Worldsheet Properties of Extremal Correlators in AdS/CFT

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work to appear with

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Plan of talk

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- 4 Steps to the Worldsheet Correlator
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- Worldsheet Correlator at High Energy (large J)
- An explicit example and Crossing Symmetry
- Work in Progress

Introduction

ADS/CFT correspondence implies

$$< O_{l_1}(x_1)..O_{l_n}(x_n) >_g = \int d^d t \mathcal{G}^g_{l_1..l_n}(t_1..t_d; (x_i - x_j), \lambda) \qquad (1)$$

where, $\mathcal{G}^g_{l_1,...,l_n}$ is the genus g worldsheet correlator of vertex operators corresponding to O_i.

d= dimension of moduli space of n-punctured Riemann surface of genus g = 6g-6+2n

A Proposal

Proposal to construct World Sheet Correlators in $\lambda \to 0$ limit (Gopakumar)

The methodology is to convert:
$\sum_{worldlines} \to \sum_{surfaces}$.
Now $\sum_{worldlines} = \sum_{graphs} \Pi[D\sigma]$,these σ's are the inverses of the proper lengths of the edges of the graph.
Also,$\sum_{surfaces} = \sum_{g} \Pi[Dt]$,these t's are the coordinates on the moduli space of punctured genus g surfaces.

Can explicitly map σ's into t's using Strebel differentials.

Strategy

- Identify field theory correlators (e.g. 4 pt. functions) which are independent of t'hooft coupling
- Find the corresponding worldsheet correlators for high energies in the limit $\lambda \to 0$.
- Check whether the correlators match with expectations at large λ from SUGRA analysis.

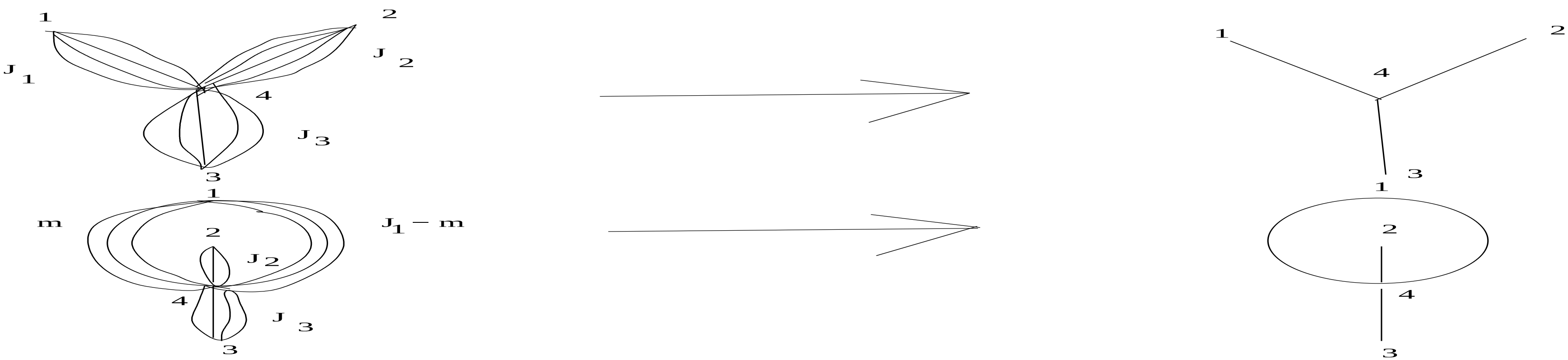
ExtremalCorrelators

- The particular case of study would be the one with four chiral primary operators:
$< Tr(Z^{J_1}(x))Tr(Z^{J_2}(y))Tr(Z^{J_3}(z))Tr(\overline{Z}^J(0))) >^{\lambda \rightarrow 0}_{\sim} \frac{C}{(x^2)^{J_1}(y^2)^{J_2}(z^2)^{J_3}}$
with $J = J_1 + J_2 + J_3$.
- Chiral primary operators are dual to the supergravity Kaluza-Klein modes (J being dual to the angular momentum on S^5)
- These do not receive corrections in λ (Freedman, et. al)
By analyzing supergravity diagrams it was shown that at strong coupling such correlators retain their free field behaviour
$< Tr(Z^{J_1}(x))Tr(Z^{J_2}(y))Tr(Z^{J_3}(z))Tr(\overline{Z}^J(0))) >^{\lambda \gg 1}_{\sim} \frac{C}{(x^2)^{J_1}(y^2)^{J_2}(z^2)^{J_3}}$
Arguments using nature of OPEs in a CFT show that C is independent of λ
- Will describe the construction of such worldsheet correlators

4 steps to the worldsheet correlator

STEP 1: Glue homotopic edges

- After gluing homotopic edges the planar graphs for the extremal correlator are the Y and the Lollipop



- Using $\frac{1}{(x^2)^J} \sim \int d\sigma \sigma^{J-1} e^{-\sigma x^2}$

- Thus one Schwinger parameter for each edge of the Skeleton graph

- For the lollipop we have the following Schwinger parametric representation:

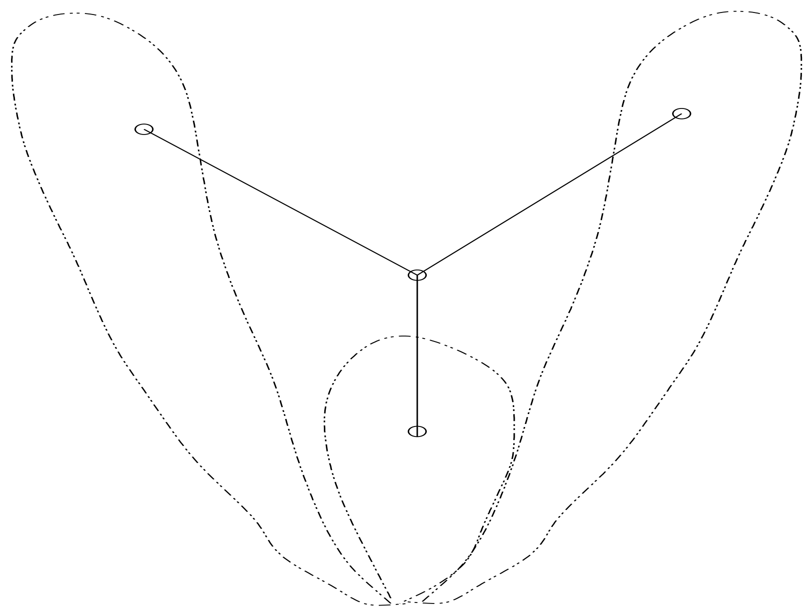
$$\sum_{m=1}^{J_1-1} \frac{1}{(J_1-m-1)!(m-1)!(J_2-1)!(J_3-1)!} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\sigma_1^{'} d\sigma_1^{''} d\sigma_2 d\sigma_3 \qquad (2)$$

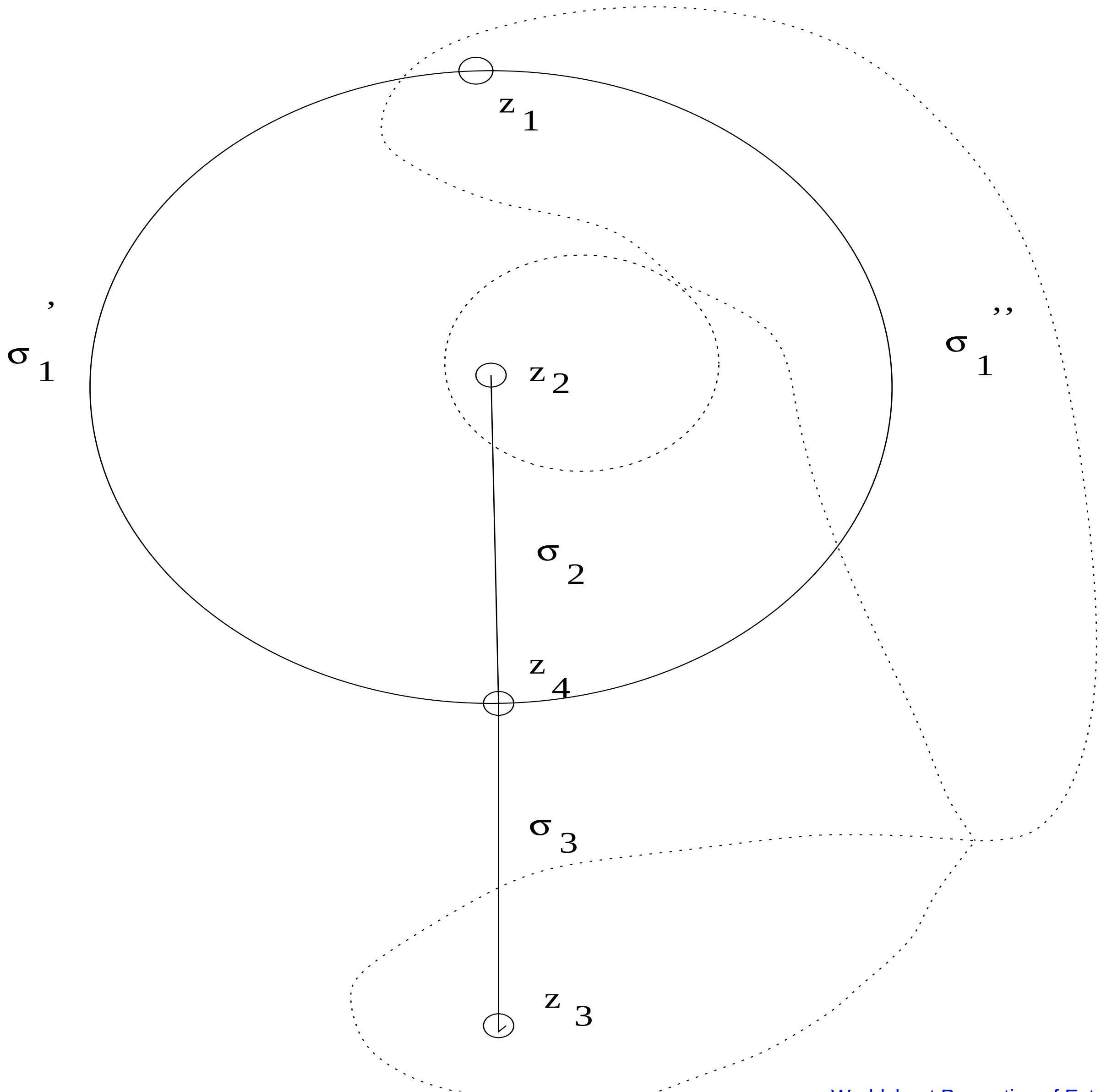
$$\sigma_1^{'\,m-1} \sigma_1^{''\,J_1-m-1} \sigma_2^{J_2-1} \sigma_3^{J_3-1} e^{-(\sigma_1^{'}+\sigma_1^{''})x^2-\sigma_2 y^2-\sigma_3 z^2}$$

m=0 (or equivalently $m=J_1$) gives the Y.

STEP 2: Constructing the Strebel differential

- The dual for the Y and the lollipop are:




z 1
σ 1 '
z 2
σ 1 ''
σ 2
z 4
σ 3
z 3

Strebel Differentials : A quadratic differential $\phi(z)dz^2$ (providing an invariant line element $\sqrt{\phi(z)}dz$) which could be associated with a critical graph

- The vertices of the graph $\leftrightarrow$ zeroes of the Strebel differential
  with valence of the vertex $\leftrightarrow$ multiplicity of the zero
- The faces of the graph $\leftrightarrow$ double poles
  It is here where the vertex operators are inserted
- The edges of the graph $\leftrightarrow$ trajectories along which the line element $\sqrt{\phi(z)}dz$ is real.

For the lollipop we have a Strebel differential of the form

$$\phi(z)dz^2 = -C\frac{z^2}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2}dz^2 \qquad (3)$$

- There are 2 double zeroes at 0 and ∞.
- There are 4 double poles
- Our goal is now to find the $\phi(z)$ (i.e. find C, z_1, z_2, z_3, z_4)in terms of the Schwinger parameters for a given η

STEP 4: From Strebel Differential to Schwinger Parameters

The explicit map suggested by Gopakumar is Schwinger parameter of an edge of a graph = Strebel length of the dual edge

$\sigma_{ij} = \int_{k_i}^{k_j} \sqrt{\phi(z)}dz = l_{ij}$ (l_{ij} is the Strebel length of the edge connecting zeroes k_i and k_j and σ_{ij} is the Schwinger parameter of the corresponding edge in the field theory graph).

- For the lollipop we have:

$$\begin{aligned} \sigma_1^{'} + \sigma_1^{''} &= p_1 \\ \sigma_2 &= p_2 \\ \sigma_3 &= p_3 \end{aligned} \tag{4}$$

p_1, p_2, p_3 are the residues of the poles of $\sqrt{\phi(z)}dz$ at z_1, z_2 and z_3 respectively.

Obtaining the Correlator

- Using the steps above we can convert the change of variables in the Schwinger parametric representation $\int d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4() = \int d(\frac{p_2}{p_1}) d(\frac{p_3}{p_1}) d\eta d\overline{\eta} f(\frac{p_2}{p_1}, \frac{p_3}{p_1}, \eta, \overline{\eta})$
- We now integrate away $p_2/p_1 = s_2$ and $p_3/p_1 = s_3$ to obtain the worldsheet correlator. $\int d(\frac{p_2}{p_1}) d(\frac{p_3}{p_1}) d\eta d\overline{\eta} f(\frac{p_2}{p_1}, \frac{p_3}{p_1}, \eta, \overline{\eta}) = \int d\eta d\overline{\eta} \mathcal{G}(\eta, \overline{\eta})$

Climbing 4 steps quickly

It turns out that the equations determining the Strebel differential are as follows:

$$
\begin{aligned}
v_1 + s_2 v_2 + s_3 v_3 &= s_2 + s_3 + 1 \\
\frac{1}{v_1} + \frac{s_2}{v_2} + \frac{s_3}{v_3} &= s_2 + s_3 + 1 \\
v_1 v_2^{s_2} v_3^{s_3} &= e^{is}
\end{aligned} \qquad (5)
$$

where , $s_2 = p_2/p_1$, $s_3 = p_3/p_1$ and $v_1 = z_1/z_4$, $v_2 = Z_2/z_4$, $v_3 = z_3/z_4$.

The connection with the Schwinger parameters are:

$s = \frac{\sigma_1^{'}}{\sigma_1}, s_2 = \frac{\sigma_2}{\sigma_1}, s_3 = \frac{\sigma_3}{\sigma_1}$

with $\sigma_1 = \sigma_1^{'} + \sigma_1^{''}$.

The crossratio is :

$$\eta = \frac{(z_2 - z_4)(z_3 - z_1)}{(z_3 - z_4)(z_2 - z_1)} = \frac{(v_2 - 1)(v_3 - v_1)}{(v_3 - 1)(v_2 - v_1)} \tag{6}$$

- So we first solve for v_1, v_2, v_3 as functions of s_2, s_3, s.
- We then substitute this in the expression for η and invert s as a function of s_2, s_3, η

This completes the promised change of variables.

Schwinger parametric representation at large J (high energy)

It is useful to first simplify the Schwinger parametric representation

- **Simplification 1** The change of variable is independent of the overall scale $\sigma_1 = \sigma_1^{'} + \sigma_1^{''}$. So we can simplify the representation by integrating it away. The representation now is

$$\sum_{m=1}^{J_1-1} \frac{1}{(J_1-m-1)!(m-1)!(J_2-1)!(J_3-1)!} \int_0^1 \int_0^\infty \int_0^\infty ds ds_2 ds_3 \tag{7}$$
$$\frac{s^{m-1}(1-s)^{J_1-m-1} s_2^{J_2-1} s_3^{J_3-1}}{(x^2+s_2 y^2+s_3 z^2)^{J_1+J_2+J_3}}$$

Here $s_2 = \sigma_2/\sigma_1$ and $s_3 = \sigma_3/\sigma_1$.

- **Simplification 2** The change of variables depend only on the topology of the graph. So before changing variables we can perform the sum over m:

$$\frac{1}{(J_1-2)!(J_2-1)!(J_3-1)!}\int_0^1 ds\int_0^\infty ds_2\int_0^\infty ds_3 \frac{s_2^{J_2-1}s_3^{J_3-1}}{(x^2+s_2y^2+s_3z^2)^J} \tag{8}$$

where $J = J_1 + J_2 + J_3$.

- It would be convenient for us to take out a dimensionful $d^2 = x^2 + y^2 + z^2$ from the denominator. We can achive this by introducing l, m and n, such that, $x^2/d^2 = l, y^2/d^2 = m, z^2/d^2 = n$:

$$\int_0^1 ds\int_0^\infty ds_2\int_0^\infty ds_3 \frac{s_2^{J_2-1}s_3^{J_3-1}}{(l+s_2m+s_3n)^J} \tag{9}$$

- **Simplification 3:** (Aharony, David, Gopakumar,et al.), In the Schwinger parametric representation J behaves like ($1/h$).
In our case, s is a flat direction (since the integrand is independent of it), while, the saddle points for s_2 is $J_2 l/J_1 m$ and the saddle point for s_3 is $J_3 l/J_1 n$.
In order to expand around the saddle point, we write $s_2 = 1 + \epsilon_2$ and $s_3 = 1 + \epsilon_3$.Keeping only quadratic fluctuations we observe that we have:

$$e^{-J(Ln(J)+\alpha Ln(\frac{l}{\alpha})+\beta Ln(\frac{m}{\beta})+\gamma Ln(\frac{n}{\gamma})-\frac{m^2\alpha^2}{2l^2}\frac{1-\beta}{\beta}\epsilon_2^2-\frac{n^2\alpha^2}{2l^2}\frac{1-\gamma}{\gamma}\epsilon_3^2-\frac{mn\alpha^2}{l^2}\epsilon_2\epsilon_3)} \tag{10}$$

- Now in the large J limit the Gaussian fluctuations could be interpreted as delta functions. With the understanding that we are going to do away with ϵ_3 integral first, we could write the expression above in a still more simpler manner as below:

$$e^{-J(Ln(J)+\alpha Ln(\frac{l}{\alpha})+\beta Ln(\frac{m}{\beta})+\gamma Ln(\frac{n}{\gamma}))}(\frac{2\pi}{J})(\frac{l^2\sqrt{\beta\gamma}}{mn\alpha^{\frac{5}{2}}})\delta(\epsilon_2)\delta(\epsilon_3) \tag{11}$$

What we require to do now is to convert the following integral $\int ds \int d\epsilon_2 \int d\epsilon_3 \delta(\epsilon_2)\delta(\epsilon_3)$ to an integral over $\eta, \overline{\eta}$ (the moduli space) and ϵ_3 and then integrate ϵ_3 out which should be easy due to the δ function.

Worldsheet Correlator at high energy (large J)

From the above discussion it is clear what we should be doing in the high energy limit. We should be expanding the equations determining v_1, v_2, v_3 in terms of s_2, s_3, s around the saddle points of s_2, s_3, which are $(\beta l)/(\alpha m), (\gamma l)/(\alpha n)$ respectively.

So we may write $s_2 = (\beta l)/(\alpha m) + \epsilon_2$, $s_3 = (\gamma l)/(\alpha n) + \epsilon_3$. In terms of ϵ_2 and ϵ_3 we have a perturbation expansion:

$$v_1 = v_{10} + \epsilon_2 v_{11} + \epsilon_3 v_{12} + ... \tag{12}$$
$$v_2 = v_{20} + \epsilon_2 v_{21} + \epsilon_3 v_{22} + ...$$
$$v_3 = v_{30} + \epsilon_2 v_{31} + \epsilon_3 v_{32} + ...$$

In the above v_{11}, v_{12}, v_{21}, etc are just functions of s since the value of s_2, s_3 are fixed at the saddle point. If we plug this in the expression for η, we get:

$$\eta = \eta_o(s) + M(s)\epsilon_2 + N(s)\epsilon_3 \tag{13}$$

In the above M(s) and N(s) are extremely complicated functions.

So now the field theory integrand upto some factors (which we will later reinstate) has the following form : $\int ds \int d\epsilon_2 \int d\epsilon_3 \delta(\epsilon_2)\delta(\epsilon_3)$. We have to now change from $s, \epsilon_2, \epsilon_3$ to $\eta, \overline{\eta}, \epsilon_3$ and then integrate out ϵ_3. So we have in our present approximation (which would be exact at high energy):

$$\begin{aligned} \eta &= \eta_o(s) + M(s)\epsilon_2 + N(s)\epsilon_3 \\ \overline{\eta} &= \overline{\eta_o}(s) + \overline{M}(s)\epsilon_2 + \overline{N}(s)\epsilon_3 \\ \epsilon_3 &= \epsilon_3 \end{aligned} \tag{14}$$

where M(s) and N(s) are extremely complicated functions.

For the change of variables it would be very useful to keep the following in mind:

- With the understanding that we are integrating out ϵ_3 first we can work out the change of variables for the case $\epsilon_3 = 0$ because of the presence of $\delta(\epsilon_3)$. Moreover, we have $\delta(\epsilon_2)$ too. So when we work out the Jacobian for s, ϵ_2 to $\eta, \overline{\eta}$ we can put ϵ_2 to zero as well.
- Now we would also intrerpret $\delta(\epsilon_2)$ as a "primary" function of $\overline{\eta}$ (which means if we would like to integrate we should do the $\overline{\eta}$ integral first).

Another simplification arises here. It turns out that after change of variables the function is independent of M(S) and N(S), the first order perturbations!

- So it turns out that:

$$\int ds \int d\epsilon_2 \int d\epsilon_3 \delta(\epsilon_2)\delta(\epsilon_3) = \int d\eta \int d\overline{\eta} \int d\epsilon_3 \frac{\delta(\overline{\eta} - \overline{\eta_o}(\eta_o^{-1}(\eta)))\delta(\epsilon_3)}{\eta_o^{'}(\eta_o^{-1}(\eta))} \tag{15}$$

It is very easy to see the equality directly (the $\overline{\eta}$ and the ϵ_3 integrals give 1 while $ds = d\eta/\eta_o^{'}(\eta_o^{-1}(s))$), but that we indeed end up in this simplest possible form could be shown only after considerable algebra.!

- Now we can integrate ϵ_3 out and claim that our correlator is:

$$\mathcal{G}(\eta,\overline{\eta}) = \frac{\delta(\overline{\eta} - \overline{\eta_o}(\eta_o^{-1}(\eta)))}{\eta_o^{'}(\eta_o^{-1}(\eta))}(const) \tag{16}$$

The (const)is :$e^{-J(2LnJ+\alpha Ln(l)+\beta Ln(m)+\gamma Ln(n))}(\frac{2\pi}{J})(\frac{l^2\sqrt{\beta\gamma}}{mn\alpha^{\frac{5}{2}}})$

An explicit example and crossing symmetry

We can write down an explicit expression for the world sheet correlator when $x^2/J_1 = y^2/J_2 = z^2/J_3$. The saddle points for s_2, s_3 are both 1. In that case our equations at zeroth order are:

$$\begin{aligned} v_1 + v_2 + v_3 &= -3 \\ \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} &= -3 \\ v_1 v_2 v_3 &= e^{is} \end{aligned} \tag{17}$$

In other words v_1, v_2, v_3 are the three roots of the cubic equation $v^3 - 3v^2 + 3e^{is}v - e^{is} = 0$. It turns out that the right choice of the roots is:

$$\begin{aligned} v_1 &= \omega^2 A + \omega B + 1 \\ v_2 &= A + B + 1 \\ v_3 &= \omega A + \omega^2 B + 1 \\ A = \beta^{1/3}(1 + \sqrt{1-\beta})^{1/3}, B = \beta^{1/3}(1 - \sqrt{1-\beta})^{1/3}, \beta &= 1 - e^{is} \end{aligned} \tag{18}$$

With this choice of v's we get:

$$\eta = \eta_o(s) = \frac{-\omega(1 - \frac{B^2}{A^2})}{1 - \omega^2 \frac{B^2}{A^2}}, \frac{B^2}{A^2} = \omega^{-\frac{1}{2}} (tan(\frac{s}{4}))^{\frac{2}{3}} \tag{19}$$

Therefore when s becomes zero, η becomes $-\omega$. This actually justifies our choice of roots, because when s vanishes our lollipop graph goes to the Y graph. We can compare our result with the Y in this case.

- It is not hard to see that we have $\eta\overline{\eta} = 1$ on the saddle line $\eta_o(s)$! So our "saddle line" is just the unit circle

- The final explicit expression for our worldsheet correlatorturns out to be:

$$\mathcal{G}(\eta,\overline{\eta}) = (const)\delta(\overline{\eta} - \frac{1}{\eta})6(\omega - 1)\omega^{\frac{3}{4}}(\eta\omega^2 + 1)^{-\frac{5}{2}}(\eta + \omega)^{\frac{1}{2}} \frac{1}{1 - (\frac{\eta+\omega}{\eta\omega^2+\omega})^3} \tag{20}$$

We observe we have a rational function of η.
Also,

$$(const) = e^{-J(2LnJ+\alpha Ln(\alpha)+\beta Ln(\beta)+\gamma Ln(\gamma))}(\frac{2\pi}{J\sqrt{\alpha\beta\gamma}}) \tag{21}$$

Crossing Symmetry

- Crossing symmetry is a non-trivial check of the Gopakumar proposal. (David and Gopakumar)
- The planar lollipop graph is equivalent under exchange of vertices 2 and 3. So a crucial requirement is that the SL(2,C) transformation exchanging v_2 and v_3 should keep the saddleline invariant. This transformation is $\eta \to 1/\eta$. It does keep the unit circle invariant.

- It is also straightforward to check that:

$$\mathcal{G}(\frac{1}{\eta}, \frac{1}{\overline{\eta}}) = |\eta|^4 \mathcal{G}(\eta, \overline{\eta}) \tag{22}$$

The (const)multiplying the correlator is also invariant under simultaneous exchange of x_2, x_3 and J_2, J_3.

Work in Progress

- We are trying to reproduce the behaviour of these worldsheet correlators at high energy from the classical sigma model of the string on $ADS_5 \times S^5$. In particular we need to understand from this point of view why the correlator gets localised on a curve on the moduli space rather than a point in contrast to high energy string scattering in flat space as analysed by Gross and Mende.
- We also saw that the curve is just the unit circle when the distances of the operators are adjusted appropriately. We hope to reproduce the result for this specific case as well.