Black Holes in $\mathcal{N}=4$ Supersymmetric String Theories

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One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH} = S_{stat}$$

Strominger, Vafa

$S_{BH} = A/4G_N$, $\qquad A =$ Area of event horizon

$$S_{stat} = \ln(\text{degeneracy})$$

$=$ ln (dimension of Hilbert space of supersymmetric states carrying given set of charges)

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

In this limit the curvature at the horizon is small and hence we can ignore string theoretic higher derivative corrections to the effective action in computing the black hole entropy.

On the microscopic side we can use appropriate asymptotic formula for the degeneracy of states to calculate the statistical entropy.

Given this success, it is natural to carry out our study of this correspondence to finer details.

What are the effects of higher derivative corrections in the action on S_{BH}?

What are the finite size corrections to S_{stat}?

Does the agreement between S_{BH} and S_{stat} continue to hold in some form even after taking into account the effect of these higher derivative corrections?

In order to study this problem we need to address two separate issues.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

We also need to know how to calculate the statistical entropy to greater accuracy.

– calculate corrections suppressed by inverse powers of various charges.

We shall address these issues for quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories.

Our goal:

1. Find exact formula for the degeneracy of quarter BPS states in a class of $\mathcal{N}=4$ supersymmetric string theories.

2. Compare the result with black hole entropy including higher derivative corrections.

By degeneracy we shall always mean the index that counts:

No. of bosonic supermultiplets – No. of fermionic supermultiplets

Underlying assumption: Once we take into account all interactions, the bosonic and fermionic supermultiplets would combine to become non-supersymmetric states.

$\longrightarrow$ Only the index worth of states will remain supersymmetric.

CHL models based on $\mathbb{Z}_N$ orbifolds

Choudhury, Hockney, Lykken

1. Begin with heterotic string theory on

   $$T^4 \times S^1 \times \widehat{S}^1$$

   T^4: A four torus

   S^1, $\widehat{S}^1$: two circles with period 2π

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an order N internal symmetry preserving $\mathcal{N} = 4$ supersymmetry. N: 2,3,4,5,6,7,8

Dual description

1. Begin with type IIB string theory on

$$K3 \times S^1 \times \widetilde{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an appropriate order N internal symmetry of type IIB string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

Special choices of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

The number of U(1) gauge fields in these theories is

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

For $N = 1$ we have $k = 10$ and $r = 28$.

States are characterized by (r dimensional) electric charge vector Q and magnetic charge vector P.

Duality Symmetries

$$Define: \qquad L = \begin{pmatrix} I_6 & \\ & -I_{r-6} \end{pmatrix}$$

Heterotic T-duality transformation is generated by a set of matrices Ω satisfying

$$\Omega L \Omega^T = L$$

and preserving the charge lattice.

$$Q \to \Omega Q, \qquad P \to \Omega P$$

T-duality invariants

$$P^2 = P^T L P, \qquad Q^2 = Q^T L Q, \qquad P{\cdot}Q = P^T L Q$$

The heterotic S-duality group of this theory can be found by studying the T-duality group in type IIB description. Vafa, Witten

Result: It is $\Gamma_1(N)$ subgroup of SL(2,Z), consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

$$a, b, c, d \in \mathbb{Z}, \quad a, d = 1 \bmod N, \quad c = 0 \bmod N$$

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

Moduli fields:

$\mathcal{N}=4$ supersymmetric string theories contain $6(r-6)+2$ moduli scalar fields.

1. Set of $r\times r$ matrix valued scalar M satisfying

$$M^T = M, \qquad MLM^T = L$$

2. An axion-dilaton field

$$\tau = a + iS, \qquad S > 0$$

The vacuum is parametrized by arbitrary values of these moduli scalar fields.

Although for simplicity we shall focus on $\mathbb{Z}_N$ orbifolds of heterotic on $T^4 \times S^1 \times \widehat{S}^1$ with prime values of N, the analysis can be generalized to

1. Other CHL orbifolds with non-prime values of N

2. $\mathcal{N} = 4$ supersymmetric asymmetric $\mathbb{Z}_N$ orbifolds of type IIA on $T^4 \times S^1 \times \widehat{S}^1$ – with all supersymmetries coming from the right-moving sector

$\mathcal{N} = 4$ supersymmetric models for which the analysis has not been generalized so far:

1. Begin with heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$

2. Take a quotient by $\mathbb{Z}_M \times \mathbb{Z}_N$ group with

– $\mathbb{Z}_N$ generated by $2\pi/N$ shift along S^1 together with an internal symmetry of order N

— $\mathbb{Z}_M$ generated by $2\pi/M$ shift along $\widehat{S}^1$ together with an internal symmetry of order M.

Allowed (M, N): (2,2), (2,4), (2,6), (3,3), (4,4)

Question: What is the entropy of an extremal black hole in this theory, carrying charge vectors (Q, P)?

– can be answered using the entropy function method.

1. Evaluate the Lagrangian density in the most general near horizon geometry preserving symmetries of $AdS_2 \times S^2$ and integrate it over S^2.

2. Take the Legendre transform of this with respect to the near horizon electric fields e_i and identify the conjugate variables as electric charges Q_i.

3. Extremize the resulting function with respect to the near horizon values of the scalar fields, and sizes of AdS_2 and S^2.

4. $2\pi\times$ Result: Black hole entropy

In the class of $\mathcal{N} = 4$ supergravity theories under consideration, the leading supergravity action gives

$$S_{BH} = \pi\sqrt{Q^2P^2 - (Q \cdot P)^2}$$

At the four derivative level the Lagrangian density has a Gauss-Bonnet term:

$$\phi(a,S)\,\sqrt{-\det g}\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\right\}$$

$$\begin{aligned}\phi(a,S) &= -\frac{1}{64\pi^2}\Big((k+2)\ln S \\ &\quad +\ln f^{(k)}(a+iS)+\ln f^{(k)}(-a+iS)\Big)\\ k &= \frac{24}{N+1}-2, \qquad r=2k+8\\ f^{(k)}(\tau) &= \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}\end{aligned}$$

After taking into account the Gauss-Bonnet term, the black hole entropy is given by

$$\frac{\pi}{2}\Big[\left(\frac{Q^2}{S}+\frac{P^2}{S}(S^2+a^2)-2\,\frac{a}{u_S}\,Q\cdot P\right)+128\,\pi\,\phi(a,S)\Big]$$

at its extremum with respect to a, S.

a, S: Near horizon values of axion-dilaton fields.

Our goal: Find the exact degeneracy $d(Q,P)$ of dyons of charge (Q,P) and compare $\ln d(Q,P)$ with S_{BH}.

We shall describe the computation for a specific class of (Q,P).

Description in IIB on $K3\times S^1\times \widetilde{S}^1/\mathbb{Z}_N$:

1) Q_5 D5-brane wrapped on $K3\times S^1$

2) Q_1 D1-branes wrapped on S^1

3) $-k/N$ units of momentum along S^1

4) One Kaluza-Klein monopole along $\widetilde{S}^1$

5) J units of momentum along $\widetilde{S}^1$

– BMPV black hole at the center of Taub-NUT

After translated to the heterotic description, this gives

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2k/N, \quad Q \cdot P = J$$

Once we have calculated the degeneracy for these charges, we can extend the formula to many other charge vectors using duality symmetries of the theory / collective coordinate excitations.

In the weakly coupled type IIB description the low energy dynamics of the system is described by three weakly interacting pieces:

1) The closed string excitations around the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

Taking the coupling $\to 0$ limit we can make the three pieces non-interacting.

The generating function of the spectrum of BPS states is given by the product of the generating function of each of these three different systems.

Note: Individual pieces can be interacting.

e.g. D1-D5 system binds strongly to the Kaluza-Klein monopole.

We shall denote the generating function by:

$$-1/\Phi_k(\rho, \sigma, v)$$

If we define $g(m,n,p)$ through

$$-\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} (-1)^p e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,,$$

then

$$d(Q,P) = g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q\cdot P\right)$$

Contribution to the generating function from individual pieces

D1-D5 center of mass motion in KK monopole background:

$$-\,e^{-2\pi iv}\,(1-e^{-2\pi iv})^{-2}$$

$$\times\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{4}\,(1-e^{2\pi in\sigma+2\pi iv})^{-2}\,(1-e^{2\pi in\sigma-2\pi iv})^{-2}\right\}$$

Red: zero-mode contribution

$\rightarrow$ comes from studying spectrum of bound states of a superparticle to the Taub-NUT space

Pope; Gauntlett, Kim, Park, Yi

Black: non-zero mode oscillators

Dynamics of KK monopole:

$$e^{-2\pi i\sigma/N}\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma/N})^{-\frac{24}{N+1}}(1-e^{2\pi in\sigma})^{-\frac{24}{N+1}}\right\}$$

Relative motion between the D1 and D5 branes:

$$e^{-2\pi i\rho}\prod_{r=0}^{N-1}\prod_{\substack{l,b\in Z,k'\in Z+\frac{r}{N}\\ k'\geq 0,l>0}}\left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{-\sum_{s=0}^{N-1}e^{-2\pi ils/N}c^{(r,s)}(4lk'-b^2)}$$

Done by Dijkgraaf, Moore, Verlinde, Verlinde for $N=1$.

$c^{r,s}(n)$: known coefficients, given in terms of jacobi ϑ-functions and Dedekind η-functions.

Product of three pieces $=-1/\Phi_k(\rho,\sigma,v)$

Definition of $c^{(r,s)}(n)$:

$$A = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]$$

$$B = \eta(\tau)^{-6}\vartheta_1(\tau,z)^2$$

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right]$$

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A\,,$$

$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}\,A - \frac{2}{N+1}\,B\,E_N(\tau)\,,$$

$$F^{(r,rk)}(\tau,z) = \frac{8}{N(N+1)}\,A + \frac{2}{N(N+1)}\,E_N\left(\frac{\tau+k}{N}\right)\,B\,,$$

for $1 \leq s \leq (N-1)$, $1 \leq r \leq (N-1)$, $0 \leq k \leq (N-1)$,

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbf{Z},n} c^{(r,s)}(4n-b^2)\,q^n\,e^{2\pi izb}$$

Φ_k transforms as a modular form of weight k of a subgroup of the modular group of genus two Riemann surfaces.

For $N = 1$, $k = 10$, i.e. heterotic on T^6, the corresponding Φ_{10} is a weight 10 modular form of the full $Sp(2, \mathbb{Z})$ group.

– Igusa cusp form.

$$-\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p\\ m\ge -1, n\ge -1/N}} (-1)^p\, e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,.$$

There are two ways of carrying out this expansion.

$e^{-2\pi iv}/(1-e^{-2\pi iv})^2$ can be expanded in powers of $e^{2\pi iv}$ or $e^{-2\pi iv}$.

$\rightarrow$ for fixed m, n, the sum over p can either be bounded from above or bounded from below.

Which one gives the correct $g(m,n,p)$?

Will be discussed later.

Equivalent description

$$d(Q,P) = \frac{1}{N}(-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2+\sigma Q^2+2vQ\cdot P)\right],$$

ρ, σ, v: complex parameters

$\mathcal{C}$: a three real dimensional subspace:

$$0 \le Re\,\rho \le 1, \quad 0 \le Re\,\sigma \le N, \quad 0 \le Re\,v \le 1\,.$$

$$Im\,\rho = M_1, \quad Im\,\sigma = M_2, \quad Im\,v = M_3,$$
$$M_1, M_2 >> |M_3| >> 0$$

Sign of M_3 is related to whether we expand $1/\Phi_k$ in positive or negative powers of $e^{2\pi iv}$.

The results can be generalized to other charge vectors in the same theory using

1. Duality symmetries

2. Collective coordinate excitation

However the result is not valid for arbitrary pair (Q, P).

An open question: What is the maximal set of (Q,P) for which the result is valid?

Example: $N = 1$ case

The results are valid for arbitrary charge vectors of the form:

$$Q = \begin{pmatrix} \alpha \\ \beta \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \qquad P = \begin{pmatrix} \gamma \\ \delta \\ \cdot \\ \cdot \\ \cdot \end{pmatrix},$$

$$\alpha\delta - \beta\gamma = 1, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{Z}$$

The results are not valid for (Q, P) for which

$$g.c.d.\{Q_i P_j - Q_j P_i\} \neq 1$$

Walls of marginal stability

$\mathcal{N} = 4$ supersymmetric string theories contain $6(r-6)+2$ moduli scalar fields.

1. Set of $r \times r$ matrix valued scalar M satisfying

$$M^T = M, \qquad MLM^T = L$$

2. An axion-dilaton field

$$\tau = a + iS, \qquad S > 0$$

Far away from the black hole the vacuum is parametrized by arbitrary values of these moduli scalar fields.

As we move in the asymptotic moduli space we encounter walls of marginal stability

– codimension 1 supspaces on which a quarter BPS dyon becomes marginally unstable against decay into a pair of **half BPS states**.

Across these lines of marginal stability the dyon spectrum can change.

Our results are valid for weakly coupled type IIB string theory.

— corresponds to some fixed region of the moduli space.

How does the degeneracy change as we move away from this region crossing the walls of marginal stability?

Results for walls of marginal stability

For a fixed charge vector (Q, P) there are many walls corresponding to the possiblity of decay into various pairs.

$$m(Q,P) = m(Q_1,P_1) + m(Q-Q_1,P-P_1)$$

$$Q_1 \parallel P_1, \qquad (Q-Q_1) \parallel (P-P_1)$$

– codimension 1 subspaces of asymptotic moduli space.

Results for the location of the walls

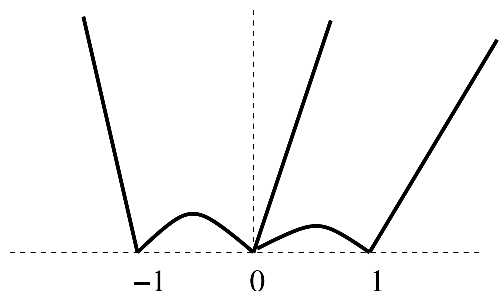
1. For fixed values of the moduli M, the walls of marginal stability are **circles** or **straight lines** in the $a + iS$ plane.

2. These curves never intersect in the interior of upper half plane.

3. They can intersect on the real axis but only at **rational points**.

$\rightarrow$ different domains bounded by walls of marginal stability have vertices at rational points or $i\infty$.

Example: $N=1$ case, i.e. heterotic on T^6



The straight line passing through 0 represents the decay $(Q,P)\to(Q,0)+(0,P)$

The straight line passing through 1 represents the decay $(Q,P)\to(Q-P,0)+(P,P)$

The circle passing through 0 and 1 represents the decay $(Q,P)\to(Q,Q)+(0,P-Q)$ etc.

Results for the degeneracy

1. The degeneracies in different domains are given by the same integral formula, but with different integration contours $\mathcal{C}$.

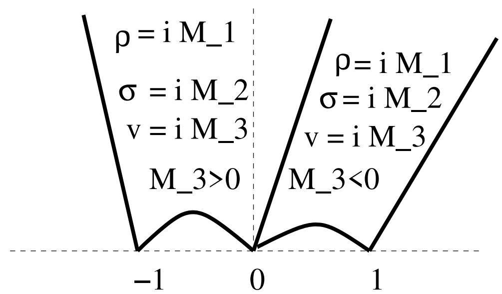
2. As we deform one contour into another we pick up residues from the poles of $1/\Phi_k$.

– reflects change in the degeneracy across the walls of marginal stablity.

3. The change across a wall $(Q,P) \to (Q_1,P_1) + (Q_2,P_2)$:

$$|\Delta d(Q,P)| = |Q_1 \cdot P_2 - Q_2 \cdot P_1| |d(Q_1,P_1)| |d(Q_2,P_2)|$$

Example: $N = 1$ case, i.e. heterotic on T^6



In other domains we have different choices of the three dimensional integration contour $\mathcal{C}$.

Comparison with black hole entropy:

For this we need to study the behaviour of $d(Q,P)$ for large charges.

$$
\begin{aligned}
d(Q,P) &= \frac{1}{N}(-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \\
&\quad \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right],
\end{aligned}
$$

a) Do the v integral by picking up residues from the poles of $1/\Phi_k$.

Result:

$$
d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}
$$

for some function $F(\rho,\sigma)$.

b) Do the ρ and σ integral using saddle point approximation.

For this we can treat

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

as we would treat a path integral and develop a Feynman diagram approach for evaluating the integral.

$\rightarrow$ 1PI effective action $-\Gamma(\rho,\sigma)$

$\ln d(Q,P)$ is the value of $\Gamma(\rho,\sigma)$ at its extremum.

$\Gamma(\rho, \sigma)$ can be calculated using Feynman diagram expansion.

Loop expansion parameter: Inverse charge

To 'one loop order'

$$\Gamma = \frac{\pi}{2}\Big[\left(\frac{Q^2}{S} + \frac{P^2}{S}(S^2 + a^2) - 2\,\frac{a}{S}\,Q \cdot P\right) + 128\,\pi\,\phi(a, S)\Big] + \mathcal{O}(Q^{-2}, P^{-2})$$

$$\rho = i/(2NS), \qquad \sigma = iN(a^2 + S^2)/(2S)$$

$\phi(a, S)$: the same function that appeared in the computation of the black hole entropy.

Thus to this order the black hole entropy computed with Gauss-Bonnet correction agrees with the statistical entropy.

How good is the asymptotic formula?

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S_{stat}^{(0)}$	$S_{stat}^{(1)}$
2	2	0	50064	10.82	6.28	10.62
4	4	0	32861184	17.31	12.57	16.90
6	6	0	16193130552	23.51	18.85	23.19
6	6	1	11232685725	23.14	18.59	22.88
6	6	2	4173501828	22.15	17.77	21.94
6	6	3	920577636	20.64	16.32	20.41
6	6	4	110910300	18.52	14.05	18.40

What is the role of the walls of marginal stability in this comparison?

In the large charge limit the change in the degeneracy across walls of marginal stability is exponentially suppressed compared to the leading term.

Thus we would expect that the asymptotic expansion of the black hole entropy should not change as we move across the walls of marginal stability.

Consistent with the generalized attractor mechanism.

However there is still an exponentially suppressed change in $d(Q, P)$ across walls of marginal stability.

Can we explain this on the black hole side?

For a given set of charges (Q, P) we have, besides single centered black holes, also two centered black hole configurations.

Typically as we cross a wall of marginal stability, a particular 2-centered black hole (dis)-appears, causing a change in entropy.

Denef; Denef, Moore

This change is precisely in accordance with the prediction of the exact formula for $d(Q, P)$.

$$|\Delta d(Q,P)| = |Q_1 \cdot P_2 - Q_2 \cdot P_1| |d(Q_1, P_1)| |d(Q_2, P_2)|$$

Speculations for dyon spectrum in $\mathcal{N}=2$ supersymmetric string theories

$$d(\vec{Q},\vec{P}) = \int_{\mathcal{C}} dM\, f(\vec{Q},\vec{P},M)$$

$$f(\vec{Q},\vec{P},M) = \exp\Big(f^{(1)}_{ij}(M)Q_iQ_j + f^{(2)}_{ij}(M)P_iP_j + f^{(3)}_{ij}(M)Q_iP_j\Big) \times g(M)$$

M: A set of complex variables

$f^{(1)}_{ij}$, $f^{(2)}_{ij}$, $f^{(3)}_{ij}$: Simple functions of M

$g(M)$: some complicated function of M that encodes non-trivial information about the spectrum

As we cross walls of marginal stability the integration contour $\mathcal{C}$ will change.

Jump in the spectrum: obtained from residues at the poles picked up during contour deformation.

Denef-Moore wall crossing formula gives

$$\Delta d(\vec{Q},\vec{P}) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,|Q_1\cdot P_2 - Q_2\cdot P_1|\; d(\vec{Q}_1,\vec{P}_1)d(\vec{Q}_2,\vec{P}_2)$$

across a wall corresponding to the decay $(Q,P)\to(Q_1,P_1)+(Q_2,P_2)$.

Thus the wall crossing formula will relate the residues at the poles in $f(Q, P, M)$ to $f(Q_1, P_1, M)$ and $f(Q_2, P_2, M)$.

$\longrightarrow$ a set of bootstrap relations.

Can we find a consistent solution?