Revisiting perturbed CFTs coupled to 2d gravity through the Majorana fermion

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1. Motivations

2d quantum gravity:
- toy model for 4d gravity (computable, enlarged symmetry, less dofs)
- cross-fertilization with (non-) critical string theory
- spontaneous dimensional reduction (Carlip, 1705.05417)

Historically : study of CFT matter → Liouville theory = gravity
Real-world → non-conformal matter
Richer structure but more complicated

Massive Majorana fermion:
- free theory
- next simplest case after massive scalar
- conformal perturbation of a CFT → compare with DDK ansatz (finite-temperature Ising model)

2. Functional integral for 2d gravity

KEEP!

Classical action:

$$S[g,\psi] = S_c[g] + S_m[g,\psi]$$

metric ↵   ↳ matter fields

Cosmological constant action:

$$S_c = \mu \int d^2x \sqrt{g}$$

Matter action:

$$S_m[g,\psi] = S_{cft}[g,\psi] + S_p[g,\psi]$$

↳ Weyl invariant: $S_{cft}[g e^{2\omega},\psi] = S_{cft}[g,\psi]$

Partition function (for fixed metric):

$$Z[g] = e^{-S_c[g]} Z_m[g] \qquad Z_m[g] = \int d_g\psi \; e^{-S_m[g,\psi]}$$

Hard to compute exactly $Z_m[g]$

⟶ work in conformal gauge to disentangle gravity / matter dynamics

$$g_{\mu\nu} = e^{2\varphi} \hat{g}_{\mu\nu} \qquad \varphi : \text{Liouville mode}$$

↳ fixed

Introduce gravitational (WZW) action

$$Z_m[g] = e^{-S_{grav}[\hat{g},\varphi]} Z_m[\hat{g}]$$

Total action:

conformal: $S_{grav} = S_L$ (Liouville)

$$S_{tot}[\hat{g},\varphi,\psi] = \underbrace{S_c[\hat{g},\varphi] + S_{grav}[\hat{g},\varphi]}_{\text{gravity}} + \underbrace{S_m[\hat{g},\psi]}_{\text{matter}}$$

Emergent Weyl symmetry: $\hat{g} \longrightarrow e^{2\omega}\hat{g}$, $\varphi \longrightarrow \varphi - \omega$

3. Spectral analysis

* compute $\ln Z_m \propto -\ln \det D$ (ill-defined)

Goal: * compute effective action for the operator D associated to a field ψ

$$D = -g^{\mu\nu}\nabla_\mu \nabla_\nu + Q(x)$$

Eigenvalue equation: $D\psi_n = \lambda_n \psi_n$ note: ignore zero-modes

Zeta functions:

$$\zeta(s,x,y) = \sum_n \frac{\psi_n(x)\,\psi_n(y)^\dagger}{\lambda_n^s}$$

pole with residue $1/4\pi$ at $s=1$

Heat kernel $K(t)$:

$$\zeta(s) = \int d^2x \sqrt{g}\, \operatorname{tr} \zeta(s,x,x) = \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1} K(t)$$

Green function: $G(x,y) = \zeta(1,x,y)$

$\rightarrow$ regularization at coincident points

$$G_\zeta(x) = \lim_{s\to 1}\left(\zeta(s,x,x) - \frac{M^{2(s-1)}}{4\pi(s-1)\Gamma(s)} \right)$$

M : mass scale (otherwise terms with $\neq$ dim.)

Effective action:

$$\ln Z_m[g] \propto -\ln \det \frac{D}{M^2} = \zeta'(0) + \ln M^2\, \zeta(0)$$

In practice: compute $\delta\zeta, \delta\zeta'$ for small $\delta\varphi$ and integrate to get S_{grav}.

Note: fixed area formalism, $\mu \longrightarrow A$ ($\exists$ subtleties).

4. Massive scalar KEEP!

$$S_m = \frac{1}{2\pi}\int d^2x \sqrt{g}\left(\frac{1}{2}(\partial X)^2 + \frac{m^2}{2} X^2\right)$$

Complete action [Bilal-Leduc, '16], [Bilal-De Lacroix, '17] → include boundaries

→ no zero-mode

$$S_{grav} = -\frac{c_m}{6} S_L + \frac{1}{2}\ln\frac{A}{A_0} + \frac{m^2}{2}\int d^2x\left(\sqrt{g}\,\tilde{G}_\zeta(x;g) - \sqrt{\hat{g}}\,\tilde{G}_\zeta(x;\hat{g})\right)$$

$c_m = 1$ ↵ zero-mode ↵

(Weyl anomaly)

$$+\frac{1}{2}\int_0^\infty \frac{dt}{t}\left(e^{-m^2t} - m^2t - 1\right)\left(\tilde{K}(t;g) - \tilde{K}(t,\hat{g})\right)$$

Small mass expansion:

$$S_{grav} = -\frac{c_m}{6} S_L + \beta^2 S_M + \zeta^2 S_{AY} + \dots$$

Mabuchi ↵ ↳ Aubin-Yau

Here: $\beta^2 = \frac{m^2 A}{2}$ $\zeta^2 = \underbrace{g}_{\text{genus}}\beta^2$

STOP

Other expansions: + modified zero-mode

- massive with non-minimal coupling ($c_m = 1 + 6q^2$, $\beta^2 = q^2\chi/4$, $\zeta^2 = 0$)
- massless " " " " and linear term

Mabuchi action:

$$S_M = \frac{1}{4\pi}\int d^2x\sqrt{\hat{g}}\left[-2\pi\chi\,\hat{g}^{\mu\nu}\partial_\mu K\partial_\nu K + \left(\frac{4\pi\chi}{\hat{A}_0} - \hat{R}\right)K + \frac{4}{A}\varphi e^{2\varphi}\right]$$

$$e^{2\varphi} = \frac{A}{A_0}\left(1 + \frac{A_0}{2}\Delta_0 K\right)$$

$K \sim$ Kähler potential

Aside: minisuperspace approx: $H_M = H_L$

6. Comparison with DDK

Note: regularity used to describe massive matter and gravity

Consider deformed CFT (action $S = S_{cft} + S_p$)

$$S_p[g,\psi] = \sum_i \lambda^i \int d^2x \sqrt{g}\, \mathcal{O}_i(\psi)$$

$\hookrightarrow$ primary in CFT, h_i

DDK ansatz for S_{tot}:

$$S_{DDK}[\hat{g},\varphi,\psi] = Q^2 S_L[\hat{g},\varphi] + S_{cft}[\hat{g},\psi] + S^{(p)}_{DDK}[\hat{g},\varphi,\psi] \qquad Q^2 = \frac{26-c_m}{6}$$

$$S^{(p)}_{DDK} = \sum_i \lambda^i \int d^2x \sqrt{\hat{g}}\, e^{2a_i Q\varphi}\, \hat{\mathcal{O}}_i(\psi)$$

$= $ conformal perturbation of $S_{grav,cft} = S_L + S_{cft}$

Require emergent Weyl invariance:

$$a_i = \frac{Q}{2} - \sqrt{\frac{Q^2}{4} + h_i - 1} \underset{Q\to\infty}{\approx} \frac{1}{Q}(1-h_i)$$

Massive Majorana:

$\mathcal{O}(\psi) = \bar{\psi}\psi = \varepsilon$

$\lambda \sim m$

$h(\varepsilon) = (1/2, 1/2)$, OPE: $\quad \varepsilon(z,\bar{z})\,\varepsilon(w,\bar{w}) \sim \dfrac{1}{|z-w|^2}$

Problems with DDK ansatz:

- not proper conformal gauge gravitational action (matter/gravity sectors not independent, matter "sees" the ghosts)
- conformal gauge requires exact emergent Weyl invariance of the total action
  beta function: $\hat{\beta}(\lambda) = O(\lambda^2)$
- direct computation by Zamolodchikov: ignore divergence due to colliding insertions
- different from our computation

5. Massive Majorana fermion

$$S_m = \frac{1}{4\pi}\int d^2x \sqrt{g}\; \bar{\psi}\left(i\nabla\!\!\!/ + m\gamma_*\right)\psi \qquad\qquad \psi^* = \psi$$

Spectral analysis for

$$D = -g^{\mu\nu}\nabla_\mu \bar{\nabla}_\nu + \frac{R}{4} + m^2$$

Action:

$$S_{grav} = -\frac{\overset{\rightarrow 1/2}{c_m}}{6} S_L - \frac{m^2}{4}\int d^2x\, \mathrm{tr}\left(\sqrt{g}\, \overset{\tilde{G}_5^{(m^2=0)}(x;g) + O(m^2)}{\overset{\shortparallel}{\tilde{G}_5(x;g)}} - \sqrt{\hat{g}}\, \tilde{G}_5(x;\hat{g})\right)$$

$$-\frac{1}{4}\int_0^\infty \frac{dt}{t}\, \underbrace{\left(e^{m^2 t} - m^2 t - 1\right)}_{\shortparallel \atop O(m^4)}\left(\tilde{K}(t;g) - \hat{K}(t;\hat{g})\right) + \text{zero-mode term}$$

In progress:

- small mass expansion - recover Mabuchi?
- zero-modes

7. Conclusion

- conformal perturbation / coupling to gravity do not commute
- regularize Zamolodchikov computation by introducing a connection on CFT space $\longrightarrow$ corrections $= S_H + \cdots$?

More generally: properties of S_H not known.

- conformal properties ?
- critical exponent ? ($\gamma_{1\text{-loop}}$)
- universality ?