}^{ij} \varepsilon_{ij})^2 = -4w^2; \\
\hat{F}^{-ab} &= -16\,\mathcal{R}(M)_{cd}{}^{ab} T^{klcd}\, \varepsilon_{kl} \\
\hat{C} &= 64 \mathcal{R}(M)^{-cd}{}_{ab}\, \mathcal{R}(M)^{-ab}_{cd} - 32 T^{ab\,ij} D_a D^c T_{cb\,ij}
\end{aligned}
$$

- It is useful to express the entropy function computed from the Lagrangian in terms of a set of scale invariant variables:

$$Y^I = \frac{1}{4} v_2 \bar{w}\, X^I, \quad \Upsilon = \frac{1}{16} v_2^2\, \bar{w}^2\, \hat{A} = -\frac{1}{4} v_2^2 |w|^4$$

$$U = \frac{v_1}{v_2}, \tilde{D} = v_2\, D + \frac{2}{3}(U^{-1} - 1), \quad \tilde{\chi} = v_2 \chi$$

We compute the relevant quantities appearing in the Lagrangian for the near-horizon background specified in terms of the parameters given above.

$$
\begin{aligned}
R &= 2\left(v_1^{-1} - v_2^{-1}\right), \\
f_i{}^j &= [\frac{1}{2}v_1^{-1} - \frac{1}{4}(D + \frac{R}{3}) - \frac{1}{32}|w|^2]\,\delta_i{}^j , \\
f_\alpha^\beta &= [-\frac{1}{2}v_2^{-1} - \frac{1}{4}(D + \frac{R}{3}) + \frac{1}{32}|w|^2]\,\delta_\alpha^\beta , \\
\mathcal{R}(M)_{ij}^{kl} &= (D + \frac{R}{3})\,\delta_{ij}{}^{kl}, \mathcal{R}(M)_{\alpha\beta}^{\gamma\delta} = (D + \frac{R}{3})\,\delta_{\alpha\beta}^{\gamma\delta} \\
\mathcal{R}(M)_{i\alpha}^{j\beta} &= \frac{1}{2}(D - \frac{R}{6})\,\delta_i^j\,\delta_\alpha^\beta \\
\hat{A} &= -4w^2, \hat{F}_{\underline{rt}}^- = -\mathrm{i}\hat{F}_{\underline{\theta\varphi}}^- = -16w(D + \frac{R}{3}) \\
\hat{C} &= 192D^2 + \frac{32}{3}R^2 - 16|w|^2(v_1^{-1} + v_2^{-1}) + 2|w|^4
\end{aligned}
$$

Indices i, j refer to the AdS_2 coordinates r, t, whereas indices α, β refer to S^2 coordinates θ, φ.

The reduced Lagrangian is then obtained as sum of $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\mathcal{F}_1 = \frac{1}{8}N_{IJ}\Big[U^{-1}e^I e^J - U p^I p^J\Big] - \frac{1}{4}(F_{IJ} + \bar{F}_{IJ})e^I p^J$$
$$+\frac{i}{2}e^I\Big[F_I + F_{IJ}\bar{Y}^J + 8F_{I\Upsilon}\sqrt{-\Upsilon}\tilde{D} - \text{h.c.}\Big]$$
$$-\frac{1}{2}Up^I\Big[F_I - F_{IJ}\bar{Y}^J - 8F_{I\Upsilon}\sqrt{-\Upsilon}\tilde{D} + \text{h.c.}\Big],$$

where, $N_{IJ} = \mathrm{i}(\bar{F}_{IJ} - F_{IJ})$

$$
\begin{aligned}
\mathcal{F}_2 = {} & \frac{4i}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I)(\tilde{D}U + U - 1) + \frac{1}{4}\tilde{\chi}\tilde{D}U \\
& + \mathrm{i}U\left[F - Y^I F_I - 2\Upsilon F_\Upsilon + \frac{1}{2}\bar{F}_{IJ}Y^I Y^J - \mathrm{h.c.}\right] \\
& + \mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[48U\tilde{D}^2 + 64\tilde{D}(U-1) + 32(U + U^{-1} - 2) \\
& - 8(1+U)\sqrt{-\Upsilon}\Big] \\
& + 32\mathrm{i}U\left[\tilde{D}^2\Upsilon F_{\Upsilon\Upsilon} - \frac{1}{4}\tilde{D}\bar{Y}^I F_{I\Upsilon}\sqrt{-\Upsilon} - \mathrm{h.c.}\right]
\end{aligned}
$$

These results refer to a general function $\boxed{F(Y, \Upsilon)}$.

One then obtains the expression for the entropy function $\mathcal{E}$ consistent with electric/magnetic duality :

$$
\begin{aligned}
\mathcal{E} = {} & \frac{1}{2}U\,\Sigma(Y,\bar{Y},p,q) \\
& +\frac{1}{2}U\,N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L) \\
& +8\mathrm{i}U\,\tilde{D}\sqrt{-\Upsilon}\Big[F_{I\Upsilon}N^{IJ}(\mathcal{Q}_J - \bar{F}_{JK}\mathcal{P}^K) - \text{h.c.}\Big] \\
& -\frac{4\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I\bar{F}_I)(\tilde{D}U + U - 1) - \frac{1}{4}\tilde{\chi}\tilde{D}U - \\
32\mathrm{i}U\,\tilde{D}^2 & \Big[\Upsilon F_{\Upsilon\Upsilon} + \frac{1}{2}\mathrm{i}\Upsilon\,N^{IJ}(F_{I\Upsilon} - \bar{F}_{I\Upsilon})(F_{J\Upsilon} - \bar{F}_{J\Upsilon}) - \text{h.c.}\Big] \\
& -\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[48U\,\tilde{D}^2 + 64\tilde{D}(U-1) - 2U\Upsilon \\
& +32(U + U^{-1} - 2) - 8(1+U)\sqrt{-\Upsilon}\Big]
\end{aligned}
$$

Requiring that $\mathcal{E}$ be stationary with respect to variations of $\tilde{D}$ and $\tilde{\chi}$, we obtain the conditions (we assume $U \neq 0$),

$$\begin{aligned}
\tilde{D} &= 0, \\
\tilde{\chi} &= -\frac{16\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I) - 256\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)(1 - U^{-1}) \\
&\quad + 32\mathrm{i}\sqrt{-\Upsilon}\Big[F_{I\Upsilon} N^{IJ}(\mathcal{Q}_J - \bar{F}_{JK}\mathcal{P}^K) - \mathrm{h.c.}\Big]
\end{aligned}$$

These conditions lead to a much simpler version of the entropy function $\mathcal{E}$:

$$
\begin{aligned}
\mathcal{E}(Y,\bar{Y},\Upsilon,U) = {}& \frac{1}{2}U\,\Sigma(Y,\bar{Y},p,q) + \\
& \frac{1}{2}U\,N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L) \\
& - \frac{4\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I)(U-1) \\
& -\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[- 2U\Upsilon + 32(U + U^{-1} - 2) - 8(1+U)\sqrt{-\Upsilon}\Big]
\end{aligned}
$$

- This is the "generalized entropy function" depending on the variables U, Υ and Y^I.
- The macroscopic entropy is proportional to the entropy function evaluated at the attractor values,

$$
\mathcal{S}_{\mathrm{macro}}(p,q) = 2\pi\mathcal{E}\Big|_{\mathrm{attractor}}
$$

VARIATIONAL EQUATIONS

Let us first consider the variation eqn. without R^2 correction. In the absence of R^2-interactions, the function F does not depend on Υ. So the entropy function reduces to

$$\mathcal{E}(Y,\bar{Y},\Upsilon,U) = \frac{1}{2}U\,\Sigma(Y,\bar{Y},p,q)+ \frac{1}{2}U\,N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L) -\frac{4\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I)(U-1)$$

- Varying with respect to Υ yields $U=1$.
- Value of Υ is determined in terms of Y^I from the variation with respect to U,

The effective entropy function is then given by,

$$\mathcal{E}(Y,\bar{Y},\Upsilon,1) = \frac{1}{2}\Sigma(Y,\bar{Y},p,q)+$$
$$\frac{1}{2}N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L)$$

which is independent of Υ. The attractor value of Y^I is obtained from the variation of the entropy function under infinitesimal changes of Y^I and $\bar{Y}^I$:

$$\delta\mathcal{E} = 0 = \mathcal{P}^I\,\delta(F_I + \bar{F}_I) - \mathcal{Q}_I\,\delta(Y^I + \bar{Y}^I)$$
$$+\frac{1}{2}\mathrm{i}\left(\mathcal{Q}_K - \bar{F}_{KM}\,\mathcal{P}^M\right)N^{KI}\,\delta F_{IJ}\,N^{JL}\left(\mathcal{Q}_L - \bar{F}_{LN}\,\mathcal{P}^N\right)$$
$$-\frac{1}{2}\mathrm{i}\left(\mathcal{Q}_K - F_{KM}\,\mathcal{P}^M\right)N^{KI}\,\delta\bar{F}_{IJ}\,N^{JL}\left(\mathcal{Q}_L - F_{LN}\,\mathcal{P}^N\right)$$

where $\delta F_I = F_{IJ}\delta Y^J$ etc. This equation determines the horizon value of Y^I in terms of black hole charges (p^I, q_I). One then finds the attractor value of Υ to be,

$$\sqrt{-\Upsilon} = \frac{8\,\Sigma}{\Sigma + N^{IJ}\left(\mathcal{Q}_I - F_{IK}\,\mathcal{P}^K\right)\left(\mathcal{Q}_J - \bar{F}_{JL}\,\mathcal{P}^L\right)}\ .$$

The macroscopic entropy is then given by,

$$\mathcal{S}_{\mathrm{macro}}(p,q) = 2\pi\,\mathcal{E}\Big|_{\mathrm{attractor}} = \frac{8\pi\,\Sigma}{\sqrt{-\Upsilon}}\Big|_{\mathrm{attractor}}\ .$$

- For a BPS black hole, $\mathcal{Q}_I = \mathcal{P}^J = 0$ and $\Upsilon = -64$, so that $\mathcal{S}_{\rm macro} = \pi\,\Sigma\big|_{\rm attractor}$, which agrees with the previous result.

NON-BPS BLACK HOLES WITH R^2-INTERACTIONS

- We consider extremal black holes in the presence of R^2-terms and compute the extremization equations for the fields U, Υ and Y^I following from the entropy function.
- Varying with respect to U gives,

$$\Sigma + \left(\mathcal{Q}_I - F_{IK}\mathcal{P}^K\right) N^{IJ} \left(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L\right) - \frac{8\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I) - \mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[-4\Upsilon + 64(1-U^{-2}) - 16\sqrt{-\Upsilon}\Big] = 0$$

One may verify that the BPS conditions $\mathcal{P}^I = \mathcal{Q}_I = 0$ and $\Upsilon = -64$ leaves only the term proportional to $(1-U^{-2})(F_\Upsilon - \bar{F}_\Upsilon)$ which vanishes as a result of $U = 1$.

The $\boxed{Y^I}$ attractor equation is obtained from the variation of th entropy function:

$$
\begin{aligned}
&U\left(\mathcal{Q}_I - F_{IJ}\,\mathcal{P}^J\right) \\
-\frac{i}{2}U&\left(\mathcal{Q}_K - \bar{F}_{KM}\mathcal{P}^M\right) N^{KP} F_{PIQ}\, N^{QL}\left(\mathcal{Q}_L - \bar{F}_{LN}\,\mathcal{P}^N\right) \\
&+4\mathrm{i}(-\Upsilon)^{-1/2}(U-1)\left[F_I - \bar{F}_I - F_{IJ}(Y^J - \bar{Y}^J)\right] \\
&-\mathrm{i}\left[2U\,\Upsilon - 32(U+U^{-1}-2) + 16\sqrt{-\Upsilon}\right] F_{\Upsilon I} = 0
\end{aligned}
$$

Another result is obtained by taking the variation $\delta = Y^I\partial/\partial Y^I + \bar{Y}^I\partial/\partial\bar{Y}^I + 2\Upsilon\partial/\partial\Upsilon$ and using the homogeneity properties of the various quantities involved :

$$U\Sigma - \mathrm{i}(\bar{Y}^I F_I - \bar{F}_I Y^I)\left[U + 4(-\Upsilon)^{-1/2}(U-1)\right]$$
$$+2\mathrm{i}U\left[\Upsilon F_{I\Upsilon}N^{IJ}(\mathcal{Q}_J - \bar{F}_{JK}\mathcal{P}^K) - \mathrm{h.c.}\right]$$
$$+2\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\left[2U\Upsilon + 4\sqrt{-\Upsilon}(1+U)\right] = 0$$

The above equations are indeed satisfied in the BPS case.

Example

As an application of the above formalism to non-BPS black hole, let us consider the example discussed by Sahoo and Sen (0603149). The prepotential is given by,

$$F(Y,\Upsilon) = -\frac{Y^1 Y^2 Y^3}{Y^0} - C\,\frac{Y^1}{Y^0}\Upsilon$$

We take the non-supersymmetric black hole to carry non-vanishing charges, namely,

$$q_0 = p^1 = Q \quad , \quad p^2 = p^3 = P \quad , \quad Q, P > 0$$

In the absence of R^2 interactions, i.e. setting $C = 0$, the attractor equations are solved by,

$$Y^0 = \frac{1}{4}P \ \ , Y^1 = \frac{1}{4}i\,Q\ , Y^2 = \frac{1}{4}i\,P \ \ , Y^3 = \frac{1}{4}iP$$

and we obtain $\boxed{\Upsilon = -4}$.

Evaluation the entropy yields

$$\mathcal{S}_{\text{macro}} = 2\pi\, Q\, P > 0$$

These values are subject to R^2-corrections.
Computing these corrections to first order in C, we find that the genealized attractor equations are satisfied provided that

$$\begin{aligned} U - 1 &= -16\,\frac{C}{P^2}\,, \ \Upsilon = -4 \\ Y^0 &= \frac{P}{4} + 24\frac{C}{P}\,,\ Y^1 = i\left(\frac{Q}{4} + 10C\,\frac{Q}{P^2}\right) \\ Y^2 &= Y^3 = i\left(\frac{P}{4} + 4\,\frac{C}{P}\right) \end{aligned}$$

To first order in C, the entropy is computed to be,

$$\mathcal{S}_{\mathrm{macro}} = 2\pi \left(P\,Q + 40\,C\,\frac{Q}{P} \right) \tag{1}$$

whereas the microstate counting gives,

$$\mathcal{S}_{\mathrm{micro}} = 2\pi \left(P\,Q + 64\,C\,\frac{Q}{P} \right)$$

However, one has to take into account that other higher derivative interactions associated with matter multiplets which are likely to contribute to the entropy of the non-supersymmetric black holes. The presence of additional higher derivative terms in higher dimensions have also been discussed by Sahoo and Sen (0608182).

NON-HOLOMORPHIC CORRECTIONS

(G. L. Cardoso, B. de Wit and S. Mahapatra, to appear)

- The OSV conjecture for the BPS black holes implies that the black hole degeneracies can be expressed in terms of an inverse Laplace transform over a partition function $Z(p, \phi)$ associated with a mixed ensemble, which is equal to the modulus of the square of the topological string partition function.
- Since the issue of electric-magnetic dualty symmetry is somewhat obscure in the use of a mixed ensemble, a proposal has been made to start with a canonical ensemble involving both electric and magnetic charges and then the degeneracy can be obtained by an inverse Laplace transform.
- The arguments based on electric/magnetic duality indicate the presence of a nontrivial measure factor $\sqrt{\Delta^-}$ whose form depends on the free energy modified by possible non-holomorphic corrections (G.L.Cardoso, B.de Wit, J.Kaepeli and T.Mohaupt, hep-th/0601108).

The modified version of the OSV integral consistent with electric/magnetic duality is given by,

$$d(p,q) \sim \int d\phi \sqrt{\Delta^-} e^{\pi[\mathcal{F}_E(p,\phi) - q_I \phi^I]}$$

where,

$$\Delta^- = |det[ImF_{KL} + 2Re(\Omega_{KL} - \Omega_{K\bar{L}}]|$$

and,

$$\mathcal{F}_E = 4[ImF(Y,\Upsilon) + \Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})]_{Y^I=(\phi^I+ip^I)/2}$$

We consider the type II realization of the FHSV model which corresponds to the compactification on the Enriques Calabi-Yau three-fold which is described as an orbifold $(T^2 \times K3)/Z_2$ and the theory has $N = 2$ supersymmetry.

The $N = 2$ description is based on the holomorphic function modified with Υ dependent terms and possibly non-holomorphic corrections encoded in the function Ω.

$$F = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + 2i\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

We explicitly consider the T and S duality transformations of the period vector (Y^I, F_I) and solutions of these equations give the form of Ω with non-holomorphic contributions which can be compared with the results from the holomorphic anomaly equations for the genus-g partition functions $F^{(g)}$ of the topological string. At genus-1, we compute the measure factor by obtaining the expression for Ω.

SUMMARY

- The entropy and the attractor equations for extremal black hole solutions follow from a variational principle based on an entropy function. In the general case, such an entopy function can be derived from the reduced action evaluated in a near horizon geometry.
- BPS black holes constitute special solutions of this variational principle, but can also be derived directly from a different entropy function based on supersymmetry enhancement at the horizon.

- We have clarified the relation between the two entropy functions in $N = 2, D = 4$ supergravity with higher derivative interactions.
- The computation of the non-holomorphic corrections for the class of models with $N = 2$ supersummety seems to be giving nontrivial measure factors.
- It will be interesting to consider the non-holomorphic corrections to the more general entropy function where the presence of non-holomorphic terms will affect the supersymmetry of the original action and the construction of such a supersymmetric action is expected to be quite challenging.

THE END