Flavoured Twistor Strings

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Motivation

Twistor String Theory: Proposed correspondence between the open string sector of the topological B-model on $\mathbb{CP}^{3|4}$ and perturbative $\mathcal{N} = 4$ SYM. [Witten]

Simplicity of gauge theory amplitudes in fixed helicity basis. The MHV n-point gluon amplitude is proportional to

$$A = g^{n-2} \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle}$$

where $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ and $\langle \lambda_i, \lambda_j \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$.

These amplitudes are reproduced by nonperturbative effects in a holomorphic Chern-Simons theory with supertwistor space as the target space. The isometries of $\mathbb{CP}^{3|4}$ linearly encode the superconformal symmetry of $\mathcal{N} = 4$.

Unfortunately, the correspondence doesn't hold beyond tree level. Contributions from closed string B-model sector and appearance of conformal supergravity modes in the gauge theory loops. [Berkovits-Witten]

Development of MHV-formalism on gauge theory side even for less or non-susy YM. [Cachazo-Svrček-Witten]

But the twistor-inspired MHV-rules *can* be extended at loop level for $\mathcal{N} = 4$. Also for $\mathcal{N} = 1$ and pure YM.
[Brandhuber-Spence-Travaglini]
[Bedford-Brandhuber-Spence-Travaglini]

What is the quantum completion of Witten's twistor string? Some (non-topological?) B-model extension with modified target space?

Towards this end study the range of 4d gauge theories with less susy, which admit a tree-level twistor string description.

Most obvious candidates should be theories that preserve conformal invariance at loop level, order-by-order: UV-finite

- ☑ $\mathcal{N}=1$ exactly marginal deformations of $\mathcal{N}=4$
  [Kulaxizi-Zoubos]

- ☑ $\mathcal{N}=1,2$ quiver gauge theories as discrete $\mathbb{C}\mathrm{P}^{3|4}$ orbifolds
  [Park-Rey]
  [Giombi-Kulaxizi-Ricci-Robles-Llana-Trancanelli-Zoubos]

Look at $\mathcal{N}=2$ finite theories with fundamental matter:

$$\mathcal{N}=2\ \mathrm{Sp}(N_c) \text{ gauge theory with } N_f=4$$

$$\mathcal{N}=2\ \mathrm{SU}(N_c) \text{ gauge theory with } N_f=2N_c$$

Outline

- Review of the $\mathcal{N} = 4$ theory
- The $\mathcal{N} = 2$ $\mathrm{Sp}(N_c)$ theory with $N_f = 4$
- Introducing topological ‘flavour’-branes
- Conclusions and Outlook

Review of the $\mathcal{N} = 4$ theory

Write null momenta in terms of $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. Twistor space is a copy of $\mathbb{CP}^3$ defined by the homogeneous coordinates $Z^I = (\lambda^\alpha, \mu^{\dot{\alpha}})$, where

$$\widetilde{\lambda}_{\dot{\alpha}} \to i \frac{\partial}{\partial \mu^{\dot{\alpha}}}, \quad -i \frac{\partial}{\partial \widetilde{\lambda}^{\dot{\alpha}}} \to \mu_{\dot{\alpha}}.$$

Witten showed that holomorphic λ dependence of the MHV amplitudes means they are supported on genus zero, degree one curves in twistor space, $\mathbb{CP}^1 \subset \mathbb{CP}^3$.

Adding four fermionic co-ordinates ψ^I plus conjugates turns $\mathbb{CP}^3$ into a super-CY: $\mathbb{CP}^{3|4}$.

This is now a suitable target space for the B-model.

The B-model open string d.o.f. are described by the hCS action

$$S = \frac{1}{2}\int_{\mathrm{D5}} \Omega \wedge \mathrm{Tr}\left(\mathcal{A}\cdot\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

defined on a 'D5'-brane sitting at the locus $\bar{\psi}^{\bar{I}} = 0$. The superfields $\mathcal{A}$ can be expanded as

$$\mathcal{A} = A + \psi^I \lambda_I + \frac{1}{2!}\psi^I\psi^J\phi_{IJ} + \frac{1}{3!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\tilde{\lambda}^L + \frac{1}{4!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\psi^L G\ .$$

Via the Penrose transform the component fields get mapped to specific helicity particles in Minkowski space:

$\mathcal{N} = 4$ spectrum

How about interactions? These correspond to those of *self-dual* $\mathcal{N} = 4$. The full interactions arise non-perturbatively through D1–instantons wrapping the holomorphic genus zero, degree one curves with $\mathbb{C}\mathrm{P}^{3|4}$ embedding [Witten],[Nair]

$$\mu_{\dot\alpha} + x_{\alpha\dot\alpha}\lambda^\alpha = 0 \ \text{ and } \ \psi^I + \theta^I_\alpha \lambda^\alpha = 0\ .$$

These are the same curves onto which the MHV amplitudes are localised. By integrating over the moduli space (x, θ) we get the amplitude prescription

$$A_{(n)} = g^2 \int d^4x\, d^8\theta\, \langle \int_{\mathbb{C}\mathrm{P}^1} J_1 w_1 \cdots \int_{\mathbb{C}\mathrm{P}^1} J_n w_n \rangle$$

with the J's being D1-instanton world-volume currents and the w_i's the external particle wavefunctions.

Recover all MHV amplitudes - Extension to next-to-MHV etc.

The $\mathcal{N}=2$ Sp(N) theory with $N_f=4$

Physical string realisation: N D3's living at an O7 plane with 4 D7 branes. The near horizon geometry on the D3's is $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathbb{Z}_2$. [Sen], [Banks-Douglas-Seiberg], [Fayyazuddin-Spalinski], [Aharony-Fayyazuddin-Maldacena]

The massless open string d.o.f can be summarised in [Gava-Narain-Sarmadi]

Component	SO(1,3)	$SU(2)_a$	$SU(2)_A$	$U(1)_R$	Sp(N)	SO(8)
A, G	$(2,2)$	1	1	0	$N(2N+1)$	1
ϕ	$(1,1)$	1	1	$+2$	$N(2N+1)$	1
$\phi^\dagger$	$(1,1)$	1	1	-2	$N(2N+1)$	1
$\lambda_{\alpha,a}$	$(2,1)$	2	1	$+1$	$N(2N+1)$	1
$\bar\lambda_{\dot\alpha,a}$	$(1,2)$	2	1	-1	$N(2N+1)$	1
z_{aA}	$(1,1)$	2	2	0	$N(2N-1)-1$	1
$\zeta_{\alpha,A}$	$(2,1)$	1	2	-1	$N(2N-1)-1$	1
$\bar\zeta_{\dot\alpha,A}$	$(1,2)$	1	2	$+1$	$N(2N-1)-1$	1
q_a^M	$(1,1)$	2	1	0	$2N$	8
η_α^M	$(2,1)$	1	1	-1	$2N$	8
$\bar\eta_{\dot\alpha}^M$	$(1,2)$	1	1	$+1$	$2N$	8

The gauge theory Lagrangian is in $\mathcal{N}=1$ superfield formulation:

$$\begin{aligned}\mathcal{L} =& \frac{1}{8\pi}\mathrm{Im}\ \mathrm{Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\ e^{2V}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta d^2\bar\theta\ Q^{\dagger I} e^{-2V} Q_I \\ &+ \int d^2\theta d^2\bar\theta\ Q'^I e^{2V} Q_I'^\dagger + \mathrm{Tr}\left(\int d^2\theta d^2\bar\theta\ e^{2V} Z^\dagger e^{-2V} Z + \int d^2\theta d^2\bar\theta\ e^{-2V} Z' e^{2V} Z'^\dagger\right) \\ &+ \sqrt{2}\left(\int d^2\theta (Q'^I \Phi Q_I + \mathrm{Tr}\,(Z'[\Phi, Z])) + h.c.\right)\ .\end{aligned}$$

The flavour symmetry group in this notation is given under the maximal embedding $\mathrm{U}(1) \times \mathrm{SU}(4) \subset \mathrm{SO}(8)$. An additional series of helicity-dependent rescalings leads to the self-dual truncation:

$$\begin{aligned}\mathcal{L} =& \mathrm{Tr}\left[-\frac{1}{2}GF + D\phi^\dagger D\phi + i\bar\lambda^a\, \not{D}\lambda_a - \lambda^a\lambda_a\phi^\dagger\right] \\ &- \mathrm{Tr}\left[\frac{1}{2}Dz^{aA}Dz_{Aa} + i\bar\zeta^A\, \not{D}\zeta_A + z^{aA}[\lambda_a, \zeta_A] + \zeta^A\zeta_A\phi\right] \\ &- \frac{1}{2}Dq^a{}_M Dq^M{}_a - i\bar\eta_M\, \not{D}\eta^M + q^a{}_M\lambda_a\eta^M - \frac{1}{2}\eta_M\phi\eta^M\ .\end{aligned}$$

On the twistor side we perform a super-orientifold

$$(a) \quad \psi^a \to \psi^a \quad , \quad \psi^A \to -\psi^A \; ; a = 1,2 \; , \; A = 3,4$$

$$(b) \quad \mathcal{A}^i{}_j \to \Omega^{ik} (\mathcal{A}^T)_k{}^l \Omega_{lj} = (\mathcal{A}^T)^i{}_j \equiv \mathcal{A}_j{}^i$$

The superfield $\mathcal{A}$ decomposes into an adjoint and an anti-symmetric $\mathrm{Sp}(N)$ piece yielding part of the spectrum

$$\begin{aligned} \hat{\mathcal{A}} &= (A + \psi^a \lambda_a + \psi^1 \psi^2 \phi + \psi^3 \psi^4 \phi^\dagger + \epsilon_{cd} \psi^3 \psi^4 \psi^c \tilde{\lambda}^d + \psi^1 \psi^2 \psi^3 \psi^4 G) \\ &\quad + \psi^A (\zeta_A + \psi^a z_{Aa} + \epsilon_{AB} \psi^1 \psi^2 \tilde{\zeta}^B) \\ &= \mathcal{V} + \psi^A Z_A \\ &= \mathcal{V} + \mathcal{Z} \; , \end{aligned}$$

We still need to recover the fundamental d.o.f.

Introducing topological 'flavour'-branes

These are objects localised at the super-orientifold fixed plane (the locus $\psi^A = 0$, $\bar{\psi}^{\bar{a},\bar{A}} = 0$). We will drop the 'D5' terminology in favour of $\mathrm{D_c}$ (colour) and $\mathrm{D_f}$ (flavour).

The boundary conditions for open strings stretching between the $\mathrm{D_c}$ and $\mathrm{D_f}$ branes can be summarised as

Direction	D_c–D_c	D_c–D_f	D_f–D_f
Z,$\bar{Z}$	NN	NN	NN
ψ^a	NN	NN	NN
ψ^A	NN	ND	DD
$\bar{\psi}^{\bar{a}}$,$\bar{\psi}^{\bar{A}}$	DD	DD	DD

The $c - f$ $(f - c)$ strings will yield the required fundamental hypermultiplets.

We will first understand the $f-f$ strings, obtained by a fermionic analogue of dimensional reduction from the $c-c$ strings.

This is related to the problem of understanding states on sub-supermanfolds of supermanifolds and is achieved by considering the dependence on the fermionic coordinates only in certain combinations. This, in turn, can be implemented in terms of a set of integral constraints.
[Lechtenfeld-Popov],[Sämann]

For our purposes, we will take the $f-f$ states to be depending on the nilpotent coordinate $y=\psi^3\psi^4$, with ψ^1,ψ^2 unrestricted. The integral constraints realising this on a given state $\mathcal{K}$ are

$$\int \mathrm{d}^4\psi\psi^1\psi^2\psi^A\mathcal{K}=\int \mathrm{d}^4\psi\psi^a\psi^A\mathcal{K}=\int \mathrm{d}^4\psi\psi^A\mathcal{K}=0\ .$$

The ψ dependence of $\mathcal{K}$ is then restricted to

$$\mathcal{K}^X{}_Y = \mathrm{d}\bar{Z}^{\bar{m}} \left(K(Z,\bar{Z},\psi^a)_{\bar{m}}{}^X{}_Y + \psi^3\psi^4 L(Z,\bar{Z},\psi^a)_{\bar{m}}{}^X{}_Y \right) \, .$$

However, because of the holomorphic DD b.c.'s along ψ^A one can also write down $\mathcal{B}^A(Z,\bar{Z},\psi^a,\psi^A)\partial/\partial\psi^A$ as physical vertex operators. We will consider the subset of these constrained by the *complement* of the NN integral equations.

$$\mathcal{B}^A(Z,\bar{Z},\psi)^X{}_Y \frac{\partial}{\partial\psi^A} = \psi^B B^A_B(Z,\bar{Z},\psi^a)^X{}_Y \frac{\partial}{\partial\psi^A} \, .$$

This completes our proposal for the fermionic dimensional reduction.

NB: The total number of d.o.f before and after the reduction is the same, with the d.o.f from $\mathcal{B}$ and $\mathcal{K}$ giving the expected counting of states of the 8d $\mathcal{N}=1$ D7–brane theory.

Return to considering the $c - f$ $(f - c)$ states. These are expressed as

$$\mathcal{Q}^i{}_X = P(Z, \bar{Z}, \psi^a)^i{}_X + \psi^A Q_A(Z, \bar{Z}, \psi^a)^i{}_X + \psi^3 \psi^4 R(Z, \bar{Z}, \psi^a)^i{}_X$$

The super-orientifold action extends naturally both to these and the $f - f$ strings. It imposes the reality condition

$$\mathcal{Q}^X_i = \Omega_{ij} \mathcal{Q}^j{}_Y \Omega^{YX}$$

and restricts the superfield expansion to

$$\mathcal{Q}_X = \psi^A Q_{AX} = \psi^A (\eta_{AX} + \psi^a q_{aAX} + \psi^1 \psi^2 \tilde{\eta}_{AX}).$$

This is precisely matches the form obtained for the antisymmetric hypermultiplet.

By choosing to introduce 2 pairs of $\mathrm{D_f}$ and mirror-$\mathrm{D_f}$s, X is an index of Sp(2). This is associated with the maximal embedding $\mathrm{Sp}(2) \times \mathrm{SU}(2) \subset \mathrm{SO}(8)$. We can then append the hCS action

$$
\begin{aligned}
S &= \frac{1}{2}\int_{\mathrm{D}_c} \mathbf{\Omega} \wedge \left(\mathrm{Tr}[\hat{\mathcal{A}} \cdot \bar{\partial}\hat{\mathcal{A}} + \frac{2}{3}\hat{\mathcal{A}} \wedge \hat{\mathcal{A}} \wedge \hat{\mathcal{A}}] + \mathcal{Q}^X \cdot \bar{\partial}\mathcal{Q}_X + \mathcal{Q}^X \wedge \hat{\mathcal{A}} \wedge \mathcal{Q}_X \right) \\
&= \frac{1}{2}\int_{\mathrm{D}_c} \mathbf{\Omega} \wedge \left(\mathrm{Tr}[\mathcal{V} \cdot \bar{\partial}\mathcal{V} + \frac{2}{3}\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V} + \mathcal{Z} \cdot \bar{\partial}\mathcal{Z} + 2\mathcal{Z} \wedge \mathcal{V} \wedge \mathcal{Z}] \right. \\
&\qquad \left. + \mathcal{Q}^X \cdot \bar{\partial}\mathcal{Q}_X + \mathcal{Q}^X \wedge \mathcal{V} \wedge \mathcal{Q}_X \right) .
\end{aligned}
$$

while in component form

$$
\begin{aligned}
S_{hCS} = \int_{\mathbb{C}\mathrm{P}^3} \mathbf{\Omega}' \wedge \Big(& \mathrm{Tr}[G \wedge F + \phi^\dagger \wedge \bar{D}\phi - \tilde{\lambda}^a \wedge \bar{D}\lambda_a + \lambda^a \wedge \lambda_a \wedge \phi^\dagger] \\
& + \mathrm{Tr}[-\frac{1}{2}z^{aA} \wedge \bar{D}z_{aA} - \tilde{\zeta}^A \wedge \bar{D}\zeta_A - z^{aA} \wedge \lambda_a \wedge \zeta_A + \zeta^A \wedge \zeta_A \wedge \phi] \\
& +\tilde{\eta}_{AX} \wedge \bar{D}\eta^{AX} - \frac{1}{2}q_{aAX} \wedge \bar{D}q^{aAX} - q_{aAX} \wedge \lambda^a \wedge \eta^{AX} + \frac{1}{2}\eta_{AX} \wedge \phi \wedge \eta^{AX} \Big)
\end{aligned}
$$

We have recovered the spectrum. To compare interactions derive Feynman rules for gauge theory and calculate MHV amplitude ratios with the ones evaluated from Witten's prescription on the twistor side.

We find agreement for a large set of amplitudes up to the same constant normalisation factor:

$$\langle\lambda^a,\phi^\dagger,\bar\lambda^b,\phi\rangle \quad \langle\phi^\dagger,{z^a}_A,{z^b}_B,\phi\rangle \quad \langle\lambda^a,\lambda^b,\zeta_A,\zeta_B\rangle \quad \langle\lambda^a,{z^b}_B,{z^c}_C,\lambda^d,\phi^\dagger\rangle$$

$$\langle\lambda^a,\zeta_A,\bar\zeta_B,\bar\lambda^b\rangle \quad \langle\phi^\dagger,{q^a}_A,{q^b}_B,\phi\rangle \quad \langle\phi_1,\phi_2,\phi_3^\dagger,\phi_4^\dagger\rangle \quad \langle\phi,{q^a}_A,{q^b}_B,\eta_C,\eta_D\rangle$$

$$\langle\eta_A,\lambda^a,\eta_B,\lambda^b\rangle \quad \langle{z^a}_A,\zeta_C,\bar\zeta_D,{z^b}_B\rangle \quad \langle\eta_A,\lambda^a,\bar\lambda^b,\bar\eta_B\rangle$$

$$\langle{z^a}_A,{z^b}_B,{z^c}_C,{z^d}_D\rangle \quad \langle{q^a}_A,{q^b}_B,{q^c}_C,{q^d}_D\rangle$$

These include 4-point amplitudes with fundamental, anti-symmetric and adjoint external particles and two 5-point.

NB1: On the twistor side we have encoded part of the flavour symmetry geometrically. Amplitude agreement forces identification $\mathrm{SU}(2) \subset \mathrm{SO}(8)$ with $\mathrm{SU}(2)_A$. Twistor string describes $\mathcal{N} = 2$ theory with global $\mathrm{SU}(2)_A \times \mathrm{Sp}(2)$.

NB2: A priori nothing is fixing the number of $\mathrm{D_f}$–branes. It would be intriguing if there existed a topological analogue of the RR cancellation condition precisely requiring the introduction of 4 $\mathrm{D_f}$s.

NB3: We have used $\mathrm{Sp}(N_c)$ group properties while constructing the gauge theory and twistor actions but the amplitudes are colour-stripped. One can straightforwardly obtain similar results for the $\mathcal{N} = 2$ theory with gauge group $\mathrm{SU}(N_c)$ and $N_f = 2N_c$.

Conclusions

- We established a tree–level correspondence between $\mathcal{N} = 2$ gauge theories with fundamental matter and twistor string theory.

- This required the introduction of flavour-branes in the B-model on $\mathbb{CP}^{3|4}$.

- We provided a prescription for the fermionic analogue of dimensional reduction for topological branes on sub-supermanifolds.

... and Outlook

- It would be interesting to obtain these objects through an analysis of boundary conditions in the B–model.

- This could allow the study of branes separating in the fermionic directions and exploration of Coulomb and Higgs branches.

- This would in turn take us away from $\mathbb{CP}^3$ and towards quantum twistor string?