Large D Black Hole Membrane Dynamics

Parthajit Biswas

NISER Bhubaneswar

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References

The talk is mainly based on

- S. Bhattacharyya, P. Biswas, B. Chakraborty, A. Dinda and Y. Dandekar, arXiv:1704.06076

The other references are

- S. Bhattacharyya, A. De, S. Minwalla, R. Mohan, and A. Saha, arXiv:1504.06613
- S. Bhattacharyya, M. Mandlik, S. Minwalla, and S. Thakur, arXiv:1511.03432
- R. Emparan, R. Suzuki and K. Tanabe, arXiv:1302.6382, 1502.02820
- R. Emparan and K. Tanabe, arXiv:1401.1957

Introduction/Motivation:

- The evolution of spacetime is governed by Einstein equation. And it is very hard to solve Einstein equation in general.
- Our aim in this talk is to find new perturbative solution to Einstein equation.
- We will use the number of space time dimensions as perturbation parameter.
  Hence, the solution would be a series in $\frac{1}{D}$ expansion.

How large D simplifies the problem

- Emparan Suzuki and Tanabe ('2013) observed the following:
- Schwarzchild black hole in Kerr-Schild coordinate is given by

$$dS^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{D-2} + \left(\frac{r_0}{r}\right)^{D-3} (dt + dr)^2 \quad (1)$$

Now, if we take $r > r_0$, and keep the ratio $\frac{r_0}{r}$ fixed, then

$$\lim_{D\to\infty} \left(\frac{r_0}{r}\right)^{D-3} = 0$$

So, in $D \to \infty$, space time outside the horizon becomes flat.

- But if we take, $r = r_0 \left(1 + \frac{R}{D-3}\right)$

Now if we keep R fixed instead of r_0 and take $D \to \infty$

$$\lim_{D\to\infty} \left(\frac{r_0}{r}\right)^{D-3} = e^{-R}$$

This means that if we go closer to the horizon as we take $D \to \infty$ we will find something nontrivial from flat space.

- So, in large dimension (D) Schwarzchild Black hole becomes a thin shell of thickness $\mathcal{O}\left(\frac{1}{D}\right)$ around the horizon propagating in flat space background or in general AdS/dS background.

Setup

- Our background is AdS/dS or more generally any asymptotic background that solves Einstein equation.
- We will work with Einstein-Hilbert action with cosmological constant

$$S = \int d^D x \sqrt{-G}\ [R + \Lambda]$$

- The Evolution of space-time will be governed by the Einstein equation

$$E_{AB} \equiv R_{AB} - \left(\frac{R + \Lambda}{2}\right) G_{AB}$$

- AdS-Schwarzchild Black Hole in D dimension in Kerr-Schild form is given by

$$dS^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{D-2}^2 + \frac{1}{1+r^2}\left(\frac{r_0}{r}\right)^{D-3}\left(-\sqrt{1+r^2}dt + \frac{dr}{\sqrt{1+r^2}}\right)^2$$

- Now we will consider more general metric which we will call "Ansatz Metric"

$$G_{AB}^{(0)} = g_{AB}^{(AdS)} + \psi^{-D}(n_A - u_A)(n_B - u_B) \qquad (2)$$

Where, ψ is any arbitrary function. and u_A is an arbitrary one form velocity field.

$$n_A = \frac{\partial_A \psi}{\sqrt{\partial\psi \cdot \partial\psi}}, \;\; n \cdot n = 1, \;\; u \cdot u = -1, \;\; n \cdot u = 0 \qquad (3)$$

Where all the '·' is with respect to $g_{AB}^{(\mathrm{AdS})}$

- Two nice thing about this metric is that
  - For $\psi - 1 \sim \mathcal{O}(1)$ it becomes pure AdS.
  - And, $\psi = 1$ surface is a null surface, which can be argued as the horizon of the black hole.

$$\left(n_A n_B G^{AB}_{(0)} = 1 - \frac{1}{\psi^{-D}} \right)$$

- As $D \to \infty$ the metric blows up for $\psi << 1$.
- But $\psi < 1$ region is causally disconnected from the $\psi > 1$ region.
- Therefore for our construction we will be concerned about only the $\psi > 1$ region. And neglect the region $\psi < 1$.

Scaling with D

- As D$\rightarrow \infty$ both the number of equations to be solved and the variables goes to infinity. So, no perturbation technique will work.
- To get rid of this complication we shall assume that a large part of the geometry is fixed by some symmetry and the metric is dynamical only along some finite number(p) of directions.
  $dS^2 = G_{AB} dX^A dX^B = \tilde{G}_{ab}(x^a) dx^a dx^b + f(x^a) d\Omega^2_{D-p}$
- If $T_{A_1,A_2,\ldots A_n}$ is a tensor of order $\mathcal{O}\left(\frac{1}{D}\right)^k$ maintaining the symmetry then its divergence is of order $\mathcal{O}\left(\frac{1}{D}\right)^{k-1}$.

$$T_{A_1,A_2,\ldots,A_n} \sim \mathcal{O}\left(\frac{1}{D}\right)^k \Rightarrow g^{A_p A_q} \nabla_{A_p} T_{A_1,A_2,\ldots,A_q,\ldots} \sim \mathcal{O}\left(\frac{1}{D}\right)^{k-1}$$

- However, we should emphasize that for our calculation we do not need any details of the decomposition. The only aspect of it that will be used for our calculation is the above **scaling law**

Solving Einstein equation at leading order

- Now we will calculate Einstein tensor on the ansatz metric $G^{(0)}_{AB}$.
- Einstein equation can be written as

$$E_{AB} \equiv R_{AB} - (D-1)\lambda G_{AB} = 0$$

- Ricci tensor evaluated on $G^{(0)}_{AB} = \eta^{\text{AdS}}_{AB} + \psi^{-D} O_A O_B$ is given by

$$R_{AB} = \tilde{R}_{AB} + \psi^{-D}\left(\frac{DN}{2}\right)\left\{\left[DN - (\nabla\cdot O)\right](n_A O_B + n_B O_A) + (K - DN) O_A O_B\right\} + \left(\frac{\psi^{-2D}}{2}\right)\left\{DN\left[DN - (\nabla\cdot O)\right] O_A O_B\right\} + \mathcal{O}(D) \tag{4}$$

Where $\tilde{R}_{AB}$ is the Ricci tensor evaluated on g_{AB} and is of the order $\sim \mathcal{O}(D)$

- Einstein equation to be satisfied at the leading order $(\mathcal{O}(D^2))$ the following conditions has to be satisfied

$$(\nabla\cdot O - DN)_{\psi=1} = \mathcal{O}(1) \text{ and } (\nabla\cdot u)_{\psi=1} = \mathcal{O}(1)$$

Subsidiary Conditions

- So, we have found some conditions on ψ and O. But only at the $\psi = 1$ surface.
- Therefore there is a large ambiguity in the construction of ψ and O. We will fix this ambiguity by some convenient choices referred as 'Subsidiary Conditions'
  - $\nabla^2 \psi^{-D} = 0$ everywhere.
  - $O \cdot O = 0$ and $O \cdot n = 1$ everywhere.
  - $P^{AB}(O \cdot \nabla) O_A = 0$ everywhere

Where $P^{AB} = g^{AB} - n^A n^B + u^A u^B$

Einstein Equation at First sub-leading order

- After imposing the conditions mentioned in the previous slides $\mathcal{E}_{AB}$ now becomes of order $\mathcal{O}(D)$
- To cancel this order $\mathcal{O}(D)$ piece we will add $\frac{1}{D}G^{(1)}_{AB}$ with $G^{(0)}_{AB}$
- Then at order $\mathcal{O}(D)$ $\mathcal{E}_{AB}$ has two pieces one coming from $\left(\frac{1}{D}\right)G^{(1)}_{AB}$ which we will call Homogeneous part (H_{AB}) and the other part coming from $G^{(0)}_{AB}$ which we will call Source part (S_{AB}). Schematically,

$$\mathcal{E} \sim H_{AB} + S_{AB}$$

- We can choose $G^{(1)}_{AB}$ in such a way that the order $\mathcal{O}(D)$ term will be cancelled.
- So we will get a solution of Einstein equation in the $\frac{1}{D}$ expansion

$$G_{AB} = G^{(0)}_{AB} + \frac{1}{D}G^{(1)}_{AB} + \cdots \tag{5}$$

Gauge Condition & Explicit ψ Dependence

- We choose a gauge such that $O^A G^{(1)}_{AB} = 0$.
- Under this gauge choice the structure of most general correction is

$$G^{(1)}_{AB} = \mathcal{S}_1 O_A O_B + \frac{1}{D}\mathcal{S}_2 P_{AB} + [O_A \mathcal{V}_B + O_B \mathcal{V}_A] + \mathcal{T}_{AB}$$

Where
$u^A \mathcal{V}_A = n^A \mathcal{V}_A = 0; \quad u^A \mathcal{T}_{AB} = n^A \mathcal{T}_{AB} = 0; \quad g^{AB}\mathcal{T}_{AB} = 0.$

- We can define $\mathcal{S}_1,\ \mathcal{S}_2,\ \mathcal{V}_A,\ \mathcal{T}_{AB}$ as

$$\mathcal{S}_1 = \sum_n f_n(R)\mathfrak{s}_n,\ \mathcal{S}_2 = \sum_n h_n(R)\mathfrak{s}_n$$

$$\mathcal{V}_A = \sum_n v_n(R)\mathfrak{v}_n]_A,\ \mathcal{T}_{AB} \sum_n t_n(R)[\mathfrak{t}_n]_{AB}$$

where $R = D(\psi - 1)$, and $\mathfrak{s}_n, \mathfrak{v}_n, \mathfrak{t}_n$ are different scalar vector and tensor structure of $\mathcal{O}(1)$

Source

- The source (S_{AB}) at first subleading order is given by

$$\psi^{-D}\left(\frac{K}{2}\right)\left\{\psi^{-D}\left[\frac{(u\cdot\nabla)K}{K}+(\nabla\cdot u)\right]O_B O_A+2(n\cdot\nabla)\left(O_A O_B\right)-\frac{\nabla^2(O_A O_B)}{K}\right.$$
$$\left.+\left[\frac{(n\cdot\nabla)K}{K}+\nabla\cdot u-u^C(O\cdot\nabla)n_C\right](n_B O_A+n_A O_B)\right\} \tag{6}$$

Solution at first subleading order

- The variation along the radial direction is $\mathcal{O}(D)$ times higher than the other directions. So we will get some second order inhomogeneous ordinary differential equations.
- We can decouple scalar vector and tensor structure and can uniquely determine them given some suitable boundary conditions.

- Different structure will be decoupled under the following combination

$$\sum_n \frac{d}{dR}\left[\left(e^R-1\right)\dot{t}_n\right]\ [\mathfrak{t}_n]_{AB}=\left(\frac{2\ e^R}{DN^2}\right)\left[P_A^C P_B^{C'}-P_{AB}\left(\frac{P^{CC'}}{D}\right)\right]S_{CC'}$$

$$(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R\dot{v}_n\right]\ [\mathfrak{v}_n]_A=\left(\frac{2\ e^R}{DN^2}\right)\left[u^B S_{BC}P_A^C\right]$$

$$(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R\dot{f}_n-\frac{h_n}{2}\right]\ \mathfrak{s}_n=\left(\frac{2\ e^R}{DN^2}\right)\left(u^A S_{AB}u^B\right)-\sum_n v_n\left(\frac{\nabla\cdot\mathfrak{v}_n}{DN}\right)$$

$$\sum_n \ddot{h}_n\mathfrak{s}_n=\left(\frac{2}{N^2}\right)\left[O^A\ S_{AB}\ O^B\right] \tag{7}$$

Boundary Condition and solution

- we need two sets of boundary conditions to fix the integration constants.
- $f_n(R), v_n(R), t_n(R), h_n(R)$ decays exponentially as $R \to \infty$.
- It fixes one set of integration constants.
- It turns out that trace correction will be fixed by this normalization constant.
- For f_n and v_n it will be fixed by the requirement that the $\psi = 1$ is the horizon and u_A is its null generator. This will give the following condition

$$f_n(R=0) = 0; \quad v_n(R=0) = 0; \tag{8}$$

- For $t_n(R)$ the other integration constant could be fixed by demanding the solution is regular at the horizon
- It turns out with the above boundary condition, and with our choice of subsidiary conditions correction at the first sub leading order vanishes.

- After determining all the correction we see that Einstein equation will be satisfied upto first subleading order provided the following constraint equation will be satisfied
- The constraint equation at the first subleading order.

$$P^{\nu}_{\mu}\left[\frac{\nabla^2 u_\nu}{\mathcal{K}} - \frac{\nabla_\nu \mathcal{K}}{\mathcal{K}} + u^\alpha \mathcal{K}_{\alpha\nu} - u^\alpha \nabla_\alpha u_\nu\right] = \mathcal{O}\left(\frac{1}{D}\right)$$

$$\nabla \cdot u = \mathcal{O}\left(\frac{1}{D}\right)$$

- Here $\mathcal{K}_{\mu\nu}$ is the extrinsic curvature, $\mathcal{K}$ is its trace and u is the velocity field on the membrane.

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - n^\mu n^\nu$$

Constraint Equation

- Now we count the no of equation and no of variable

$$P_\mu^\nu \left[\frac{\nabla^2 u_\nu}{\mathcal{K}} - \frac{\nabla_\nu \mathcal{K}}{\mathcal{K}} + u^\alpha \mathcal{K}_{\alpha\nu} - u^\alpha \nabla_\alpha u_\nu \right] = \mathcal{O}\left(\frac{1}{D}\right)$$

$$\nabla \cdot u = \mathcal{O}\left(\frac{1}{D}\right)$$

- The vector equation is projected equation perpendicular to n and u, so, it gives (D-2) equations. And we have one scalar equation. So, total number of equation we have is (D-1)
- The number of variables we have: one shape function($\psi(x)$). And one velocity field on the membrane i.e. (D-1) functions but it satisfies $u \cdot u = -1$ that gives (D-2) function. So, total number of variables we have (D-1).
- So, we have well posed initial value problem.

Second Subleading Order

- Provided the leading order membrane equations have been satisfied Einstein tensor will be $\mathcal{O}(1)$ quantity.
- Now we can do the second subleading order calculation.
- The fact that the first subleading order correction vanishes simplifies the next order calculation a lot.
- For the source part we need to calculate Einstein tensor on the ansatz metric upto second subleading order.
- Homogeneous part has universal structure at all order, we don't need to calculate it.
- Exactly following the same steps of leading order we have determined the metric correction at second subleading order, and also the correction to the membrane equation.

2 nd order corrected equation

- Scalar equation upto second subleading order becomes

$$\nabla \cdot u = \frac{1}{2\mathcal{K}} \left(\nabla_{(\alpha} u_{\beta)} \nabla_{(\gamma} u_{\delta)} \mathcal{P}^{\beta\gamma} \mathcal{P}^{\alpha\delta} \right) + \mathcal{O}\left(\frac{1}{D}\right)^2$$

- Vector membrane equation upto second subleading order becomes

$$\begin{aligned}
&\left[\frac{\nabla^2 u_\alpha}{\mathcal{K}} - \frac{\nabla_\alpha \mathcal{K}}{\mathcal{K}} + u^\beta \mathcal{K}_{\beta\alpha} - u \cdot \nabla u_\alpha \right] \mathcal{P}^\alpha_\gamma \\
&+ \Bigg[-\frac{u^\beta \mathcal{K}_{\beta\delta} \mathcal{K}^\delta_\alpha}{\mathcal{K}} + \frac{\nabla^2 \nabla^2 u_\alpha}{\mathcal{K}^3} - \frac{(\nabla_\alpha \mathcal{K})(u \cdot \nabla \mathcal{K})}{\mathcal{K}^3} - \frac{(\nabla_\beta \mathcal{K})(\nabla^\beta u_\alpha)}{\mathcal{K}^2} - \frac{2\mathcal{K}^{\delta\sigma} \nabla_\delta \nabla_\sigma u_\alpha}{K^2} - \frac{\nabla_\alpha \nabla^2 \mathcal{K}}{\mathcal{K}^3} \\
&+ \frac{\nabla_\alpha (\mathcal{K}_{\beta\delta} \mathcal{K}^{\beta\delta} \mathcal{K})}{K^3} + 3\frac{(u \cdot \mathcal{K} \cdot u)(u \cdot \nabla u_\alpha)}{\mathcal{K}} - 3\frac{(u \cdot \mathcal{K} \cdot u)(u^\beta \mathcal{K}_{\beta\alpha})}{\mathcal{K}} - 6\frac{(u \cdot \nabla \mathcal{K})(u \cdot \nabla u_\alpha)}{\mathcal{K}^2} + 3\frac{u \cdot \nabla u_\alpha}{D-3} \\
&+ 6\frac{(u \cdot \nabla \mathcal{K})(u^\beta \mathcal{K}_{\beta\alpha})}{\mathcal{K}^2} - 3\frac{u^\beta \mathcal{K}_{\beta\alpha}}{D-3} - \frac{(D-1)\Lambda}{\mathcal{K}^2} \left(\frac{\nabla_\alpha \mathcal{K}}{\mathcal{K}} - 2u^\sigma \mathcal{K}_{\sigma\alpha} + 2(u \cdot \nabla) u_\alpha \right) \Bigg] \mathcal{P}^\alpha_\gamma = \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}$$

Matching with Known Result

- We have linearized the membrane equation around AdS Schwarzchild black hole and calculated Quasinormal mode frequencies.

$$\omega_s r_0 = \pm\sqrt{l\left(1+\frac{r_0^2}{L^2}\right)-1} - i(l-1)$$

$$\omega_v r_0 = -i(l-1)$$

This matches with the QNM frequencies calculated by EST from gravitational analysis.

- We have checked that Known exact solutions namely AdS-Schwarzchild Black hole, AdS Black Brane and rotating black hole in AdS satisfy the membrane equation.

Connection with Fluid Gravity(Ongoing Work)

- We can try to match the result of 'Large-D' expansion with another perturbative expansion namely 'Fluid-Gravity Correspondence'.
- We will use 'Large D' expansion over the metric corrected upto first subleading order in derivative expansion(which is known).

Once we know the metric upto first subleading order in derivative expansion, we have to follow the steps of large D expansion.

- Calculate the horizon $r = r_0(x^\mu)$.
- Calculate the normal of the horizon (n_A), the extrinsic curvature K_{AB} and it's trace K.
- Calculate the null generator of the horizon. On the horizon $u^A = G^{AB} n_B$, where G^{AB} is the inverse of the full metric.
- We have checked that membrane equations will be true provided ideal fluid equation has been satisfied.

$$P^\nu_\mu \left[\frac{\nabla^2 u_\nu}{\mathcal{K}} - \frac{\nabla_\nu \mathcal{K}}{\mathcal{K}} + u^\alpha \mathcal{K}_{\alpha\nu} - u^\alpha \nabla_\alpha u_\nu \right] = \mathcal{O}\left(\frac{1}{D}\right)$$

$$\nabla \cdot u = \mathcal{O}\left(\frac{1}{D}\right)$$

- This implies that with field redefinition we should be able to write the fluid dynamics equation as a membrane equation.

Future Directions

- To explore the relation between ‘Fluid-Gravity’ and ‘Large-D’ upto next subleading order is one immediate future direction. Which may help to understand the connection between these two approach better.
- To calculate the stress tensor in presence of Cosmological constant and to derive the equation of motion from its conservation is another future direction.

Thank You