Giant gravitons in AdS_3

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based on:

1) `arxiv0709.1168` – Gautam Mandal, S.R. and Mikael Smedbäck

2) `arxiv0709.1171` – S.R

Introduction

1. In the past few years, progress has been made in understanding string theory on curved spacetime backgrounds by enumerating and quantizing classical supersymmetric solutions. (eg. Biswas et al., Mandal et al., Grant et al., Martelli et al. ...)
2. The moduli space of supersymmetric solutions is a subset of the classical phase space and can be quantized using canonical methods (Dedecker 1953 and others, imprecisely known as ‘CWZ’ formalism!)
3. Here we will first list and then quantize all classical supersymmetric probe solutions in $AdS_3 \times S^3$ to obtain a description of the low energy 1/2 and 1/4 BPS sectors of string theory on this background. This helps us resolve some old puzzles but also throws up some new ones!

Overview

1. Setup
2. Classical Solutions
   (a) supersymmetric D-Strings in diverse backgrounds(D1-D5, D1-D5p, global AdS, Lunin-Mathur geometries)
   (b) supersymmetric D1-D5 bound states
3. Moving off the special point in Moduli Space
4. Quantizing probes in the N.H. of the D1-D5 system

Overview II

4. Quantizing probes in global AdS
   (a) 'Polyakov' approach to Classical Solutions
   (b) Semi-Classical Quantization
   (c) Exact spacetime partition function of the D-string
5. Comparison to the symmetric product
6. Discussion

Setup

1. We will consider probes in 4 geometries – D1-D5, D1-D5p, global AdS and the smooth solutions of Lunin-Mathur.
2. We will focus on the case where we have a RR 3-form and 7-form flux, but all other RR-fluxes and the NS-NS fluxes are set to zero.
3. This is the most commonly studied point in the D1-D5 moduli space, but the system can be generalized by turning on NS-NS fields and other RR-fluxes
4. At this point in moduli space, the boundary theory is singular because the stack of D1-D5 branes can separate at no cost in energy. This will have important implications.

Classical Solutions: D1-D5 background

1. Consider the D1-D5 system with metric:

$$ds^2 = f^{\frac{-1}{2}}\left(-dt^2 + (dx_5)^2\right) + f^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_3^2\right) + ds_{\mathrm{int}}^2$$

   with $x_5 \sim x_5 + 2\pi$.

2. This geometry has a BPS bound

$$E = P_5$$

   In the near-horizon ‘boundary CFT’, this corresponds to $\bar{L}_0 = 0$.

3. D-strings that preserve this relation are just those that keep the vector $\frac{\partial}{\partial t} - \frac{\partial}{\partial x_5}$ tangent to their worldvolume.

D1-D5 system II

4. Given an initial shape of the brane at one time we can translate it along the integral curves of this vector field to generate the entire brane worldvolume.
5. Here, the solutions are(as functions of worldsheet τ, σ)

$$\begin{aligned} t &= \tau \\ x_5 &= w\sigma + \tau \\ r &= r(\sigma) \quad \Omega_3 = \Omega_3(\sigma) \quad M_{\text{int}} = M_{\text{int}}(\sigma) \end{aligned}$$

6. Hence, the set of all supersymmetric D-strings is parameterized by the set of all initial shapes.

Global AdS

1. The system that will be most important to us is global AdS with metric

$$ds^2 = -\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2 + \sqrt{\frac{Q_1}{Q_5 v}} ds^2_{\mathrm{M_{int}}}$$

2. Here, the boundary theory is a (4,4) SCFT with fermions in the NS sector. The BPS bound is $\bar{L}_0 = \bar{J}_0^z$ where J is the $SU(2)$ R-symmetry current.
3. As one might expect, D-strings that maintain $\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}$ tangent to their worldvolume saturate the BPS bound.

Global AdS II

4. In terms of the boundary theory, these are states of the form $|\mathrm{anything} > \otimes |\mathrm{chiral-primary} >$
5. They preserve $4 = \frac{1}{4}16$ supersymmetries and are counted by the elliptic genus.

The Lunin-Mathur Geometries

1. In both supergravity and the boundary CFT we can perform an operation called ‘spectral flow’ that takes us from the NS to the R sector
2. Performing spectral flow on global AdS takes us, to the near horizon region of one of the solutions of Lunin and Mathur. This is an S^3 fibered on AdS_3 via

$$d\Omega_3^2 \rightarrow d\zeta^2 + \cos^2\zeta(d\phi_1 + d\theta)^2 + \sin^2\zeta(d\phi_2 + dt)^2$$

3. We find further evidence for the claim that the 0 mass BTZ black hole is not the correct background dual to the R-sector ground state. Our solutions behave almost identically in global AdS and the background above but very differently in the background of the 0 mass BTZ black hole.

Bound States

1. In both global AdS and the near-horizon region of the Lunin-Mathur geometries, our probes are ‘bound’ to the center of AdS for generic charges
2. In global AdS consider the difference $E - L$ where E is the energy and L the angular momentum in AdS:

$$\frac{Q_5}{2\pi}\int \frac{\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \rho'^2 + G_{ab}X^{a'}X^{b'}}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}d\sigma$$

3. The strict $\rho \to \infty$ limit(very close to the boundary), forces $E - L = Q_5 w$. If we want other values of $E - L$, we must sit at finite ρ. Suggests that these solutions correspond to ’discrete’ states.
4. These bound states are not seen in the 0 mass BTZ black hole.

Composite probes

1. Supersymmetric probes can also be bound states of p $D1$ branes and q $D5$ branes. These correspond to $D5$ branes that wrap the internal manifold, maintain the special killing vector tangent to their worldvolume and have worldvolume gauge fields with a second Chern class.
2. The symplectic structure on the moduli-space of supersymmetric classical solutions to this 6 dimensional theory is identical to the $1+1$ theory above, with the modifications:
   (a) $Q_5 \to p(Q_5 - q) + q(Q_1 - p)$.
   (b) M_{int} becomes the moduli space of p instantons in a $U(q)$ theory.
3. So, there is a zoo of probes corresponding to different value of p, q.

Moving in Moduli Space

1. One can move off the special manifold in moduli space and generalize the simplest D1-D5 sytem by turning on a NS-NS B field. An explicit supergravity solution of this kind was found by Dhar et. al.
2. When this is done the probes above do not remain supersymmetric.
3. We interpret this to mean that the BPS partition function jumps as we move off this point in moduli space.

Quantization: 0 mass BTZ BH

1. In the 0 mass BTZ geometry, the symplectic form on our solutions becomes

$$\omega = \delta P_r \wedge \delta r + \delta P_{\Omega_3} \wedge \delta \Omega_3 + \delta P_{M_int} \wedge \delta M_{int}$$

2. This leads to the left-moving sector of a $U(1) \times SU(2) \times U(1)^4$ WZW model. Adding a linear dilaton term to the U(1) allows a realization of the (small) $N = 4$ algebra on 'long'strings(Seiberg-Witten).
3. These states are at the bottom of a continuum! The wave-function is flat in y which means 'most' of it is at $y \longrightarrow \infty$.

Global AdS via the WZW model

1. In the interior of AdS, the symplectic form does not decouple nicely as it did above. So, we must find a different approach to classical solutions. Once we quantize solutions in global AdS, we can merely spectral flow to obtain the Ramond sector answer.
2. **Classically** in the Born-Infeld action, $\int \sqrt{g} + \int B$ we can introduce a worldsheet metric as a lagrange multiplier to recast this as $\int (G_{\mu\nu} + B_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu$.
3. On $AdS_3 \times S^3 \times T^4$, this yields a $SL(2,R) \times SU(2) \times U(1)^4$ WZW model.
4. Classical solutions of the WZW model are equivalent to classical solutions of the DBI action after imposing the **Virasoro constraints,** $T(x^+, x^-) = 0$.

Classical SUSY WZW Solutions

1. Classical solutions of a WZW model can be written as

$$g(x^+, x^-) = g^+(x^+)g^-(x^-);$$

2. Supersymmetry allows us to completely solve the left-moving sector of the theory and obtain a one-parameter family of solutions.

$$g(x^+, x^-)_{SL(2,R)} = \exp\left\{i\frac{j}{Q_5}\sigma_2 x^+\right\} g_1(x^-)$$

$$g(x^+, x^-)_{SU(2)} = \exp\left\{i\frac{j}{Q_5}\sigma_3 x^+\right\} g_2(x^-)$$

3. Our solutions have a spectral flow symmetry which takes $j \to j + Q_5 w$.

Chiral-Chiral Primaries

1. In the DBI approach, 1/2 BPS states have two special tangent vectors:

$$n_{\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} \pm \frac{\partial}{\partial \phi_2}$$

   This fixes $\theta = w\sigma \;\; \phi_2 = w\sigma$ and everything-else = constant. The charges, then are quantized in units of Q_5. Generic (1/2) BPS solutions cannot be obtained as limits of smooth giant graviton solutions!

2. In the Polyakov approach, (1/2) BPS solutions are geodesics passing through the center of AdS

Chiral-Chiral Primaries II

1. This clarifies several issues! For example, Argurio et al. conjectured from a study of F-string excitations in the dual F1-NS5 system that the bdry theory could be a deformation of $(M^{Q_5}/S_{Q_5})^{Q_1}/S_{Q_1}$. However, the solutions above cannot distinguish different kinds of probes. This restores democracy between Q_1 and Q_5
2. Gaberdiele/Kirsch and Dabholkar/Pakman recently calculated 2 point functions for low energy chiral-primaries. The insight above suggests that similar results would be obtained by repeating their calculations for higher energies too!
3. The fact that at special charges, chiral-chiral primaries can be described by giant gravitons at infinity corresponds to the the small instanton singularity which makes some chiral primaries vanish.

SL(2,R) WZW model: Background

1. The $SL(2,R)$ model was widely studied and finally clarified by Henningson, Hwang, Roberts, Sundborg and then by Maldacena, Ooguri. There are two kind of representations, discrete and continuous.
2. Recall in the $SU(2)$ model, we have 'affine null-vectors'. In the $SL(2,R)$ model, instead we have to add in 'winding-sectors' of states which are **not** in lowest-weight representations. Under spectral flow:

$$J^z \longrightarrow J^z + \frac{kw}{2}$$

3. This is the quantum analogue of the classical symmetry we mentioned earlier

Linking Classical/Quantum WZW models

1. The WZW model may be studied by canonical methods(Gawedzki, Kupianen, Chu, Goddard, Halliday, Schwimmer).
2. Using this, we can identify spacetime $\frac{1}{4}$ BPS states on the worldsheet. For charges smaller than Q_5 they are
   (a) SL(2,R): |affine-primary> on the left and |anything> on the right
   (b) SU(2): |affine-primary> on the left and |anything> on the right
3. For charges larger than Q_5, they are the spectral flows of these states.
4. For charges quantized in Q_5, semi-classically these states are at the bottom of the continuous representations.

Identifying SUSY states(II)

1. All (1/4) BPS states in spacetime are:

   $$|j+Q_5w>|anything>$$

   $|anything>$ must also satisfy the virasoro constraints.

2. Solving the mass-shell condition gives

   $$E=2(j+Q_5w)+\frac{N_{SL(2,R)}+N_{SU(2)}+h_{\mathrm{int}}}{w}$$

   This gives us the energy as a function of other charges for 1/4 BPS states.

3. For a composite probe, we can just replace $Q_5\to pQ_5+qQ_1$.

Exact Analysis of the D-String

1. For the D-string, we can dualize to a F1-NS5 frame and use the studies of strings in AdS_3.
2. When we generalize the spacetime partition function calculated by Maldacena and Ooguri, we find precise agreement with the formulae for discrete $\frac{1}{4}$ BPS states and $\frac{1}{2}$ BPS states above(although fermion zero modes modify degeneracies)
3. We also find that for chiral primaries that we expected would lie in a continuum, the measure vanishes

Comparison to the Symmetric Product

1. This theory is believed to be a deformation of the symmetric product $(T^4)^{Q_1 Q_5}/S_{Q_1 Q_5}$.
2. The $\frac{1}{4}$ BPS partition function is not protected but can we compare the elliptic genus and the $\frac{1}{2}$ BPS partition function?
3. In a nice paper, de Boer calculated the elliptic genus and $\frac{1}{2}$ BPS partition function of the symmetric product and found that upto an energy $\frac{Q_1 Q_5}{4}$ for the elliptic genus and $\frac{Q_1 Q_5}{2}$ for the 1/2 BPS partition function they were completely described by multi-particles of gravitons with a suitable exclusion principle!

2 Puzzles and their Resolutions!

1. **Puzzle 1:** Wouldnt we expect stringy contributions after the energy Q_5? **Resolution:** At a generic point in moduli space, there are no supersymmetric brane probes. So $\frac{1}{4}$ BPS states at a generic point in moduli space are complete described by multi-gravitons.

2. **Puzzle 2:** At this point in moduli space, why doesnt the $\frac{1}{2}$ BPS partition function see giant graviton contributions? **Resolution:** $\frac{1}{2}$ BPS states are described by geodesics even after the energy where we expect string contributions to start showing. So, the $\frac{1}{2}$ BPS spectrum of supergravity agrees with the $\frac{1}{2}$ BPS spectrum of the string theory till an energy $\frac{Q_1 Q_5}{2}$ except for some missing chiral primaries due to the singularities of the theory at this point in moduli space.

Subleading terms in the Elliptic Genus

1. **Puzzle 3:** At this point in moduli space, why doesnt the elliptic genus match?
2. If we carefully consider the terms that contribute to the elliptic genus, we find that this sum runs over states that are at the bottom of a continuum. This invalidates the index theorems that protect the elliptic genus. So, at this point in moduli space there are indeed corrections to the elliptic genus of the orbifold.
3. Due to modular invariance, the high energy behaviour of the elliptic genus is dominated by the lowest energy terms. Since the giant graviton contributions appear after an energy gap, these additional contributions are exponentially subleading and do not affect entropy calculations.

Discussion

1. It would be interesting to understand the physical significance of these subleading corrections to the elliptic genus in the spirit of the Farey Tail.
2. They may correspond to multi black-holes. These have previously been suggested by Sundborg and de Boer.
3. It would also be of interest to perform an exact analysis of composite probes. This reamains another direction for future work.

nothing

1. For SU(2), the essential result is that quantizing solutions with a given monodromy ν gives rise to states in the representation $j \sim \frac{\nu}{Q_5}$.
2. This argument may be repeated for $SL(2, R)$. If the monodromy is 'real'(elliptic conjugacy class), we get discrete representations $j \sim \frac{\nu}{Q_5}$. If the monodromy is 'imaginary'(hyperbolic conjugacy class), we get continuous representations $j \sim \frac{i\nu}{Q_5}$

Classical SUSY WZW Solutions

1. Classical solutions of a WZW model can be written as

$$g(x^+, x^-) = g^+(x^+)g^-(x^-);$$

$$g^+(x^+ + 2\pi) = g^+(x^+)M; \qquad g^-(x^- + 2\pi) = M^{-1}g^-(x^-)$$

2. These give rise to conserved currents(a pair for each generator I)

$$J^I(x^+) = \frac{k}{2}\mathrm{Tr}(iT^I(\partial_+ g)g^{-1}); \quad J^I(x^-) = \frac{k}{2}\mathrm{Tr}(iT^I g^{-1}(\partial_- g))$$

3. The spectral flow symmetry

$$g \longrightarrow e^{iw\sigma_3 x^+} g e^{-iw\sigma_3 x^-}$$

takes $j \to j + Q_5 w$ and 'puffs' up geodesics.

SUSY WZW solutions

1. The BPS bound, in this language is very neat:

$$\int (J^z)_{SL(2,R)}(x^+)dx^+ = \int (J^z)_{SU(2)}(x^+)dx^+$$

2. Supersymmetry allows us to completely solve the left-moving sector of the theory and obtain a one-parameter family of solutions.

$$(g(x^+, x^-))_{SL(2,R)} = \exp\left\{i\frac{j}{Q_5}\sigma_2 x^+\right\} g_1(x^-)$$

$$(g(x^+, x^-))_{SU(2)} = \exp\left\{i\frac{j}{Q_5}\sigma_3 x^+\right\} g_2(x^-)$$

3. This corresponds to the state $|chiral - primary > \otimes |anything >$

Semi-Classical Analysis: Global AdS

We are unable to quantize the entire moduli space of solutions in global AdS in the DBI approach. But consider the restricted solution space:

$$t = \tau, \theta = \theta_0, \zeta = const, \phi_2 = w_2\sigma + \tau, \phi_1 = \tau$$

The symplectic form reduces to that of a SHO

$$E = -P_0 = wQ_5 + 2n + 1, P_\theta = 2n + 1$$

The wavefunction is peaked at

$$\sinh^2 \rho = \frac{2n+1}{wQ_5}$$

More evidence, that these solutions give rise to 'discrete' states.

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3. This argument may be repeated for $SL(2,R)$. If the monodromy is 'real'(elliptic conjugacy class), we get discrete representations $j \sim \frac{\nu}{Q_5}$. If the monodromy is 'imaginary'(hyperbolic conjugacy class), we get continuous representations $j \sim \frac{i\nu}{Q_5}$

Identifying SUSY states(I)

1. Consider a state $|j>$ that is an **affine primary** of weight $j < Q_5$ in SL(2,R)[Discrete Representation] and of weight j in SU(2). This satisfies the physical state conditions because

$$L_0|j> \sim \frac{-j^2}{Q_5} + \frac{j^2}{Q_5} = 0$$

and is spacetime supersymmetric

$$(j_0^z)_{SL(2,R)}|j> = (j_0^z)_{SU(2)}|j>$$

2. We can spectral flow **both** in $SU(2)$ and $SL(2,R)$ to obtain a state, $|j + Q_5 w>$ that is **not** an affine primary but spacetime supersymmetric.

Exact Analysis of a D-string

1. Argurio, Giveon, Shomer have constructed exact vertex operators, including the fermions, that correspond to chiral states in discrete representations.
2. Alternately, one can write down a spacetime partition function and take a supersymmetric limit, which zeroes in on the discrete states above.
3. The composite probes correspond to 'supercritical' strings and we cannot do much better than the semi-classical answer above.

Contribution to Elliptic Genus

1. The elliptic genus of the bulk theory is believed to be the same as the elliptic genus of the orbifold $(T^4)^{Q_1 Q_5}/S_{Q_1 Q_5}$.
2. In a beautiful paper, de Boer showed that the 'polar-part' of the orbifold elliptic genus(all energies $< \frac{Q_1 Q_5}{4}$) was totally accounted for by 'multi-gravitons'.
3. this raises a puzzle, because in the analysis above, we have string oscillator states that contribute at energies $Q_5 << \frac{Q_1 Q_5}{4}$. Do they cancel?
4. For the pure $D1$ string, we can calculate a precise spacetime partition function, including the fermions and it appears that the cancellations do not occur.

Possible Resolution 1

1. Possibility 1: The appropriate cancellations when we consider **all** probes, not only D-strings. This is possible, because the level of the $SU(2)$ model actually goes as

$$k = p(Q_5 - q) + q(Q_1 - p) \tag{1}$$

   so the larger values of p, q can cancel contributions from small values of p, q.

2. Unfortunately, the composite probes are 'supercritical' and we can only quantize them approximately(as explained earlier). so, we cannot check cancellations beyond the D-string.

Possible Resolution 2

1. The string theory has a 20 dimensional moduli space. 16 parameters correspond to the size-shape of the T^4 and are boring. The other 4 correspond to turning on various NS-NS and RR fields.
2. We are working at a point where all these 4 are set to zero. To reach the orbifold(if it is on the moduli space), one must move a finite distance in these parameters.
3. In the process, the index may jump if the CFT crosses a line of marginal stability (Denef-Moore).
4. Then, the elliptic genus would have exponentially subleading corrections corresponding to multi-black holes(Sundborg)