Lecture 1 This is the first of two parallel lectures at this school on the subject of string field theory. The other lecture series will be given by Y. Okawa, reviewing recent developments in the construction of superstring field theories. In these lectures we talk about the construction of classical solutions in SFT. From the start, we focus almost exclusively on Witten's open bosonic string field theory. This is by far the simplest string field theory, and the theory about which we know the most about classical solutions.

First look We start by outlining, at a very crude level, the kind of object we're dealing with. Open bosonic string field theory is the field theory of fluctuations of a D-brane in bosonic string theory. The fluctuations of a D-brane are characterized by the open strings which attach to that D-brane.

Consider for example a Dp-brane in bosonic string theory. An open string attached to this Dp-brane can mimick an infinite variety of particle states, depending on how the string vibrates. The lowest modes of vibration give you a spin 0 tachyon, a p+1-dimensional spin 1 photon, and 25−p massless spin 0 particles, where $25+1=26$ is the dimension of spacetime in bosonic string theory. The higher vibrations give an infinite tower of massive particle states of higher spin. We can easily infer what kind of fields should enter the SFT lagrangian to create this spectrum of particle states:

spin 0 tachyon $\longrightarrow$ tachyonic scalar field $T(x)$

spin 1 photon $\longrightarrow$ p+1-dimensional gauge field $A_\mu(x)$ $\mu=0,1,\dots,p$

25−p massless spin 0 particles $\longrightarrow$ 25−p massless scalar fields $\phi_a(x)$ $a=1,\dots,25-p$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

massive particles of higher spin $\longrightarrow$ higher rank tensor fields

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

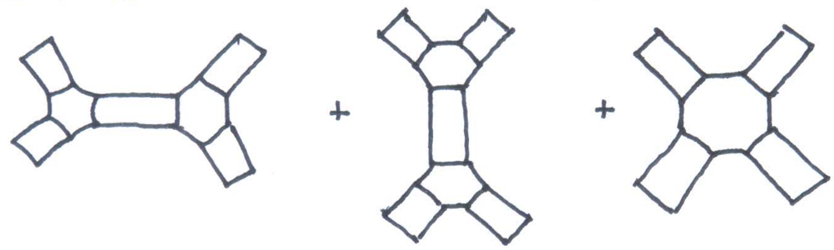
Note that the coordinate $x\in\mathbb{R}^{1,p}$ refers to a point on the worldvolume of the Dp-brane. Since open strings are attached to the D-brane, the fields do not depend on spacetime coordinates away from the D-brane worldvolume. With this we can at least begin to write an action for the fluctuations of the D-brane. Choosing $\alpha'=1$ an mostly plus signature, we have

$$S=-\int d^{p+1}x\left[\frac{1}{2}\partial^\mu T\partial_\mu T-\frac{1}{2}T^2+\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a+\text{massive modes}\right]$$

$+$ interactions.

With some more work we can write down kinetic terms for the massive fields. The form of the interactions, however, is almost impossible to guess at this level. The formulation of interactions depends heavily on the conformal field theory description of the string worldsheet, which we discuss later. In any case, the interactions must be constructed in such a way that the Feynman diagrams derived from the SFT action ~~coincide with the~~ compute open string S-matrix elements on the D-brane. The kinetic term defines a propagator; the interactions define a cubic vertex, a quartic vertex, and so on as is necessary to get the right scattering amplitudes. So, for example, the 4-string amplitude will be represented as a sum of an s-channel, t-channel, and quartic vertex contributions:

$A_4 =$



This may seem a little uncomfortable. One of the nicest things about string scattering amplitudes is that each amplitude is represented by a single diagram; the interaction is a global property of the diagram, and not a process inside vertices in a part of the diagram. There is nothing inconsistent about this, however. The three diagrams represent integration over different portions of the moduli space of disks with four boundary punctures; the single string diagram we are used to visualizing represents integration over the entire moduli space. While it is possible to slice the moduli spaces of Riemann surfaces into components representing different Feynman diagrams, this can be done in many ways, and it is not clear that there is a "natural" way to do it. Nevertheless, the formulation of SFT requires some choice of decomposition. Different decompositions correspond to different SFT actions, but since the actions produce the same scattering amplitudes, they should be related by field redefinition. The field redefinition ambiguity is not something special to SFT, but is present in all Lagrangian field theories. The reason you do not hear about it more often is that, for the field theories we're used to dealing with, there is a canonical or "best possible" formulation of the Lagrangian — or at least a finite number of useful alternatives. A central question in SFT is therefore whether there is a "best possible" formulation of the Lagrangian. Hapily, for open bosonic SFT the answer is unambiguously "yes," and this is Witten's open bosonic SFT. For closed bosonic SFT ~~the~~ one can argue that we have the best formulation, but this is less clear. For superstring field theories, the question is wide open.

Classical Solutions In this lecture we discuss the topic of classical solutions in open bosonic SFT. The interest in this topic is related to the ancient problem of background independence in string theory. The problem is as follows. The definition of string theory always starts with the Polyakov action for a relativistic string moving in some spacetime + D-brane background. We then quantize this action to obtain the string spectrum, and define the S-matrix by a path integral over all worldsheets weighted by the Polyakov action. In this way we obtain a perturbative definition of string theory around a given background. The problem is that there are an infinite number of backgrounds in which it is possible to quantize the string. Thus we have an infinite number of "versions" of string theory. How do we see that they are all manifestations of the same theory? One way to see that this may be the case is that the spectrum of the string always includes particle states which represent linearized deformations in the choice of background; the most famous example of course is the graviton, which represents linearized deformations in the shape of spacetime. To ~~really~~ show that large deformations of the background are possible, however, requires a much more powerful, nonperturbative formulation of string theory. SFT is one canditate for this formulation. In SFT, the different backgrounds of string theory are represented by different classical solutions to the field equations. This is exactly analogous to how, in general relativity, physical spacetimes are in 1-1 correspondene to solutions of Einstein's equations.

In this lecture we are concerned with open bosonic strings. So the question is whether different D-brane configurations in bosonic string theory, for a given spacetime (or closed string) background, can be described as solutions to the equations of motion of open bosonic SFT. Describing changes in the closed string background either requires closed SFT, or a much better understanding of quantum effects in open SFT. At present both approaches seem very difficult, and we will have enough work understanding shifts in backgrounds in the open string sector.

So let us return to open bosonic SFT of a Dp-brane. If we turn on the gauge field A_μ, we obtain a new background corresponding to a Dp-brane with nontrivial Maxwell field. If we give an expectation value to the massless scalars ϕ_a, we obtain a new background where the Dp-brane has been displaced from its initial position. For example, let x^a $a=1,\dots,25-p$ be coordinates transverse to the Dp-brane, and suppose it is initially located at

$$x^a = 0$$

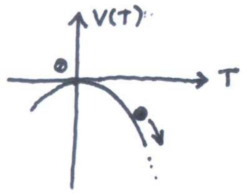
Then after giving a constant expectation value to ϕ_a, the new Dp-brane will be located at

$$x^a = \frac{1}{\sqrt{T_p}}\,\phi_a$$

to leading order in (small) ϕ_a. T_p is the tension of the Dp-brane; with our conventions and normalization of the action, the tension is given by

$$T_p = \frac{1}{2\pi^2}$$

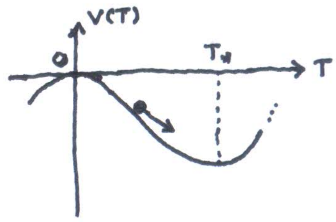
Finally, we can give an expectation value to the tachyon. Since the tachon field is pulled by an "upside down" harmonic oscillator potential $V(T) = -\frac{1}{2}T^2 + \ldots$ it cannot remain constant. Instead, it will roll down the potential with exponentially increasing expectation value. From this we see that the initial



configuration, where all fluctuations on the Dp-brane vanish, is unstable; in otherwords, the Dp-brane itself is unstable. The fate of this instability is unclear since the tachyon expectation value becomes large and the nonlinear terms in the EOM dominate; the perturbative description of the Dp-brane breaks down. This is the problem of "tachyon condensation."

A physical understanding of tachyon condensation ~~emerged~~ in open bosonic SFT emerged from the work of Sen and others in the early 2000's. The upshot is as follows:

1. Given the action of open bosonic SFT, one can define a tachyon effective potential $V(T)$ by integrating out all of the massive fields using the equations of motion. The claim is that this potential has a local minimum at $T = T_*$ representing the endpoint of tachyon condensation. This local minimum represents a highly nontrivial solution to the equations of motion of open bosonic SFT. and is called the "tachyon vacuum."



2. The tachyon vacuum represents a configuration where the Dp-brane has disappeared, and we are left with the closed string background without D-branes or open strings. This has two important consequences:
   a) The shift in the potential between the perturbative vacuum and the tachyon vacuum is given by the Dp-brane tension:
   $$V(0) - V(T_*) = T_p = \frac{1}{2\pi^2}$$
   In other words, the missing energy desity at the tachyon is precisely accounted for by the fact that the Dp-brane has disappeared.
   b) There are no physically nontrivial ~~excitations around~~ linearized excitations around the tachyon vacuum. This reflects the fact that there are no D-branes at the tachyon vacuum, and therefore no open string excitations.

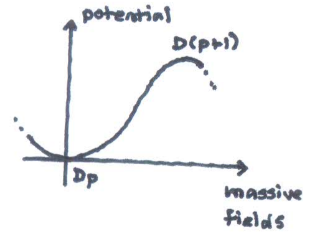
Points a) and b) are specific predictions that can be confirmed by detailed calculations in open bosonic SFT. Similar predictions can also be formulated for unstable D-branes in superstring field theory. Traditionally, these are known as "Sen's conjectures." These days they may more properly be called "Schnabl's theorem" as they have been proven to hold exactly in open bosonic SFT by M. Schnabl, using a remarkable set of analytic techniqes which will be the primary focus of these lectures.

Before Schnabl's result in 2005, the main approach to solving the open SFT equations of motion was "level truncation." The idea is to approximate the action by dropping all fields with masses above a fixed integer L, and solve the resulting EOM numerically. This traditional approach is still valued these days, as it is considered more foolproof than analytic calculations, especially when carried out to high level. Also, in level truncation one can construct backgrounds whose exact CFT description is unknown. So far, an exact construction of a classical solution using Schnabl's analytic methods requires an exact description of the CFT of interest. That being said, we will mostly not discuss level truncation in these lectures.

Besides the tachyon vacuum, other classical solutions in open bosonic SFT which have been widely studied include:

- Marginal deformations These solutions correspond to turning on finite expectation values for the massless fields on the D-brane. Such solutions can describe, for example, translations of a Dp-brane over a finite distance. From the world sheet perspective, such solutions represent deformation of the CFT by a conformal boundary interaction generated by an exactly marginal operator. Such solutions have been constructed approximately in level truncation and analytically soon after Schnabl's result for the tachyon vacuum
- Lump solutions Given a scalar field with a potential containing local maxima and minima, it is possible to construct solitonic solutions in the form of "kinks" or "lumps." The same is true for the tachyon in open bosonic SFT. In this case, the lumps of the tachyon field are believed to describe the formation of lower-dimensional D-branes from the perspective of the fluctuation fields on a higher dimensional D-brane. From the worldsheet perspective, such solutions represent the infrared fixed point of an RG flow given by perturbing the worldsheet by a boundary interaction given by a relevent operator. Lump solutions were constructed in level truncation in the early 2000's, but for a long time analytic methods had a hard time of it. An analytic construction was given only fairly recently, and in a somewhat strange manner.

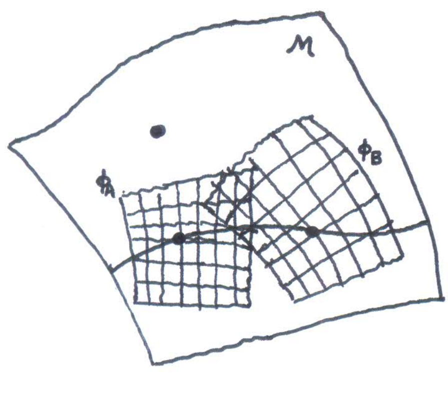
While lumps and marginal deformations cover a large class of interesting solutions, there are many open string backgrounds which cannot be described this way. For example, starting from the fluctuations of a Dp-brane, can we describe the formation of a D(p+1)-brane? If the transverse dimensions are large, the Dp+1 brane will have higher energy than the Dp; ~~therefore, such~~ turning on the tachyon would tend to lower the energy of the system, so this does not seem to be a promising way to create a D(p+1) brane. The natural thing to try is to give expectation values to the massive fields on the Dp, but massive fields feel a potential which tends to pull them back to the original Dp configuration. This corresponds to the fact that deforming the worldsheet action by an irrelevant boundary interaction leaves the theory unchanged in the infrared. However, what we might hope for is that for large expectation values the potential for massive fields may be nontrivial, and there could be stationary points representing higher energy configurations, for example the D(p+1)-brane.



So far, this has been very difficult to see, and has been one of the major outstanding problems in SFT since the formulation of Sen's Conjectures. The problem itself can be formulated as a conjecture, as follows:

<u>Open string background independence conjecture:</u> Open bosonic SFT of a given D-brane system possesess classical solutions describing all D-brane configurations in bosonic string theory on a fixed closed string background. Furthermore, there exists a field redefinition relating open bosonic SFT on one D-brane system to open bosonic SFT on any other D-brane system, as long as the two systems share the same closed string background.

Let's draw some pictures to help visualize the meaning of these conjectures.



Imagine a hypothetical manifold $\mathcal{M}$ representing the space of (off-shell and on-shell) configurations of open bosonic string theory in a given closed string background. Embedded in $\mathcal{M}$ are submanifolds representing the set of consistent open string backgrounds. Each background comes with a natural set of fluctuation fields defining an open bosonic SFT of that background. We can think of these fluctuation fields as defining a local coordinate system on $\mathcal{M}$ in the vicinity of the chosen background. The statement of background independence is that each local coordinate system defined in this way extends to cover all of $\mathcal{M}$. Furthermore, let ϕ_A represent the fluctuation fields around background A, and ϕ_B represent the fluctuation fields around background B. Then there should be a coordinate transformation

$$\phi_A = f_{AB}(\phi_B)$$

which transforms the SFT action of fluctuations ϕ_A into the SFT action of fluctuations ϕ_B:

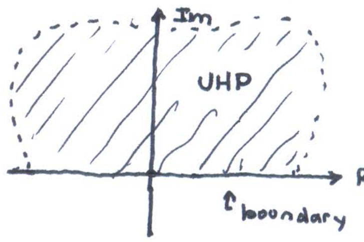
$$S_A[\phi_A] = S_A[f_{AB}(\phi_B)] = C + S_B[\phi_B]$$

where C is an addative constant.

It should be clear that, a priori, it is possible that the fluctuation fields of a given D-brane system can only rearrange themselves into configurations that are sufficiently "close" to the system we started with. If this were true, it would indicate that SFT is intrinsically limited as an approach to nonperturbative string theory. However, recently analytic methods have produced nontrivial ~~analytic~~ evidence that all open string backgrounds are acessable from the SFT on a reference D-brane. Hopefully these lectures will prepare you to understand these developments.

String Field A background of string theory is characterized by a worldsheet conformal field theory — for open strings, specifically a boundary conformal field theory (BCFT). A BCFT is a conformal field theory on a 2-dimensional manifold Σ which is topologically a disk. The boundary of Σ maps to the world lines swept out by the endpoints of an open string attached to a D-brane; the interior of Σ maps to the worldsheet swept out by the interior of the open string in spacetime. Since all disks are conformally equivalent, without loss of generality we can formulate the BCFT on the upper half plane:

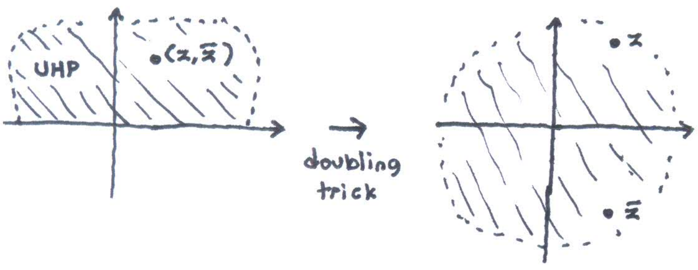
$$\text{UHP}: \quad z \in \mathbb{C} \cup \{\infty\}, \quad \text{Im}\, z \geq 0$$



The real axis is the boundary, and including the point at ∞ is topologically a circle.

A BCFT comes with two kinds of local operators: "bulk" operators $\mathcal{O}(z,\bar{z})$, which can be inserted in the interior of the UHP, and "boundary" operators $\mathcal{O}(x)$, which can be inserted on the real axis. Generally, these two kinds of operators are different; correlation functions of bulk operators $\mathcal{O}(z,\bar{z})$ diverge as $(z,\bar{z})$ approaches the real axis, and correlation functions of boundary operators $\mathcal{O}(x)$ do not have a natural analytic continuation for x not real. An important conceptual point is that the set of local operators of a QFT represents the space of possible local deformations of the theory. In our case, given a bulk operator $\mathcal{O}(z,\bar{z})$ we can deform the worldsheet action by addin a term $\int_{\text{UHP}} d^2z\, \mathcal{O}(z,\bar{z})$; given a boundary operator we can deform the worldsheet action with a boundary coupling $\int_{-\infty}^{\infty} dx\, \mathcal{O}(x)$. Generally, such deformations will not preserve conformal invariance, and therefore will not define a ~~[illegible]~~ string background. You can think of such deformations as creating a hypothetical backgound of string theory which ~~turns out~~ does not satisfy the equations of motion — an "off shell" configuration of string theory. To leading order, conformal invariance requires that $\mathcal{O}(z,\bar{z})$ is a bulk primary field of weight $(1,1)$, and $\mathcal{O}(x)$ is a boundary primary field of weight 1. In this case, the operators generate what is known as a "marginal deformation" of the BCFT. From the SFT perspective, such deformations correspond to giving expectation values for massless fluctuation fields of the background. Note that bulk operators deform the background as seen by the interior of the string; these correspond to deformations of the closed string background. Boundary operators deform the background as seen from the endpoints of the open string; these correspond to deformations of the D-brane system in a fixed closed string background. At least classically, ~~the~~ open bosonic SFT describes the later deformations, but not the former.

A point in the UHP can be described by two real coordinates (x, y), with $y \geq 0$. Equivalently, we can describe this point with a holomorphic and antiholomorphic coordinate $(z,\bar{z})$. It is often useful to consider a single point in the UHP as a pair of points on a purely holomorphic copy of the entire complex plane; z is a point above the real axis, and $\bar{z}$ is a point below the real axis. This is called the "doubling trick." Often we are interested in correlation functions of purely holomorphic or antiholomorphic operators on the UHP.



Consider a holomorphic operator $\phi(z)$, satisfying $\bar{\partial}\phi(z) = 0$. Since a correlation function

$$\langle \phi(z) \ldots \rangle_{UHP}$$

is holomorphic in z, generally it can be analytically continued to the lower half plane with $\mathrm{Im}\, z < 0$. Now we also have a corresponding correlation function with the antiholomorphic operator $\bar{\phi}(\bar{z})$ satisfying $\partial\bar{\phi}(\bar{z}) = 0$. Provided that the boundary conditions on the real axis ~~have been~~ are such that $\phi(x) = \bar{\phi}(x)$, we know that

$$\langle \bar{\phi}(\bar{z}) \ldots \rangle_{UHP} = \langle \phi(z) \ldots \rangle_{UHP}\Big|_{z\to\bar{z}}$$

The left hand side is a correlation function of an antiholomorphic operator on the UHP, and on the right is a correlation function of a holomorphic operator, analytically continued from the UHP to the lower half plane and evaluated at the point $\bar{z}$. In this way we represent the UHP with a holomorphic copy of the entire plane; we cut our work in half by discussing holomorphic operators on the entire plane instead of holomorphic and antiholomorphic operators on the UHP. Still, when working with correlation functions of operators which are niether holomorphic nor antiholomorphic, it may be more convenient to stick to the UHP visualization.

For some purposes it is useful to decribe BCFT in a state/operator formalism. The relation to correlation functions is given by

$$\langle 0| \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)|0\rangle = \langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle_{UHP}$$

$$\infty > |z_1| > \dots > |z_n| > 0$$

On the right hand side is a BCFT correlation function on the UHP of operators $\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)$; on the left hand side, $|0\rangle$ is a special state of the BCFT called the $SL(2,\mathbb{R})$ vacuum, and $\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)$ are interpreted as operators, in the sense of the canonical formalism, acting on the state space $\mathcal{H}$ of the BCFT. The operators on the right hand side are ordered from left to right in sequence of decreasing distance to the origin (radial ordering). The state $|0\rangle$ is called the $SL(2,\mathbb{R})$ vacuum since it is invariant under the $SL(2,\mathbb{R})$ subalgebra of the Virasoro algebra:

$$[L_1, L_0] = L_1 \qquad [L_1, L_{-1}] = 2L_0 \qquad [L_{-1}, L_0] = -L_{-1}$$

where L_n are the Virasoro operators, appearing in the mode expansion of the energy momentum tensor:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \qquad\qquad L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1}\, T(z)$$

The part of the contour in the lower half plane represents $\bar{T}(\bar{z})$, via doubling trick. It is easy to show that

$$L_{-1}|0\rangle = 0, \quad L_0|0\rangle = 0, \quad L_1|0\rangle = 0$$

implying invariance under $SL(2,\mathbb{R})$. More generally, let $\phi(z)$ be a holomorphic primary operator of weight h with mode expansion

$$\phi(z) = \sum_{n\in\mathbb{Z}} \frac{\phi_n}{z^{n+h}} \qquad\qquad \phi_n = \oint_0 \frac{dz}{2\pi i} z^{n+h-1}\, \phi(z)$$

where the index n labeling the modes is chosen so that

$$[L_0, \phi_n] = -n\,\phi_n$$

Then one can show that

$$\phi_n|0\rangle = 0 \qquad n > -h$$

Usually we think of a state as representing the configuration of the quantum system at $t=0$; in radial quantization, this corresponds to $|z|=1$. It is therefore natural to interpret the vacuum expectation value as an inner product between an "out" state and an "in" state

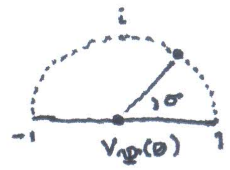
$$\underbrace{\langle 0|\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_i(z_i,\bar{z}_i)}_{\langle out|}\ \underbrace{\mathcal{O}_{i+1}(z_{i+1},\bar{z}_{i+1})\dots\mathcal{O}_n(z_n,\bar{z}_n)|0\rangle}_{|in\rangle} \qquad \begin{array}{l}|z_i|>1\\ |z_{i+1}|<1\end{array}$$

where the out state contains the operators with $|z|>1$ and the in state contains the operators with $|z|<1$. The space of "in" states defines the state space $\mathcal{H}$ of the BCFT.

Given a state $|\Psi\rangle \in \mathcal{H}$, we can define a corresponding boundary operator $V_\Psi(0)$, called the vertex operator, so that

$$|\Psi\rangle = V_\Psi(0)|0\rangle$$

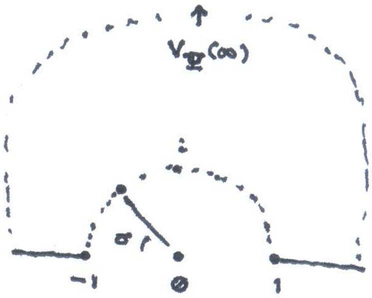
We can visualize this state as a portion of the UHP comprised of the unit disk $|z|<1$ with the vertex operator $V_\Psi(0)$ inserted at the origin. The unit half-circle at the boundary of the half-disk can be parameterized by an angle $\sigma \in [0,\pi]$.



The $SL(2,\mathbb{R})$ vacuum is a half disk with no operator insertion at all; or equivalently, is the half-disk with an insertion of the identity operator. Given a dual state $\langle\Psi| \in \mathcal{H}^*$, we can define a corresponding boundary vertex operator at infinity, so that

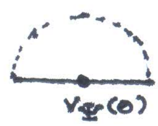
$$\langle\Psi| = \langle 0| V_\Psi(\infty)$$

We can visualize this as a portion of the UHP with the unit half-disk removed, and a vertex operator $V_\Psi(\infty)$ inserted at infinity. The unit half circle at the boundary of this region



can be parameterized by an angle $\sigma \in [0,\pi]$; this time, however it turns out to be natural to measure this angle from the negative real axis. To compute the overlap $\langle\Psi|\Phi\rangle$, ~~we~~ in this visualization, we glue the surface of $\langle\Psi|$ to the surface of $|\Phi\rangle$ along the half circles in such a way that the angle σ on the half-circle of $\langle\Psi|$ is identified with the angle σ' on the half-circle of $|\Phi\rangle$ through $\sigma = \pi - \sigma'$. This effectively patches the unit half disk to its complement in such a way as to form the entire UHP. The overlap is then given by computing the UHP correlation function: $\langle\Psi|\Phi\rangle = \langle V_\Psi(\infty)\, V_\Phi(0)\rangle_{UHP}$.

Conventionally, the vertex operator $V_\Psi(0)$ is regarded as a local operator inserted at the origin. This will be the case if we act a finite number of primary modes ϕ_n and Virasoros on the $SL(2,\mathbb{R})$ vacuum. However, in the applications we consider often $V_\Psi(0)$ will be a nonlocal operator; this will occur, for example, for states carrying operators displaced from the origin of the half-disk. The vertex operator may also contain an infinite number of energy momentum insertions, which have the cumulative effect of deforming the shape of the half-disk. A generic situation is shown to the right. What is true, however, is that we can find a basis



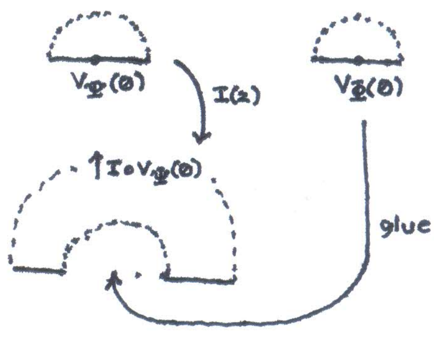
for $\mathcal{H}$ using states whose vertex operators are local at the origin. These are called "Fock space states." (In conventional usage, a Fock space state is also made of vertex operators whose L_0 eigenvalue is bounded from above, and so can be created by acting a finite number of primary mode oscillators and Virasoros on the $SL(2,\mathbb{R})$ vacuum). Nonlocal vertex operators are then only created in a limit where we allow infinite sums of Fock states. To give a toy example which captures the essential point, consider the delta function $\delta(x)$, which has support at $x=0$. All derivatives of the delta function also have support at $x=0$, but the infinite sum

$$\sum_{n=0}^{\infty} \frac{a^n}{n!}\,\delta^{(n)}(x) = \delta(x+a)$$

has support at $x=-a$. Though the vertex operator $V_\Psi(0)$ may be nonlocal, it cannot be arbitrarily nonlocal; it must still be localized within the unit half disk. What this means in practice is that correlators with $V_\Psi(0)$ in the UHP do not encounter divergence as long as other local operators in the correlator do not collide or enter the unit half-disk.

We have characterized the state space $\mathcal{H}$ of a BCFT, but we have not defined an inner product. ~~between states~~ The natural notion is the Belavin, Polyakov, Zamolodchikov inner product, or BPZ inner product, which is defined as follows. Suppose we want to compute the inner product between states $|\Psi\rangle$ and $|\Phi\rangle$, denoted $\langle\Psi|\Phi\rangle$. The idea is to map the unit half-disk of $|\Psi\rangle$ with an $SL(2,\mathbb{R})$ transformation

$$I(z) = -\frac{1}{z}$$



which maps the interior of the half disk to the exterior, and in particular send $V_\Psi(0)$ to a vertex operator $I\circ V_\Psi(0)$ at infinity. In particular, this transformation naturally associates a dual state $\langle\Psi|$ to $|\Psi\rangle$, and the BPZ inner product is given as $\langle\Psi|\Phi\rangle$. Note that the angle σ on the unit half-circle of $|\Psi\rangle$, measured w.r.t the real axis, maps to the angle σ on the unit half circle of $\langle\Psi|$, measured w.r.t the negative real axis. Therefore the BPZ inner produc is defined by gluing σ on the half-circle of $|\Psi\rangle$ to σ' on the half circle of $|\Phi\rangle$ with the identification $\sigma=\pi-\sigma'$. Since correlators on the UHP are $SL(2,\mathbb{R})$ invariant, and $I\circ I = \text{id}$, one can see that

$$\langle\Psi|\Phi\rangle = \langle I\circ V_\Psi(0)\, V_\Phi(0)\rangle_{UHP} = \langle V_\Psi(0)\, I\circ V_\Phi(0)\rangle = \pm\langle\Phi|\Psi\rangle$$

where the minus sign may appear if the vertex operators are anticommuting. It should also be clear that the BPZ inner product is nondegenerate: If $\langle\Psi|\Phi\rangle = 0$

for $\forall\ |\Psi\rangle \in \mathcal{H}$, then $|\Phi\rangle = 0$. This follows from the observation that an operator in the BCFT which has vanishing 2-point function with itself and with any other operator can itself be taken to vanish.

The worldsheet theory of an open bosonic string is a tensor product of "matter" and "ghost" BCFTs:

$$\text{BCFT} = \text{BCFT}_{\text{matter}} \otimes \text{BCFT}_{\text{ghost}}$$

The ghost factor is described by a bc system with central charge -26. It is characterized by anticommuting, holomorphic worldsheet fields $b(z), c(z)$ (with antiholomorphic counterparts which we can account for with doubling trick). satisfying:

$$b(z) = \text{primary of dimension } 2$$

$$c(z) = \text{primary of dimension } -1$$

$$b(x) = \bar{b}(x) \qquad c(x) = \bar{c}(x) \qquad x \in \mathbb{R}$$

$$b(z)\, c(w) = \frac{1}{z-w} + \dots$$

The ghost BCFT is universal; it ~~is common to all~~ takes the same form for all backgrounds of the open bosonic string. The information about the background is contained in the matter BCFT. The only necessary condition on the matter BCFT is that it should have central charge $+26$, so that the total matter + ghost BCFT has central charge $+26-26=0$. For a D-p brane in flat space, the matter BCFT consists of $p+1$ free bosons $X^\mu(z,\bar{z})$ $\mu = 0, \dots, p$ subject to Neumann boundary conditions, and $25-p$ free bosons $X^a(z,\bar{z})$ $a = 1, \dots, 25-p$ subject to Dirichlet boundary conditions:

$$\partial X^\mu(z) = \text{primary of dimension } 1$$

$$\partial X^a(z) = \text{primary of dimension } 1$$

$$\partial X^\mu(x) = \bar{\partial} X^\mu(x) \quad x \in \mathbb{R} \qquad (\text{Neumann b.c.})$$

$$\partial X^a(x) = -\bar{\partial} X^a(x) \quad x \in \mathbb{R} \qquad (\text{Dirichlet b.c})$$

$$\partial X^\mu(z)\, \partial X^\nu(w) = -\frac{1}{2} \frac{\eta^{\mu\nu}}{(z-w)^2} + \dots$$

$$\partial X^a(z)\, \partial X^b(w) = -\frac{1}{2} \frac{\delta^{ab}}{(z-w)^2} + \dots$$

We also have antiholomorphic operators $\bar{\partial} X^\mu(\bar{z})$ and $\bar{\partial} X^a(\bar{z})$, but these can be acounted for with the doubling trick.

The matter/ghost form of the open string BCFT provides additional structure and properties not present for a generic BCFT. Let us list them:

① Since the central charge in the total BCFT vanishes, the energy-momentum tensor

$$T(z) = T^{\text{matter}}(z) + T^{\text{ghost}}(z)$$

is a primary operator of dimension 2. Also, correlation functions are identically conformally invariant:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \rangle_\Sigma = \langle f\circ\mathcal{O}_1\, f\circ\mathcal{O}_2 \ldots \rangle_{f\circ\Sigma}$$

where $\langle\ldots\rangle_\Sigma$ is a correlation function on a disk Σ, $\langle\ldots\rangle_{f\circ\Sigma}$ is a correlation function on a disk $f\circ\Sigma$ defined by mapping Σ with a conformal transformation $f(z)$, and $f\circ\mathcal{O}_1$ is the conformal transform of the operator $\mathcal{O}_1$. For example, if $\mathcal{O}_1$ is a ~~holomorphi~~ primary of weight $(h,\bar{h})$, we have

$$f\circ\mathcal{O}_1(z,\bar{z}) = \left(\frac{df}{dz}\right)^h \left(\frac{\overline{df}}{dz}\right)^{\bar{h}} \mathcal{O}_1(f(z),\overline{f(z)})$$

For a BCFT with nonzero central charge, this property only holds if $f(z)$ is an $SL(2,\mathbb{R})$ mapping, which takes the UHP into itself.

(2) The set of operators in the theory has a $\mathbb{Z}_2$ grading according to whether they are commuting or anticommuting; commuting operators are said to be "Grassmann even" and anticommuting operators "Grassmann odd; the $\mathbb{Z}_2$ grading is called "Grassman parity." In addition, the set of operators carries a $\mathbb{Z}$ grading called "ghost number", which counts the number of c minus the number of b insertions contained in the operator. Hence

$$\partial X^\mu(z) = \text{Grassmann even, ghost \# } 0$$
$$b(z) = \text{Grassmann odd, ghost \# } -1$$
$$c(z) = \text{Grassmann odd, ghost \# } 1$$

For background we are concerned with, b and c are the only anticommuting operators in the worldsheet theory, which leads to an identification between Grassmann parity and ghost #:

$$\text{Grassman parity} = \text{Ghost \# mod } \mathbb{Z}_2$$

We also define the Grassman parity / ghost number of states in the BCFT according to that of the corresponding vertex operators.

(3) The theory comes with an important dimension 1 holomorphic primary field called the BRST current:

$$j_B(z) = cT^{\text{matter}}(z) + :bc\partial c(z): + \frac{3}{2}\partial^2 c(z)$$

There is also an antiholomorphic counterpart $J_B(\bar{z})$ which we deal with using the doubling trick. The integral of $j_B(z)$ around a closed contour defines the BRST operator:

$$Q = \oint_C \frac{dz}{2\pi i}\, j_B(z)$$

The action of the BRST operator on an operator $\mathcal{O}(z,\bar{z})$ is define by inserting the following object in correlation functions

$$Q\cdot\mathcal{O}(z,\bar{z}) = \oint_C \frac{dz'}{2\pi i}\, j_B(z')\,\mathcal{O}(z,\bar{z})$$

where C is a small contour around the point z; If $\mathcal{O}(z,\bar{z})$ is not a boundary operator or holomorphic, C should also contain a small contour around $\bar{z}$ in the lower half plane. The BRST operator is nilpotent:

$$Q^2 = 0$$

is Grassmann odd, and carries ghost number 1. Since Q is defined by a contour integral of a weight 1 primary, it is also conformally invariant in the sense that

$$f\circ\big(Q\cdot\mathcal{O}(z,\bar{z})\big) = Q\cdot\big(f\circ\mathcal{O}(z,\bar{z})\big)$$

We also have the properties

$$Q\cdot b(z) = T(z) \qquad Q\cdot T(z) = 0$$

The last statement follows from the first and $Q^2=0$. Since $T(z)$ is the Noether current associated to conformal symmetry, BRST invariance of $T(z)$ is in fact the same thing as conformal invariance of Q. We can define the BRST operator acting on a state in the BCFT by the action of Q on the corresponding boundary vertex operator, via the state/operator correspondence:

$$Q|\Psi\rangle \longleftrightarrow Q\cdot V_{\Psi}(0)$$

A "physical state" is a BRST invariant state of the BCFT at ghost number 1:

$$\text{Physical state:} \quad Q|\Psi\rangle = 0 \qquad gh\#(|\Psi\rangle) = 1$$

Physical states are defined to be equivalent if they differ by the BRST variation of a state at ghost number 0:

$$\text{Physical equivalence:} \quad |\Psi'\rangle = |\Psi\rangle + Q|\Lambda\rangle \qquad gh\#(|\Lambda\rangle) = 0$$

We say that the space of inequivalent physical states corresponds to the cohomology of Q at ghost number 1. Note that the distinction between "physical" and "unphysical" states does not originate in the BCFT itself; while the matter/ghost form of the open bosonic string BCFT implies the existence of Q and an associated cohomology, the meaning of this cohomology originates elsewhere. Essentially it comes from the fact that the BCFT description of the worldsheet theory arises from gauge fixing the reparameterization and Weyl symmetries of the Polyakov action. The statement that physical states of the worldsheet theory should be gauge invariant translates, after gauge fixing, to the statement that physical states of the open string BCFT should be BRST invariant.

④ The correlation functions of the ghost factor of the BCFT are nonvanishing only if the ghost number of all operator insertions adds up to 3. Using Wick's theorem, all correlation functions can be reduced to a correlator with 3 c-ghost insertions

$$\langle c(z_1)\, c(z_2)\, c(z_3)\rangle^{ghost}_{UHP} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

Correlation functions with $\bar{c}(\bar{z})$ are given by the doubling trick.

This completes our review of the worldsheet theory of an open bosonic string.

We now want to pass from the first quantized worldsheet theory to the classical field theory of fluctuations of a D-brane. The first step is to specify the nature of the fluctuation fields. It is convenient to consider the set of fluctuation fields together as a single object, called the "string field." So the first step is to define the string field. We make the following claim:

Claim: A string field is an element of the vector space $\mathcal{H}$ of quantum states of the open bosonic string BCFT.

At first this statement seems a little strange. An element of $\mathcal{H}$ is a quantum state of a 2-dimensional QFT — it is a quantum mechanical object. But now we are claiming that it represents a configuration of fluctuation fields of a classical field theory representing the dynamics of a D-brane configuration in ~~open~~ bosonic string theory. There are a couple of ways to justify this. The first is that it follows a general rule about the correspondence between first quantized theories and classical field theory: Namely, the wavefunction of a first quantized theory can be interpreted as the ~~classical~~ field of an equivalent classical field theory. Since this fact may be unfamiliar, let us give an example to show that it makes sense. Consider a free, nonrelativistic quantum particle characterized by the quantum state $|\Psi\rangle$. The state $|\Psi\rangle$ evolves in time according to the Schrodinger equation:

$$i\frac{\partial}{\partial t}|\Psi\rangle = \frac{p^2}{2m}|\Psi\rangle$$

The wavefunction is given by expressing $|\Psi\rangle$ in the position basis:

$$\psi(x,t) = \langle x|\Psi(t)\rangle$$

where the Schrodinger equation reads

$$i\frac{\partial}{\partial t}\psi(x,t) = \frac{-1}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

Now if we forget where this equation came from, there is nothing a priori contradictory about interpreting $\psi(x,t)$ as a classical complex scalar field subject to a nonrelativistic wave equation. In fact, the wave equation can be derived as equations of motion of a classical field theory action:

$$S = \int dx\, dt \left[i\psi^*(x,t)\frac{\partial}{\partial t}\psi(x,t) - \frac{1}{2m}\frac{\partial}{\partial x}\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)\right]$$

From this point of view $\psi(x,t)$ is a complex scalar field, and there is no justification for interpreting it as a probability amplitude. However, this classical field theory is equivalent to the first quantized theory in the following sense: If we start from the action for $\psi(x,t)$ and follow the usual recipe for canonical quantization in QFT, we find a Fock space of multiparticle states given by acting creation operators $\hat{\psi}^{+}(x,t)$ on a vacuum state. The QFT Hamiltonian implies that the wave function for each particle inside the multiparticle state will evolve according to the Schrodinger equation for a free, nonrelativistic particle. So we are back to where we started, only we have a formalism describing many and variable number of nonrelativistic particles. Applying the analogous procedure to the first quantized states of an open bosonic string gives a field theory formalism capable of describing many and variable number of quantum open bosonic string states.

There is a second, perhaps more physical justification for the definition of the string field. From the state-operator mapping of BCFT, we know that every state $|\Psi\rangle \in \mathcal{H}$ has a corresponding boundary vertex operator $V_\Psi(0)$. As mentioned before, the set of boundary operators corresponds to the set of possible boundary deformations of the BCFT. This, in turn, corresponds to the space of deformations (or fluctuations) of the D-brane system defining the open bosonic string BCFT.

It is important to distinguish between a generic string field, and the particular kind of string field which enters the action and equations of motion — the "dynamical string field." In a similar way, in gauge theories we have Lie algebra valued differential forms — including the 2-form field strength — but the dynamical variable of the theory is a 1-form — the gauge potential. The dynamical string field in open bosonic SFT is the same kind of state in $\mathcal{H}$ where we impose the physical state condition, namely, it is Grassmann odd and ghost number 1. Just as the Schrodinger equation of the nonrelativistic particle is interpreted as a field equation for a complex scalar, the physical state condition is interpreted as a linearized equation for the string field:

$$Q\Psi = 0 \qquad gh\#(\Psi) = 1, \quad \Psi = \text{Grassmann odd}$$

and the equivalence of physical states is interpreted as a linearized gauge invariance:

$$\Psi' = \Psi + Q\Lambda \qquad gh\#\Lambda = 0. \quad \Lambda = \text{Grassmann even}$$

Note that I am dropping the ket around Ψ; this is to emphasize that Ψ is a classical field; we will not try to interpret it as a probability amplitude.

To see that this makes sense as a field equation, it is helpful to give a more concrete presentation of the string field ~~using eigenstates of L_0~~ as an expansion in eigenstates of L_0. Let us do this for the Dp-brane. The mode expansions of the bc ghosts and free scalars are given by

$$b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}} \qquad b_n|0\rangle = 0 \qquad n \geq -1$$

$$c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}} \qquad c_n|0\rangle = 0 \qquad n \geq 2$$

$$\partial X^\mu(z) = \sum_{n\in\mathbb{Z}} \frac{\alpha_n^\mu}{z^{n+1}} \frac{-i}{\sqrt{2}} \qquad \alpha_n^\mu|0\rangle = 0 \qquad n \geq 0$$

$$\partial X^a(z) = \sum_{n\in\mathbb{Z}} \frac{\alpha_n^a}{z^{n+1}} \frac{-i}{\sqrt{2}} \qquad \alpha_n^a|0\rangle = 0 \qquad n \geq 0$$

The zeroth oscillator of ∂X^μ is related to the momentum of the string attached to the Dp:

$$\alpha_0^\mu = \sqrt{2}\, p^\mu$$

and the zeroth oscillator of ∂X^a vanishes — $\alpha_0^a = 0$ — since the open string does not carry a conserved momentum orthogonal to the brane. Since $\alpha_0^\mu|0\rangle = 0$, the $SL(2,\mathbb{R})$ vacuum carries zero momentum. To describe fields with nontrivial spacetime dependence, we need to inject some momentum in the vacuum; this can be done by "translating" in momentum space using the position zero mode x_0^μ satisfying $[x_0^\mu, p^\nu] = i\eta^{\mu\nu}$:

$$|k\rangle = e^{ik\cdot x_0}|0\rangle = e^{ik\cdot X}(0)|0\rangle$$

This state is created by acting a boundary plane-wave vertex operator $e^{ik\cdot X}(0)$ at the origin of the half-disk representing $|k\rangle$. We can then represent the string field as a sum of states ~~containing an ever la~~ created by acting an ever larger number of mode oscillators on $|k\rangle$. Arranging in sequence of increasing L_0 eigenvalue for fixed k, and recalling that the dynamical string field carries ghost number 1, we then express the string field in the form:

$$\Psi = \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \Big[\underbrace{T(k)\, c_1}_{L_0 = k^2-1} + \underbrace{A_\mu(k)\alpha_{-1}^\mu c_1 + \phi_a(k)\alpha_{-1}^a c_1 + \beta(k) c_0}_{L_0 = k^2} + \cdots \Big] |k\rangle$$

The coefficient functions $T(k), \ldots$ are an infinite list of ordinary spacetime fields — the fluctuation fields of the Dp-brane — expressed in momentum space. As you can probably anticipate, $T(x)$ is the tachyon of the Dp-brane, $A_\mu(x)$ is the Maxwell potential, $\phi_a(x)$ are the massless scalars representing transverse displacement of the Dp, and we will see the role of $\beta(x)$ in a moment. Plugging this form of Ψ into $Q\Psi = 0$ implies a set of EOM for the coefficient fields:

$$(\Box + 1)T = 0$$
$$\Box A_\mu - \partial_\mu \beta = 0$$
$$\Box \phi_a = 0$$
$$\beta - \partial^\mu A_\mu = 0$$
$$\vdots$$

The linearized gauge transformation of Ψ translates to

$$T = \text{invariant}$$
$$A'_\mu = A_\mu + \partial_\mu \lambda$$
$$\phi_a = \text{invariant}$$
$$\beta' = \beta + \Box \lambda$$
$$\vdots$$

[Excersise 1: Derive these equations by computing $Q\Psi = 0$ and $\Psi' = \Psi + Q\Lambda$]

The gauge field has the expected Maxwell gauge invariance. The field β does not carry any physical degrees of freedom — it is an auxilliary field — since we can eliminate it from the theory by solving its equation of motion algebraically: we just substitute β with $\partial^\mu A_\mu$. The EOM for A_μ becomes

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = \partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) = \partial^\nu F_{\nu\mu} = 0$$

which is Maxwell's equations. The L_0 expansion of Ψ contains an infinite number of auxilliary fields like β which do not carry physical degrees of freedom. In a sense they are not physically necessary, but any attempt to integrate them out makes the theory look formidably complicated. In any case, it is clear that a ghost number 1 state in $\mathcal{H}$ automatically incorporates the infinite tower of fluctuation fields of the reference D-brane.

There is another representation of the string field which plays a very important conceptual role in understanding the structure of the theory — the position space, or Schrodinger representation. In quantum mechanics we turn a quantum state $|\psi(t)\rangle$ into a Schrodinger wavefunction by contracting with a position eigenstate: $\psi(x,t) = \langle x|\psi(t)\rangle$. Is there something similar for the string field? For definiteness, let us concentrate on a single free boson subject to Neumann boundary conditions — the Dirichlet case is similar, and even the bc case if we allow for Grassmann odd coordinates. Consider the mode expansion of $X(z,\bar z)$:

$$X(z,\bar z) = x_0 - p \ln|z|^2 + \frac{i}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} \frac{\alpha_n}{n}\left(\frac{1}{z^n} + \frac{1}{\bar z^n}\right)$$

If we restrict to $z = e^{i\sigma}$ on the unit half-circle (corresponding to $t=0$ from the point of view of radial quantization) the mode expansion simplifies to

$$X(\sigma) = X_0 + 2\sum_{n=1}^{\infty} X_n \cos n\sigma \qquad X_n = \frac{i}{\sqrt{2}}\,\frac{\alpha_n - \alpha_{-n}}{n}$$

The appearance of cosines reflects the fact that the boundary conditions are Neumann. Now we can imagine finding a state in $\mathcal{H}$ which serves as an eigenstate for each position mode operator X_n:

$$X_n\,|x(\sigma)\rangle = x_n\,|x(\sigma)\rangle$$

where the eigenvalues x_n are related to the curve $x(\sigma)$ in the same way as X_n is related to $X(\sigma)$. Computing the overlap

$$\Psi[x(\sigma)] = \langle x(\sigma)|\Psi\rangle$$

gives a complex scalar field which depends on a curve in spacetime $x(\sigma)$. In a sense this is what we would expect; ~~that~~ an ordinary field depends on a point x in spacetime, representing a possible location of a point particle; a string field should depend on a curve in spacetime, representing a possible configuration of the string. Thus it may seem somewhat unnatural to view the string field as an infinite collection of ordinary fields; this representation is an ~~some~~ artificial consequence of the fact that a free string is indistinguishable from an infinite collection of particle species. At the interacting level the true nature of the string should emmerge, and indeed in the Schrodinger representation the interactions are simple to formulate, while in terms of the infinite collection of ordinary fields they are almost inscrutably complicated. The Schrodinger representation is also useful in giving a concrete interpretation to the geometrical picture of a state as the half-disk carrying a vertex operator. The BPZ inner product of states $|\Psi\rangle$ and $|\Phi\rangle$ can be computed as an UHP correlation function

$$\langle\Psi|\Phi\rangle = \langle I\circ V_{\Psi}(0)\; V_{\Phi}(0)\rangle_{UHP}$$

and the correlation function itself can be computed as a path integral ~~over~~ over the target space coordinate $X(z,\bar z)$ for each point in the UHP:

$$\langle\Psi|\Phi\rangle = \int [dX(z,\bar z)]\; I\circ V_{\Psi}(0)\; V_{\Phi}(0)\, e^{-S}$$

We now factorize the integration into 3 components: First over the region $|z|>1$ subject to the boundary condition $X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)$ with $X(\sigma)$ a fixed curve; Second over the region $|z|<1$ subject to the boundary condition $X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)$; and third a path integral over $X(\sigma)$ on the curve $|z|=1$ itself:

$$\langle \Psi | \Phi \rangle = \int [dX(\sigma)] \Big(\int_{\substack{|z|>1 \\ X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)}} [dX(z,\bar z)]\, I \circ V_\Psi(0)\, e^{-S} \Big) \Big(\int_{\substack{|z|<1 \\ X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)}} [dX(z,\bar z)]\, V_\Phi(0)\, e^{-S} \Big)$$

Note that ~~we can~~ $I \circ V_\Psi(0)$ only depends on the integration variables $X(z,\bar z)$ with $|z|>1$ and $V_\Phi(0)$ only depends on the integration variables $X(z,\bar z)$ with $|z|<1$. This is related to a point we made earlier: $V_\Phi(0)$, if not necessarily a completely local operator, must be localized at least within the unit half disk. Now the portion of the path integral over $|z|>1$ can also be written as a path integral over $|z|<1$ by making a conformal transformation $I(z) = -\frac{1}{z}$. Accounting for the transformation of the boundary conditions

$$\langle \Psi | \Phi \rangle = \int [dX(\sigma)] \Big(\int_{\substack{|z|<1 \\ X(z,\bar z)|_{e^{i\sigma}} = X(\pi-\sigma)}} [dX(z,\bar z)]\, V_\Psi(0)\, e^{-S} \Big) \Big(\int_{\substack{|z|<1 \\ X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)}} [dX(z,\bar z)]\, V_\Phi(0)\, e^{-S} \Big)$$

It is clear that this can be interpreted as an inner product between Schrodinger functionals:

$$\langle \Psi | \Phi \rangle = \int [dX(\sigma)]\, \Psi[X(\pi-\sigma)]\, \Phi[X(\sigma)]$$

with

$$\Psi[X(\sigma)] = \int_{\substack{|z|<1 \\ X(z,\bar z)|_{e^{i\sigma}} = X(\sigma)}} [dX(z,\bar z)]\, V_\Psi(0)\, e^{-S}$$

Thus the Schrodinger functional can be evaluated as a path integral, with vertex operator insertion, on the unit half disk with ~~some~~ fixed boundary conditions at $|z|=1$ corresponding to the configuration of a string.

To make this concrete, let us use the path integral to evaluate the Schrodinger functional of the $SL(2,\mathbb{R})$ vacuum. A similar calculation (for the closed string) is discussed in Polchinski Ch.2. We write the functional as

$$\Omega[x(\sigma)] = \langle x(\sigma) | 0 \rangle$$

since it is somewhat awkward to write $0[x(\sigma)]$. The vertex operator in this case is the identity, so we only need to evaluate

$$\Omega[x(\sigma)] = \int_{\substack{|z|<1 \\ X(z,\bar z)|_{e^{i\sigma}} = x(\sigma)}} [dX(z,\bar z)]\, e^{-S}$$

The trick is to make a change of variables inside the path integral

$$X(z,\bar z) = x(z,\bar z) + Y(z,\bar z)$$

where $x(z,\bar z)$ is a solution to the Laplace equation on the unit half disk

$$\partial\bar\partial x(z,\bar z) = 0$$

subject to the boundary conditions

~~$x(z,\bar z) =$~~

$$x(z,\bar z)\big|_{z=e^{i\sigma}} = x(\sigma) \qquad \partial x(z,\bar z)\big|_{\mathrm{Im}\, z=0} = \bar\partial x(z,\bar z)\big|_{\mathrm{Im}(z)=0}$$

The first says $x(z,\bar z)$ corresponds to the argument of the wavefunctional at $|z|=1$, and the second imposes Neumann boundary conditions on the real axis. Since $X(z,\bar z)$ is also fixed to $x(\sigma)$ at $|z|=1$, we learn that $Y(z,\bar z)$ satisfies a Dirichlet boundary condition at $|z|=1$

$$Y(z,\bar z)\big|_{z=e^{i\sigma}} = 0$$

Since $x(z,\bar z)$ is a completely fixed function, the new integration variable in the path integral is $Y(z,\bar z)$, and we have

$$\Omega[x(\sigma)] = \int_{\substack{|z|<1 \\ Y(z,\bar z)|_{e^{i\sigma}}=0}} [dY(z,\bar z)]\, e^{-S[x(z,\bar z)+Y(z,\bar z)]}$$

The worldsheet action for a free boson is

$$S = \frac{1}{2\pi}\int_{\substack{\text{half} \\ \text{disk}}} d^2z\ \partial X(z,\bar z)\,\bar\partial X(z,\bar z)$$

in $\alpha'=1$ conventions and $d^2z \equiv 2dx\,dy$ when $z=x+iy$. Plugging in our ansatz, term linear in Y drop out since $x(z,\bar z)$ satisfies the EOM. We then find

$$S[x(z,\bar z)+Y(z,\bar z)] = S[x(z,\bar z)] + S[Y(z,\bar z)]$$

and

$$\Omega[x(\sigma)] = \int_{\substack{|z|\le 1 \\ Y(z,\bar z)|_{e^{i\sigma}}=0}} [dY(z,\bar z)]\, e^{-S[x(z,\bar z)]-S[Y(z,\bar z)]}$$

$$= \mathcal{N}\, e^{-S[x(z,\bar z)]}$$

where $\mathcal{N}$ is a normalization given after evaluating the integral ove $Y(z,\bar z)$, and is independent of $x(\sigma)$. We will not attempt to fix its value. All that is left is to compute the on-shell worldsheet action. If

$$x(\sigma) = x_0 + 2\sum_{n=1}^{\infty} x_n \cos n\sigma$$

then Laplace's equation and the boundary conditions determine $x(z,\bar z)$ to be

$$x(z,\bar z) = x_0 + \sum_{n=1}^{\infty} x_n (z^n + \bar z^n)$$

Evaluating $S[x(z,\bar z)]$ then gives

$$\Omega[x(\sigma)] = \mathcal{N}\exp\Big[-\frac{1}{2}\sum_{n=1}^{\infty} n\, x_n^2\Big]$$

[Excercise 2: Do this calculation]

This is a gaussian in the space of string position modes; note that the position zero mode does not appear, consistent with our earlier observation that the $SL(2,\mathbb{R})$ vacuum is a zero momentum state. The gaussian has ~~maximum~~ a maximum when the curve $x(\sigma)$ shrinks to a point; $x(\sigma) = x_0$. The factor of n in the sum over modes says that highly irregular curves tend to be suppressed — but nevertheless $\Omega[x(\sigma)]$ does have support on unbounded string configurations. ~~however~~ The physical significance of this is not totally clear, to my knowledge.

To get the complete $SL(2,\mathbb{R})$ vacuum functional of the open bosonic string on a Dp-brane, we must include the contributions from the remaining 25 free bosons and from the ghosts. We will not discuss this further because, it turns out, the Schrodinger representation is awkward for explicit calculations. ~~In practice,~~ The main point is to give some intuition about what it means to associate a state with a region of the complex plane with operator insertions — for example, the unit half disk with vertex operator. ~~the Schrodinger~~ This is an idea we will use frequently, and the Schrodinger representation makes its meaning clear.

Witten's open bosonic SFT We are now ready to discuss Witten's open bosonic SFT. The task is to find a nonlinear extension of the linearized EOM $Q\Psi = 0$ and define an appropriate action principle. First, we note an analogy between string fields and gauge fields formulated in the language of differential forms:

rank of a form $\longrightarrow$ ghost number

exterior derivative $\longrightarrow$ BRST operator Q

gauge field $\longrightarrow$ dynamical string field Ψ

This analogy suggests a nonlinear gauge invariance of the string field:

$$\Psi' = \Psi + Q\Lambda + [\Psi, \Lambda]$$

where Λ is an infinitessimal gauge parameter. The product of string fields Ψ and Λ is defined using Witten's open string star product, which is the crux of the whole matter. For the moment let us assume we have defined this product and proceed. There is only one gauge covariant, nonlinear extension of the EOM:

$$Q\Psi + \Psi^2 = 0$$

These resemble the equations of motion in Chern-Simons theory. We can then write the action

$$S = -\frac{1}{2}\mathrm{Tr}(\Psi Q \Psi) - \frac{1}{3}\mathrm{Tr}(\Psi^3)$$

for an appropriately defined trace operation. Since Ψ is Grassmann odd and ghost # 1, the suitability of this action relies on the following "axioms:"

① Grading: $gh\#(QA) = gh\#(A) + 1$

$gh\#(AB) = gh\#(A) + gh\#(B)$

$Tr(A) \neq 0 \rightarrow gh\#(A) = 3$

Analoguos properties hold for Grassmann parity, mod $\mathbb{Z}_2$

② Nilpotency: $Q^2 = 0$

③ Integration by parts: $Tr[QA] = 0$

④ Derivation property: $Q(AB) = (QA)B + (-1)^A A(QB)$

⑤ Cyclicity: $Tr(AB) = (-1)^{AB}\, Tr(BA)$

⑥ Associativity: $A(BC) = (AB)C$

We use a notation where the symbol for a string field in the exponent of -1 denotes the Grassman parity of that string field. So, in $(-1)^{AB}$ the Grassmann parity of A should be multiplied by the Grassmann parity of B (note, in particular, that AB does not indicate the grassmann parity of the star product of A and B, which would be A+B). These properties imply that the state space $\mathcal{H}$ of the BCFT has been endowed with the structure of a cyclic, graded differential associative algebra — the same structure as matrix-valued forms on a 3-manifold.

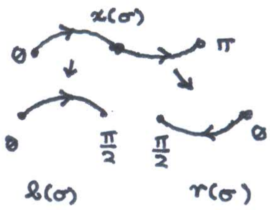
Let us try to understand how to define the product and trace. The product is associative, and all associative products are, in some way or another, matrix products. The string field in the Schrodinger representation is a functional of a curve

$$\Psi[x(\sigma)]$$

and it is natural to interpret the curve as representing matrix indices, in some sense. However, a matrix should have two indices, and there is only one curve $x(\sigma)$. We can deal with this by regarding the full curve as a pair of half-curves

$$\ell(\sigma) = x(\sigma) \qquad \sigma \in [0, \tfrac{\pi}{2}]$$

$$r(\sigma) = x(\pi - \sigma) \qquad \sigma \in [0, \tfrac{\pi}{2}]$$



$\ell(\sigma)$ is called the "left half" of the string, and $r(\sigma)$ is called the "right half." The left and right halves join at a common point:

$$\ell(\tfrac{\pi}{2}) = r(\tfrac{\pi}{2}) = x(\tfrac{\pi}{2})$$

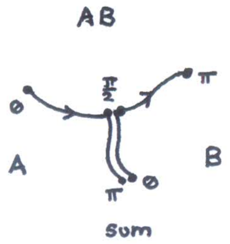
called the "midpoint." Thus we regard the string field as a functional of the left and right halves of the string:

$$\Psi \rightarrow \Psi[x(\sigma)] \rightarrow \Psi[\ell(\sigma), r(\sigma)]$$

and we have a matrix. The associative product of string fields may be defined

$$AB[\ell(\sigma), r(\sigma)] = \int [dw(\sigma)]\, A[\ell(\sigma), w(\sigma)]\, B[w(\sigma), r(\sigma)]$$

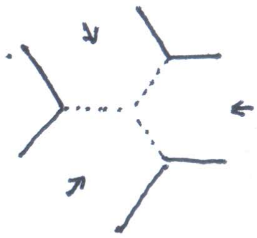
This is a functional integral version of matrix multiplication. In words,



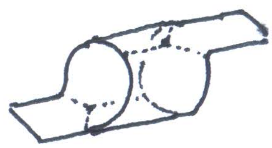
you identify the right half curve in A with the left half curve in B, and then sum over the common half-curve to derive AB. In a similar way, we can define the trace

$$\mathrm{Tr}[A] = \int [dw(\sigma)]\, A[w(\sigma), w(\sigma)]$$

The product and trace define a cubic vertex $\mathrm{Tr}[\Psi^3]$. In Feynman diagams, the cubic vertex can be visualized as a process where three incoming strings collide and join along their halves. Since the action is cubic, gluing propagators together with this vertex generates all Feynman diagrams needed for the computation of open string amplitudes. Proving that these Feynman diagrams compute open string scattering amplitudes in the form we are used to thinking about — as integrals of differential forms over the moduli spaces of Riemann surfaces — is fairly nontrivial. However, the end result is perhaps not surprising: If it wasn't the case, that would imply that we have two consistent and inequivalent theories of interacting open bosonic strings.



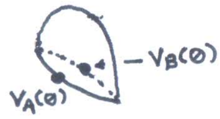
Note that, at the quantum level, open string Feynman diagrams will produce closed strings as intermediate states. This can be seen, for example, in the nonplanar 1-loop 2-point function. The corner of the moduli space where the open string propagators in the loop shrink to zero length (the UV from the open string perspective) ~~to equivalen~~ can be interpreted as the corner of moduli space where a tube of worldsheet becomes infinitely long, and the closed string states inside the tube must be on-shell. In this sense, quantum open bosonic string field theory is expected to encode closed string physics. How this is precisely accomplished is not well-understood, and remains one of the most important outstanding questions in SFT.



The Schrodinger representation captures the essential definition of the product and trace, but is not very practical for calculations. We would like to define the action in terms of BCFT correlation functions. To do this, we use the relation between the Schrodinger functional and the path integral over a half-disk with vertex operator. Suppose we want to compute

$$\mathrm{Tr}(AB)$$

The product AB instructs us to glue the right ~~half [illegible] of the~~ portion of the half circle bounding the half-disk of A to the left portion of the half-circle bounding the half-disk of B; the trace glues the left portion of the half-circle bounding the half disk of A to the right portion of the half circle bounding the half-disk of B. This defines a correlation function of the vertex operators $V_A(0)$ and $V_B(0)$ on a funny-looking "pita" shaped surface. To make this look more familiar, we can apply a conformal transformation $I(z) = -\frac{1}{z}$ to the half-disk of A before gluing to the half disk of B. It is clear that this defines a correlation function on the UHP



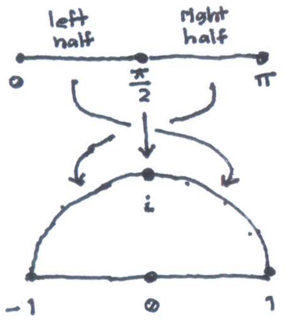
$$\langle I \circ V_A(0)\, V_B(0) \rangle$$

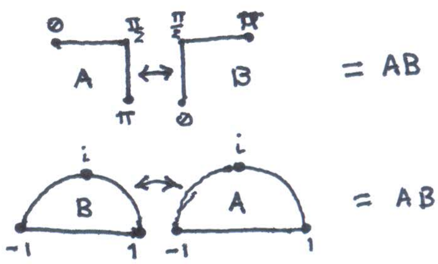
Therefore the trace of a product of string fields is identical to the BPZ inner product of the string fields

$$\mathrm{Tr}(AB) = \langle A | B \rangle$$

Note that symmetry of the BPZ inner product is equivalent to cyclicity of the trace

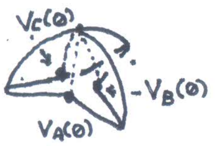
Let us mention a small visual problem which raises an important question of conventions. You might notice that the left half of the string $\sigma \in [0, \frac{\pi}{2}]$ maps to the points $\mathrm{Re}[e^{i\sigma}] > 0$ on the unit half circle, while the right half of the string $\sigma \in [\frac{\pi}{2}, \pi]$ maps to the points $\mathrm{Re}[e^{i\sigma}] < 0$. Thus it seems that the left half of the string sits on the right half of the unit half-circle, and vice-versa for the right half of the string. This leads to possible confusions in ordering when multiplying two states: when we glue the right half of the string of A to the left half of the string of B in computing AB, we must glue the corresponding half-disks in the opposite way. We avoid this potential confusion by reflecting the usual picture of the complex plane through the imaginary axis; that is, we draw the real axis so that numbers increase towards the left. In this visualization the standard orientation of complex contours is clockwise. An alternative resolution to this problem is to keep the standard picture of the complex plane, but to define a similar, but different product between string fields, where the left index of the "matrix" corresponds to the right half of the string, and the right index to the left half of the string. The theories defined with these two different multiplication conventions are related by a linear field redefinition, which ammounts to a reversal of the parameterization





of the open string: $\sigma \to \pi - \sigma$. This is analogous to a matrix transpose. Both product conventions appear in the literature, and you can decide which one you like best.

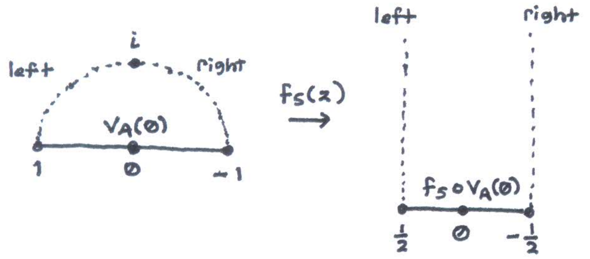
Next we want to express the cubic vertex $\mathrm{Tr}(ABC)$ in terms of correlation functions. Gluing the half-string segments appropriately gives a correlation function of three vertex operators on the surface shown to the left.



This time, it is not so easy to see how to transform this into a correlation function on the upper half plane. Part of the issue is that the unit half disk has a curved boundary, and gluing it to other curved boundaries ~~is awkward~~ gives a somewhat awkward-looking surface. For this reason it is useful to make a conformal transformation of the unit half-disk into a region whose boundaries are straight lines. This can be achieved using the so-called sliver coordinate map:

$$f_S(z) = \frac{2}{\pi}\tan^{-1} z$$

This maps the unit-half disk into a semi-infinite strip of world sheet;



The line segment $[-1,1]$ on the real axis, where we impose boundary conditions appropriate to the BCFT, is mapped to the line segment $[-\frac{1}{2},\frac{1}{2}]$. The curve representing the left half of the string $e^{i\sigma}$, $\mathrm{Re}(e^{i\sigma})>0$ is mapped to a vertical line $\frac{1}{2}+iy$ which intersects the real axis at $\frac{1}{2}$; the curve representing the right half of the string $e^{i\sigma}$, $\mathrm{Re}(e^{i\sigma})<0$ is mapped to the vertical line $-\frac{1}{2}+iy$ intersecting the real axis at $-\frac{1}{2}$. Note that the worldsheet path integral on the half-disk and on the semi infinite strip define the same Schrodinger functional provided that the boundary conditions on the unit half-circle correspond to those on the vertical lines. For example, if a free boson x takes the value $x(\sigma)$ at the angle $\sigma \in [0,\frac{\pi}{2}]$ on the half circle, it should take the same value $x(y)$ on the vertical line $\frac{1}{2}+iy$ with y related to σ throgh

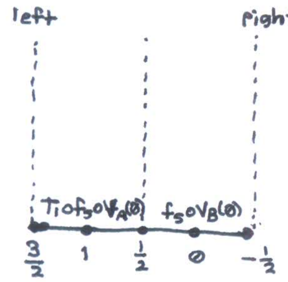
$$\frac{1}{2}+iy = f_S(e^{i\sigma}) \quad \longrightarrow \quad y = \frac{1}{4}\tanh^{-1}\sin\sigma$$

Similarly if x takes the value $x(\sigma)$ at angles $\sigma \in [\frac{\pi}{2},\pi]$, it should take the same value $x(y)$ on the vertical line $-\frac{1}{2}+iy$ with

$$-\frac{1}{2}+iy = f_S(e^{i(\pi-\sigma)}) \quad \to \quad y = \frac{1}{4}\tanh^{-1}\sin(\pi-\sigma)$$

The coordinate y on the vertical line takes values from 0 to infinity, with 0 describing the endpoint of the open string and ∞ the midpoint. We can also use the doubling trick to replace the semi-infinite strip with a holomorphic copy of the full infinite strip $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$, in which case y takes values from $-\infty$ to ∞. For the small minority who may appreciate this comment, it is interesting to note that y is the position variable conjugate to the eigenvalue κ of the midpoint-preserving reparameterization generator $L_1 + L_{-1} = K_1$. The spectrum of K_1 plays an important role in the diagonalization of the Neumann coefficients which characterize the oscillator representation of the cubic vertex.

With this visualization it is easy to glue the surfaces of different Schrodinger functionals ~~together~~ when computing products of string fields. To find the product AB, we glue the right edge of the strip of A to the left edge of the strip of B; this creates a semi-infinite strip of width 2 carrying operator insertions

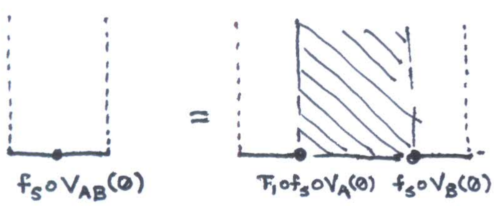


$$T_1 \circ f_S \circ V_A(0)\; f_S \circ V_B(0)$$

where T_a is a translation map

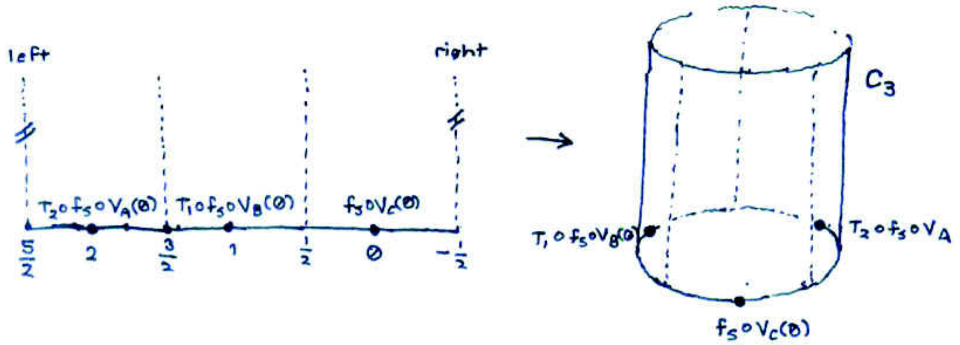
$$T_a(z) = z + a$$

Imposing the appropriate boundary conditions on the left and right edges of the doubled strip, and performing the worldsheet path integral in the interior, defines the Schrodinger functional of the product AB. Note that the vertical line where A and B have been glued is on the interior of the doubled strip. The worldsheet variables on this line are summed over, as are worldsheet variables at other interior points. This sum is precisely the sum over matrix indices. It is also worth noting that the vertex operator of the state AB is very nonlocal; it effectively inserts a whole new piece of surface between 1 and 0, in addition to the vertex operators at the edge of this region. This shows that the product of string fields is not closed on Fock space states in $\mathcal{H}$.



Now consider the 3-string vertex $\mathrm{Tr}(ABC)$. To compute this, we place the strips of A, B, C side by side to form a strip of width 3 with insertions

$$T_2 \circ f_S \circ V_A(0)\; T_1 \circ f_S \circ V_B(0)\; f_S \circ V_C(0)$$



The trace then glues the left and right edges of this strip to form a cylinder of circumference 3. The cylinder is not yet the UHP, but a cylinder of circumference L can be mapped to the UHP using

$$F_L(z) = \tan\frac{2}{L}z$$

This means that the correlation functions are related by

$$\langle \ldots \rangle_{C_L} = \langle F_L\circ(\ldots)\rangle_{UHP}$$

This gives an expression for the cubic vertex in terms of a correlation function:

$$Tr(ABC) = \langle T_2\circ f_S\circ V_A(0)\, T_1\circ f_S\circ V_B(0)\; f_S\circ V_C(0)\rangle_{C_3}$$

which can be mapped to the UHP. In a similar way, we can write the 2-string vertex (i.e. BPZ inner product) as a correlation function on a cylinder of circumference 2:

$$Tr(AB) = \langle A|B\rangle = \langle T_1\circ f_S\circ V_A(0)\; f_S\circ V_B(0)\rangle_{C_2}$$

or the 1-string vertex on a cylinder of circumference 1:

$$Tr(A) = \langle f_S\circ V_A(0)\rangle_{C_1}$$

This generalizes in the obvious way for the trace of a product of any number of string fields. Above we have implicitly defined a coordinate system on the cylinder where the vertex operator of the final string field sits at $z=0$. However, we are free to shift the origin of the coordinates somewhere else; by rotational invariance of the cylinder, correlation functions do not depend on the choice of origin.

Our definition of the product of string fields AB is still not fully concrete, since it seems we have to evaluate the Schrodinger functional of the strip of width 2 carrying the insertions $T_1\circ f_S\circ V_A(0) f_S\circ V_B(0)$. This can be remedied as follows. Consider a basis of states $|\phi_i\rangle$ for $\mathcal{H}$, for example a Fock space basis of L_0 eigenstates. Following the Gram-Schmidt procedure, we can construct a dual basis of states $|\phi^i\rangle$ with the property that

$$\langle\phi^i|\phi_j\rangle = \delta^i{}_j$$

Then the product AB can be defined

$$AB = \sum_i |\phi_i\rangle \mathrm{Tr}[\phi^i AB]$$

$$= \sum_i |\phi_i\rangle \langle T_2 \circ f_S \circ V_{\phi^i}(0)\, T_1 \circ f_S \circ V_A(0)\, f_S \circ V_B(0)\rangle_{C_3}$$

In this way, all essential operations in the theory are concretely defined in terms of correlation functions on the cylinder, which can be mapped to correlation functions on the UHP.

[Excercise 3: Show that all of the SFT axioms hold using the definition of the product and trace as correlation functions on the cylinder, assuming that ~~A and B~~ all states are represented by well-behaved e.g. Fock space vertex operators.]

[Excercise 4: The zero momentum sector of the string field can describe the translationally invariant vacua of SFT. As an approximation to the full string field in this sector, consider the zero-momentum tachyon state

$$T c_1|0\rangle$$

By substituting this into the action of Witten's open bosonic SFT, determine the resulting approximation to the tachyon potential. Note the existence of a nontrivial stationary point of the potential for $T > 0$. This is the first approximation to the tachyon vacuum in the level truncation scheme. Show that the energy density of the tachyon vacuum in this approximation is

$$E = -\frac{1}{6}\left(\frac{64}{81\sqrt{3}}\right)^2 = -\frac{2^{12}}{3^{10}}$$

Compare this to the value predicted by Sen's conjectures.]

It is worth mentioning that we have presented Witten's action in a notation which is suited to our needs, but other notations are common. The notation we use is closest in spirit to Witten's original notation, but he denotes the trace with an integral, and the star product with an explicit $*$:

Here: $\mathrm{Tr}[A] \rightarrow$ Witten: $\int A$ Here: $AB \longrightarrow$ Witten: $A*B$

We drop the star because star products in equations will appear so profusely that it becomes cumbersome to write them out. We use the trace symbol since we want to ~~distinguish~~ avoid confusion with ordinary integrals which frequently appear with string fields. There are particularly a large number of alternative notations for the BPZ inner product:

$$\mathrm{Tr}(AB) = \int A*B = \langle A|B\rangle = \langle A,B\rangle = (-1)^{A+1}\omega(A,B)$$

The notations emphasize different aspects of the theory which may be relevant in different contexts. In particular, for general string field theories

— for example closed string field theories — there is no established notion of "trace." However, it is always possible to write SFT actions using the BPZ inner product, whose definition comes automatically from CFT.

Let us discuss the gauge invariant observables of the theory. They can be categorized roughly as follows:

① The space of classical solutions modulo gauge transformations. A special case of this is the space of inequivalent linearized fluctuations around a solution Ψ_*. If we expand the string field

$$\Psi = \Psi_* + \Phi$$

where Φ is a fluctuation around Ψ_*, the action can be rewritten

$$S[\Psi_* + \Phi] = \underbrace{S[\Psi_*]}_{\text{constant}} + S_*[\Phi]$$

where $S_*[\Phi]$ takes the form

$$S_*[\Phi] = \frac{1}{2}\mathrm{Tr}[\Phi Q_{\Psi_*}\Phi] + \frac{1}{3}\mathrm{Tr}[\Phi^3]$$

and the operator Q_{Ψ_*} takes the form

$$Q_{\Psi_*} = Q + [\Psi_*, \cdot\,]$$

with the commutator graded w.r.t. Grassmann parity. It is easy to show that Q_{Ψ_*} is nilpotent due to the EOM for Ψ_*, and satisfies the same axioms as Q. From this it follows that the linearized EOM for the fluctuation field Φ is

$$Q_{\Psi_*}\Phi = 0$$

with solutions identified modulo linearized gauge transformations

$$\Phi' = \Phi + Q_{\Psi_*}\Lambda$$

Thus the spectrum of fluctuations around the solution Ψ_* is given by the cohomology of Q_{Ψ_*} at ghost number 1.

② Scattering amplitudes around the perturbative vacuum $\Psi = 0$ or a nontrivial solution $\Psi = \Psi_*$. A particularly interesting case is the closed string tadpole amplitude — the amplitude for emmission and absorbtion of a single closed string off a D-brane. This can be related to the so-called Ellwood invariant

$$\mathrm{Tr}_V[\Psi]$$

where $\mathrm{Tr}_V[\cdot]$ denotes the trace accompanied by an insertion of a BRST invariant, weight $(0,0)$ closed string vertex operator $V(z,\bar{z}) = c\bar{c}V^m(z,\bar{z})$, with $V^m(z,\bar{z})$ a weight $(1,1)$ primary of the matter CFT, ~~inserted~~ placed at the open string midpoint. Concretely, if $V_\Psi(0)$ is the vertex operator for the state Ψ on the unit half-disk, the Ellwood invariant can be computed as a correlator on the cylinder of circumference 1:

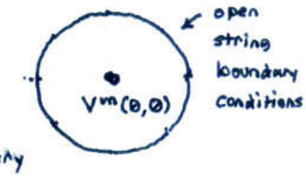
$$\mathrm{Tr}_V[\Psi] = \langle V(i\infty)\, f_S \circ V_\Psi(0) \rangle_{C_1}$$

The Ellwood invariant is related to the closed string tadpole as follows. Suppose we formulate SFT around a D-brane configuration specified by BCFT_0, and we find a classical solution Ψ_* describing BCFT_*. Then

$$\mathrm{Tr}_V[\Psi_*] = A_*(V) - A_0(V)$$

where $A_*(V)$, $A_0(V)$ are the respective closed string tadpole amplitudes in BCFT_* and BCFT_0. They can be computed as a correlation function in the matter component of the BCFT on the unit disk

$$A(V) = \frac{1}{2\pi i}\langle V^m(0,0)\rangle^m_{\mathrm{disk}}$$



The Ellwood invariant has been generalized in a couple of ways to give information about the boundary state of the BCFT represented by a classical solution.

③ The classical action. The action is a gauge invariant quantity, but typically its value when evaluated on a solution is divergent due to the infinite volume of the D-brane. However, for time independent solutions the action is equal to minus the energy of the solution times the volume of the time coordinate:

$$S = -E \cdot \mathrm{vol}_{X^0}$$

~~Dividing by the volume of space gives~~

④ Other observables? It is possible that SFT has other ~~closed~~ gauge invariant observables that have not yet been characterized. For example, we might expect invariants representing charges of topological solitons, but these are not expected to exist for the open bosonic string — perhaps the open superstring.

The most important classical solution in open bosonic SFT is the tachyon vacuum, Ψ_{TV}. Sen's conjectures makes the following prediction about the above gauge invariants:

① Since the tachyon vacuum describes a configuration without D-branes or open strings, the cohomology of $Q_{\Psi_{TV}}$ should be empty: all linearized fluctuations of the solution are pure gauge

② Since there are no D-branes at the tachyon vacuum, the closed string tadpole should vanish. Thus

$$\mathrm{Tr}_V[\Psi_{TV}] = -A_0(V)$$

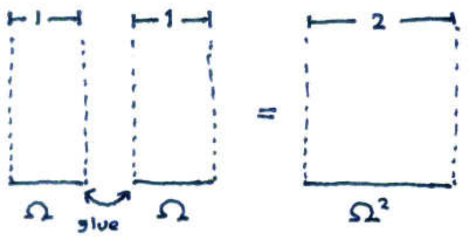
③ The action divided by the D-brane volume should give the brane tension: $\dfrac{S[\Psi_{TV}]}{\mathrm{vol}} = \dfrac{1}{2\pi^2}$

<u>Lecture 2</u> In the last lecture we outlined a general picture of what open bosonic SFT is and the kinds of questions we would like to address with it; a general understanding of the notion of a string field; and a concrete definition of the action for Witten's open bosonic SFT. With this preparation, we can begin to explore the space of string backgrounds as seen from the fluctuation fields of a reference D-brane. The backgrounds correspond to solutions of the EOM

$$Q\Psi + \Psi^2 = 0$$

The goal of this lecture is to introduce various algebraic structures which have proven to be essential in analytic solution of these equations

<u>Wedge states</u> The first thing you might try in attempting to solve these equations is to compute star products and see what kind of states you generate. The simplest state in the BCFT is the $SL(2,\mathbb{R})$ vacuum $\Omega = |0\rangle$. In the sliver frame, Ω is represented by a semi-infinite strip of worldsheet of width 1 carrying no operator insertions. We can multiply Ω with itself to give



the state Ω^2; this correspond to gluing two semi-infinite ~~vertical~~ strips of width 1 together on a vertical edge to form a semi infinite strip of width 2 (with no operator insertions). Now it may seem that a strip of width 2 is not so different from a strip of width 1: they can be related by a conformal transformation — specifically a scaling transformation by a factor of $\frac{1}{2}$ which shrinks the strip of width 2 down to a strip of width 1. The point, however is that in this conformal transformation we have to account for the boundary conditions ~~on the left and right~~ of the path integral on the left and right vertical edges, representing the left and right halves of the string in the Schrodinger functional.

We have seen how tor represent the $SL(2,\mathbb{R})$ vacuum as a functional of a path $x(\sigma)$ (for the free boson); after some relabelling of variables we have

$$\Omega[x(\sigma)] = \Omega[\ell(\sigma), r(\sigma)] \qquad \begin{aligned} \ell(\sigma) &= x(\sigma) \quad \sigma \in [0, \tfrac{\pi}{2}] \\ r(\sigma) &= x(\pi-\sigma) \quad \sigma \in [0, \tfrac{\pi}{2}] \end{aligned}$$

$$= \Omega[\ell(y), r(y)] \qquad \begin{aligned} \ell(y) &= \ell(\sigma) \quad \text{if } y = \tfrac{1}{4}\tanh^{-1}\sin\sigma \\ r(y) &= r(\sigma) \quad \text{if } y = \tfrac{1}{4}\tanh^{-1}\sin\sigma \end{aligned}$$

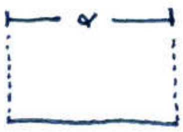
$\ell(y)$ gives the boundary condition on the path integral at a point y above the real axis on the left vertical edge of the strip. and $r(y)$ gives the corresponding boundary condition on the right edge. If we compute

Ω^2, the boundary conditions on the left and right edges are the same, but the strip over which we compute the path integral has doubled in width. If we shrink the strip by a factor of $\frac{1}{2}$, the region where we evaluate the path integral is the same as Ω, but the boundary conditions change: at a point y above the real axis on a vertical edge, the boundary condition should be respectively $\ell(2y)$ or $r(2y)$. Thus the Schrodinger functional for Ω^2 should related to that of the $SL(2,\mathbb{R})$ vacuum through

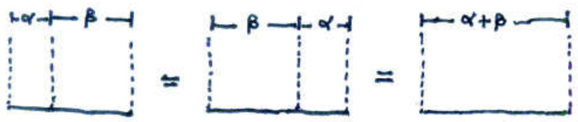
$$\Omega^2[\ell(y), r(y)] = \Omega[\ell(2y), r(2y)]$$

Note that $\ell(y), r(y)$ and $\ell(2y), r(2y)$ define the same unparameterized curve in spacetime; but as parameterized curves they are different. Since the $SL(2,\mathbb{R})$ vacuum is not fully reparameterization invariant (it is not annihilated by all Virasoro generators), the states Ω and Ω^2 are different.

Continuing, we may construct Ω^3 by gluing 3 strips of unit width side-by-side; the result is a strip of width 3. Similarly Ω^4 is a strip of width 4 and so on for any positive integer n. It is clear from this construction that there is nothing special about positive integer powers of the $SL(2,\mathbb{R})$ vacuum; we can generalize to any positive real power, defining Ω^α as a semi-infinite strip of width α containing no operator insertions:

$\Omega^\alpha =$  $\alpha \geq 0$

Ω^α is called a "wedge state," and α is often called the "wedge angle." The terminology originates from the apperance of these states when represented on the unit disk, and is mainly historical. It is immediately clear from gluing strips together



that multiplication of wedge states is abelian:

$$\Omega^\alpha \Omega^\beta = \Omega^\beta \Omega^\alpha = \Omega^{\alpha+\beta}$$

Geometrically, the restriction $\alpha \geq 0$ seems natural, but it is interesting to think about this more carefully. From the above discussion of Ω^2 it is clear that all wedge states are related to the $SL(2,\mathbb{R})$ vacuum by a reparameterization of σ. This implies that Ω^α is a Gaussian functional of $x(\sigma)$ for $\alpha \geq 0$. If we analytically continue to negative α, it turns out that Ω^α become "wrong sign" Gaussians like e^{+x^2}, and are therefore not normalizable states. We can even continue to complex α, in which case we get Gaussians with complex width. These states look normalizable for $Re(\alpha) \geq 0$, but the geometrical interpretation is not obvious. Perhaps such states can be understood in the context of a Lorentzian worldsheet theory. In any case, we will only need to think about real $\alpha \geq 0$. The absence of well-behaved states for $\alpha \leq 0$ implies that multiplication of wedge states, in a sense, cannot be undone.

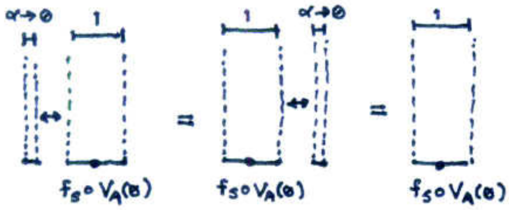
There are two singular limits of wedge states which play a fundamental role: $\alpha \to 0$ and $\alpha \to \infty$. The limit $\alpha \to 0$ defines the so-called identity string field:

$$\Omega^0 = 1$$

This corresponds to a strip of worldsheet with vanishing width, and formally acts as an identity element of open string multiplication:

$$1A = A1 = A$$

This can be seen by viewing a generic state A as a strip of width 1 with vertex operator insertion. Multiplying by 1 ammounts to attaching a strip of vanishing width to either side, and leaves the strip unchanged. Another way to see this is that the path integral on a strip of vanishing width vanishes unless the boundary conditions on the left and right vertical edges match.



Thus the identity string field must ammount to a delta functional between the left and right halves of the string:

$$1[\ell(\sigma), r(\sigma)] = \delta[\ell(\sigma) - r(\sigma)]$$

This is analogous to how the Kronecker delta acts as the identity of matrix multiplication. The existence of an identity is related to the fact that open bosonic SFT has a well-defined trace operation; given 1 and the BPZ inner product, the trace may be defined

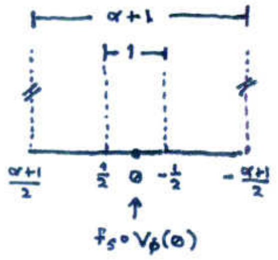
$$\langle 1 | A \rangle = \mathrm{Tr}[1A] = \mathrm{Tr}[A]$$

The identity string field is a somewhat singular state, and there is some question as to whether it should be included in the algebra of string fields. The question of whether 1 is "acceptable" depends to some degree on what you want to do with it. For the calculations we will do it is convenient and consistent to assume the existence of 1.

The opposite limit defines the so-called sliver state, Ω^∞. This is defined by a strip of infinite width. To understand what this means concretely, it is helpful to define the sliver state through its overlap with a test state. The overlap of Ω^α with a test state can be computed as a correlation function on the cylinder:

$$\langle \phi | \Omega^\alpha \rangle = \mathrm{Tr}[\phi\, \Omega^\alpha] = \langle f_s \circ V_\phi(0) \rangle_{C_{\alpha+1}}$$

The cylinder can be presented as a strip between $+\frac{\alpha+1}{2}$ and $-\frac{\alpha+1}{2}$, with opposite vertical edges identified. At the center of this strip between $+\frac{1}{2}$ and $-\frac{1}{2}$



is the strip representing the test state ϕ. In the limit $\alpha \to \infty$ this cylinder unfolds and becomes a correlation function on the UHP. Therefore the sliver state may be defined by

$$\langle \phi | \Omega^\infty \rangle = \langle f_S \circ V_\phi(0) \rangle_{UHP}$$

It is clear that the sliver state should be invariant under multiplication with other wedge states.

$$\Omega^\alpha \Omega^\infty = \Omega^\infty \Omega^\alpha = \Omega^\infty$$

and even invariant under multiplication with itself

$$(\Omega^\infty)^2 = \Omega^\infty$$

Therefore the sliver state is called a "projector" of the open string star algebra. It is clear from the presentation as a correlator on the UHP that the Schrodinger functional of Ω^∞ can be derived by path integral on the region $\mathrm{Re}(z) \geq \frac{1}{2}$ subject to the boundary condition

$$X(z,\bar{z})\big|_{z=\frac{1}{2}+iy} = \ell(y)$$

times a path integral on a region $\mathrm{Re}(z) \leq -\frac{1}{2}$ subject to the boundary condition

$$X(z,\bar{z})\big|_{-\frac{1}{2}+iy} = r(y)$$

This in particular implies that the Schrodinger functional factorizes between the left and right halves of the string

$$\Omega^\infty[\ell(\sigma), r(\sigma)] = F[\ell(\sigma)]F[r(\sigma)]$$

Viewed as an operator on the space of half-string functionals, this is a rank 1 projector — somewhat analogous to the projector $|0\rangle\langle 0|$ onto the ground state of the harmonic oscillator. The identity string field is also a projector, since we should have

$$1^2 = 1$$

This can be viewed as the identity operator on the space of half-string functionals. The analogous operator for the harmonic oscillator is

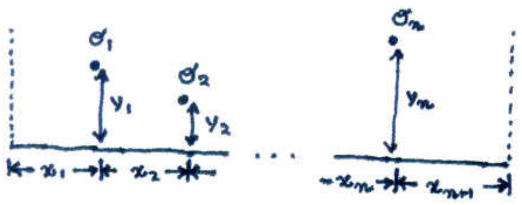
$$\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + \cdots$$

which is an infinite rank projector. Like the identity string field, the sliver state is some what singular. In fact, with the sliver state the situation is somewhat worse since generically it does not have well defined star products. Expressions such as

$$\Omega^\infty A \Omega^\infty$$

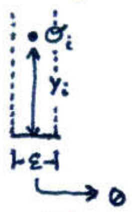
depend on how the sliver limit is taken.

While wedge states are interesting, they are not by themselves enough to get solutions to the equations of motion; this is obvious since wedge states carry ghost number 0, and a solution to the EOM must carry ghost number 1. To get a richer class of states, we consider strips of worldsheet of varying width carrying various insertions of local operators — such states are often referred to as "wedge states with insertions." It is often useful to present such states as factorized into products of wedge states and fields representing the insertions of local operators. Consider for example the state



shown to the left. Inside the semi-infinite strip is an operator $\mathcal{O}_1$ a distance x_1 from the leftmost vertical edge and a distance y_1 above the real axis; and operator $\mathcal{O}_2$ a distance $x_1 + x_2$ from the left edge and a distance y_2 above the real axis, and so on up to the operator $\mathcal{O}_n$. The idea is that for each operator $\mathcal{O}_i$ we introduce a corresponding string field $\mathcal{O}_i$ (denoted with the same symbol)

$$\mathcal{O}_i =$$



as an infinitessimally thin strip carrying the operator $\mathcal{O}_i$ a distance y_i above the real axis. The region of the surface between insertions $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$ can be described as an empty strip of width x_{i+1} — in other words, a wedge state. We can therefore represent the state as a product of wedge states and the string fields $\mathcal{O}_i$:

$$\Omega^{x_1}\mathcal{O}_1\,\Omega^{x_2}\mathcal{O}_2 \ldots \Omega^{x_n}\mathcal{O}_n\,\Omega^{x_{n+1}}$$

This is a convenient symbolic representation of the state, and is suitable for calculations.

<u>Excercise 5</u>: Show that the zero-momentum tachyon state can be written

$$c_1|0\rangle = \frac{\pi}{2}\sqrt{\Omega}\,c\,\sqrt{\Omega}$$

where the field c is defined by an infinitely thin strip with a boundary insertion of the c-ghost.

It will be of interest to compute the derivative of a wedge state w.r.t. the wedge angle. We will do this following a computation due to Okawa. Consider the overlap of Ω^α with a test state ϕ given by an insertion of an operator $\phi(0)$ at the origin of a semi-infinite strip of unit width. We assume that $\phi(0)$

has definite scaling dimension h; it is possible to construct a basis for $\mathcal{H}$ using states of this form. The overlap of Ω^α with ϕ can then be computed as a 1-point function on a cylinder of circumference $\alpha+1$:

$$\langle \phi | \Omega^\alpha \rangle = \langle \phi(0) \rangle_{C_{\alpha+1}}$$

We can scale the cylinder down to unit circumference, obtaining

$$\langle \phi | \Omega^\alpha \rangle = \left(\frac{1}{\alpha+1} \right)^h \langle \phi(0) \rangle_{C_1}$$

Now take the derivative w.r.t. α and scale the correlator back:

$$\langle \phi | \frac{d}{d\alpha} \Omega^\alpha \rangle = -h \left(\frac{1}{\alpha+1} \right)^{h+1} \langle \phi(0) \rangle_{C_1}$$

$$= -h \frac{1}{\alpha+1} \langle \phi(0) \rangle_{C_{\alpha+1}}$$

Since ϕ has scaling dimension h its OPE with the energy-momentum tensor takes the form

$$T(z)\phi(0) = \cdots + \frac{h}{z^2}\phi(0) + \frac{1}{z}\partial\phi(0) + \cdots$$

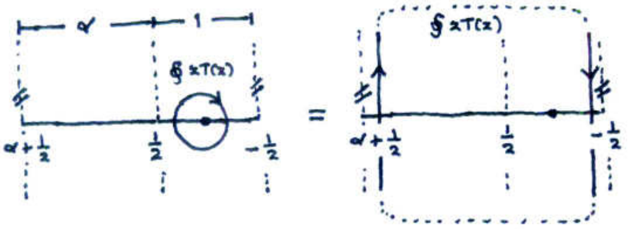
This implies

$$h\,\phi(0) = \oint_0 \frac{dz}{2\pi i} z\, T(z)\phi(0)$$

and we can write

$$\frac{d}{d\alpha} \langle \phi | \Omega^\alpha \rangle = -\frac{1}{\alpha+1} \left\langle \oint_0 \frac{dz}{2\pi i} z\, T(z)\, \phi(0) \right\rangle_{C_{\alpha+1}}$$

Next we unfold the energy momentum contour inside the cylinder. We consider the cylinder as a strip $\frac{1}{2}+\alpha \geq \mathrm{Re}(z) \geq -\frac{1}{2}$ with opposite sides identified, and use the doubling trick. Expanding the contour gives a contribution at the left vertical edge and a contribution at the right vertical edge:



$$\oint_0 \frac{dz}{2\pi i} z\, T(z) = \int_{-i\infty+\alpha+\frac{1}{2}}^{i\infty+\alpha+\frac{1}{2}} \frac{dz}{2\pi i} z\, T(z) - \int_{-i\infty-\frac{1}{2}}^{i\infty-\frac{1}{2}} \frac{dz}{2\pi i} z\, T(z)$$

In the second term we make a substitution $z \to z-(\alpha+1)$ so that both terms share a common integration variable:

$$\oint_0 \frac{dz}{2\pi i} z\, T(z) = \int_{-i\infty+\alpha+\frac{1}{2}}^{i\infty+\alpha+\frac{1}{2}} \frac{dz}{2\pi i} \Big[z\, T(z) - (z-(\alpha+1))\, T(z-(\alpha+1)) \Big]$$

The identification on the vertical edges of the cylinder implies

$$T(z) = T(z-(\alpha+1))$$

Therefore

$$\oint_0 \frac{dz}{2\pi i} z\, T(z) = \int_{-i\infty+\alpha+\frac{1}{2}}^{i\infty+\alpha+\frac{1}{2}} \frac{dz}{2\pi i} T(z) \Big[z - (z-(\alpha+1)) \Big]$$

$$= (\alpha+1) \int_{-i\infty+\alpha+\frac{1}{2}}^{i\infty+\alpha+\frac{1}{2}} \frac{dz}{2\pi i} T(z)$$

and

$$\langle \phi | \frac{d}{d\alpha} \Omega^\alpha \rangle = -\left\langle \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z)\, \phi(0) \right\rangle_{C_{\alpha+1}}$$

We can characterize the energy-momentum contour integral as a string field K:

$$K = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}$$

this is an infinitely thin strip carrying an insertion of the energy momentum tensor, integrated parallel to the imaginary axis. Therefore we have shown

$$\frac{d}{d\alpha} \Omega^\alpha = -K\Omega^\alpha$$

This implies that wedge states can be expressed in terms of the string field K:

$$\Omega^\alpha = e^{-\alpha K}$$

This equation gives us a way to derive the vertex operator representing a wedge state. Note that

$$\Omega^\alpha = \sqrt{\Omega}\, e^{-(\alpha-1)K} \sqrt{\Omega}$$

The state on the right can be viewed as a strip of width 1 containing an infinite number of vertical contour insertions of the energy-momentum tensor. To derive the vertex operator we must map this strip back to the canonical half disk. Noting that

$$f_s^{-1} \circ \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z) = \frac{\pi}{2} \int_{-i}^{i} \frac{dz}{2\pi i} (1+z^2) T(z)$$

the vertex operator of the state Ω^α is given by

$$V_{\Omega^\alpha}(0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\pi(1-\alpha)}{2} \int_{-i}^{i} \frac{dz}{2\pi i} (1+z^2) T(z) \right)^n$$

As expected, the vertex operator is nonlocal.

Given the set of wedge states, we can form an algebra by forming linear combinations — this is called the wedge algebra. Generally we can form continuous linear combinations

$$F(K) = \int_0^\infty d\alpha\, f(\alpha)\, \Omega^\alpha$$

As the notation suggests, the wedge algebra can be seen as an algebra of functions of the string field K. Since $\Omega^\alpha = e^{-\alpha K}$, $F(K)$ can be viewed as the Laplace transform of the coefficients $f(\alpha)$. It is interesting to try to give a precise definition of the wedge algebra; this point is somewhat controversial, but we will describe one proposal, due to Rastelli, which turns out to have suprising explanatory power. First, we consider the string field $F(K)$ as isomorphic to an ordinary function $F(k)$ on numbers k in the spectrum of K. The spectrum of K is given by numbers k with the property that the string field

$$K - k$$

is not invertible. We may formally determine the inverse of $K-k$ using the Schwinger parameterization

$$\frac{1}{K-k} = \int_0^\infty d\alpha \; e^{k\alpha}\,\Omega^\alpha$$

since Ω^α approaches a constant state as $\alpha \to \infty$ (the sliver), this integral is divergent for all $k \geq 0$; this gives the spectrum of K. Therefore the wedge algebra can be seen as an algebra of functions on nonnegative real numbers. The simplest possible algebra we could propose consists of bounded, continuous functions of $k \geq 0$ supplied with the norm

$$\|F(k)\| = \sup_{k\geq 0} |F(k)|$$

We have the usual axioms of a norm

$$\|F(k)\| \geq 0 \quad \text{and} \quad \|F(k)\| = 0 \rightarrow F(k) = 0$$

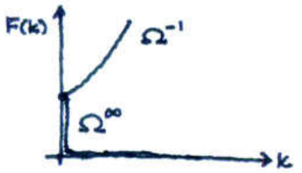
$$\|a F(k)\| = |a|\,\|F(k)\|$$

$$\|F(k) + G(k)\| \leq \|F(k)\| + \|G(k)\|$$

in addition to the property

$$\|F(k)\,G(k)\| \leq \|F(k)\|\cdot\|G(k)\|$$

The ~~last property~~ space of bounded, continuos functions of $k \geq 0$ is complete with respect to this norm, and the final property then implies that we have a Banach algebra. (In fact, we have a C^*-algebra once we account for the notion of Hermitian conjugation of a string field, which we have not discussed). We make a few observations based on this characterization of the wedge algebra. First, the identity string field and wedge states with finite, positive wedge angle are part of the algebra. Second, wedge states with negative wedge angle are excluded since they are not bounded functions of $k \geq 0$; the sliver state is excluded since it is not a continuous function:



$$\Omega^\infty = \begin{cases} 1 & \text{at } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

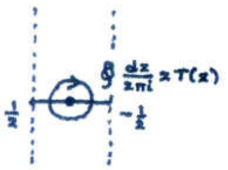
One important consequence of this is that, while $\lim_{n\to\infty} \Omega^n$ converges as a ~~state~~ as an expansion in Fock space states, it does not converge as a Cauchy sequence with respect to the norm in the wedge algebra. One can check that

$$\|\Omega^{Nn} - \Omega^n\| = N^{-\frac{1}{N-1}} - N^{-\frac{N}{N-1}}$$

which is a constant (independent of n) in the limit $n \to \infty$. The fact that the sliver limit is not convergent is significant in the study of analytic solutions.

Schnabl's $\mathcal{L}_0$ An important role in the theory is played by the dilitation generator in the sliver coordinate frame, introduced by Schnabl:

$$\mathcal{L}_0 = \oint_0 \frac{dz}{2\pi i}\, z\, T(z) \qquad \text{(sliver frame)}$$



This is different from the usual L_0 since the contour is integrated around the vertex operator on the strip of width 1, rather than the unit half disk; In particular, Virasoro charges are not conformally invariant. To relate $\mathcal{L}_0$ to the ordinary Virasoros, we must map back to the unit half-disk:

$$\mathcal{L}_0 = f_S^{-1} \circ \oint_0 \frac{dz}{2\pi i}\, z\, T(z) = \oint_0 \frac{dz}{2\pi i}(1+z^2)\tan^{-1}z\, T(z) \qquad \text{(half disk)}$$

$$= L_0 + \frac{2}{3}L_2 - \frac{2}{15}L_4 + \dots$$

Since $\mathcal{L}_0$ is made from positively moded Virasoros, we have $\mathcal{L}_0|0\rangle = 0$. This merely indicates that, in the sliver frame, we can shrink the contour without encountering poles since there is no vertex operator at the origin.

We will need to act the operator $\mathcal{L}_0$ on strips of arbitrary width. To do this, we reexpress it as follows

$$\mathcal{L}_0 = \int_{-i\infty+\frac{1}{2}}^{i\infty+\frac{1}{2}} \frac{dz}{2\pi i}\, z\, T(z) + \int_{i\infty-\frac{1}{2}}^{-i\infty-\frac{1}{2}} \frac{dz}{2\pi i}\, z\, T(z)$$

$$= \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\left(z+\tfrac{1}{2}\right)T\left(z+\tfrac{1}{2}\right) + \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i}\left(z-\tfrac{1}{2}\right)T\left(z-\tfrac{1}{2}\right)$$

In the last step we made a change of integration variable so that z is purely imaginary. Note that the energy-momentum operators are placed on the left and right vertical edges of the strip. This corresponds to the fact that mode operators are always defined on the unit circle in radial quantization. For a strip of general width, it is always the case that the left and right vertical edges correspond to the unit circle in radial quantization. Therefore, for a strip whose left vertical edge intersects the real axis at ℓ and whose right vertical edge intersects the real axis at r, the action of $\mathcal{L}_0$ is described by the contour insertions

$$\mathcal{L}_0 = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\left(z+\tfrac{1}{2}\right)T(z+\ell) + \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i}\left(z-\tfrac{1}{2}\right)T(z+r)$$

$$= \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\, z\, T(z+\ell) + \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i}\, z\, T(z+r) + \frac{1}{2}\left[\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z+\ell) + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z+r)\right]$$

The last two terms are the energy-momentum insertions defining the string field K. The first two terms define a new operator, which we will call $\frac{1}{2}\mathcal{L}^-$:

$$\frac{1}{2}\mathcal{L}^- = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\, z\, T(z+\ell) + \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i}\, z\, T(z+r)$$

The factor of $\frac{1}{2}$ here is a historical inconvenience; it derives from the fact that

$\mathcal{L}^-$ was originally defined as

$$\mathcal{L}^- = \mathcal{L}_0 - \mathcal{L}_0^*$$

where $\mathcal{L}_0^*$ is the BPZ conjugate of $\mathcal{L}_0$. The relation between $\mathcal{L}_0$ and $\frac{1}{2}\mathcal{L}^-$ can be expressed

$$\mathcal{L}_0 A = \tfrac{1}{2}\mathcal{L}^- A + (KA + AK)\tfrac{1}{2}$$

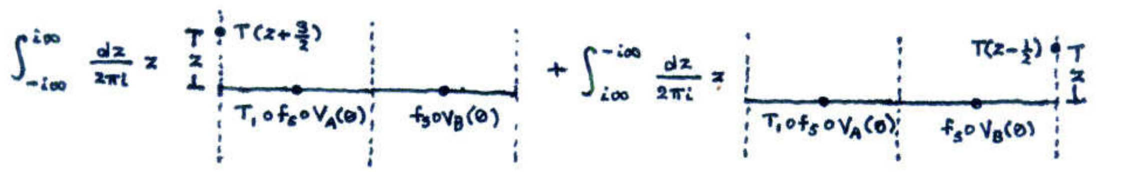
for a general string field A. The utility of this decomposition is that $\frac{1}{2}\mathcal{L}^-$ is a derivation of the star product

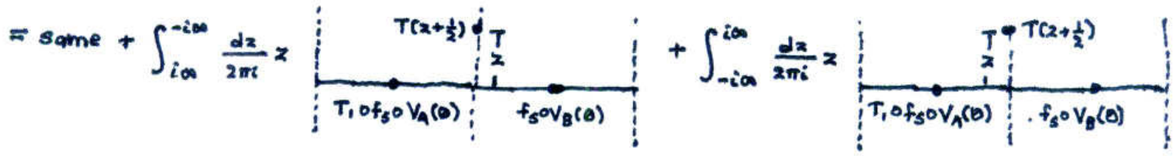
$$\tfrac{1}{2}\mathcal{L}^-(AB) = (\tfrac{1}{2}\mathcal{L}^- A)B + A(\tfrac{1}{2}\mathcal{L}^- B)$$

and leaves the trace invariant:

$$\mathrm{Tr}\left[\tfrac{1}{2}\mathcal{L}^- A\right] = 0$$

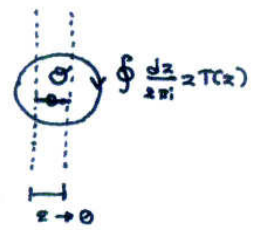
The derivation property can be seen as follows. We may represent AB as a strip of width 2 with the vertex operator of B centered at the origin. The action of $\frac{1}{2}\mathcal{L}^-$ on AB is shown below:





The point is that we can add two opposite energy-momentum contours intersecting the real axis at $\frac{1}{2}$, at the junction of the strips of A and B. Grouping terms gives the derivation property of $\frac{1}{2}\mathcal{L}^-$. A similar manipulation shows that the trace is invariant.

It is useful to understand how $\frac{1}{2}\mathcal{L}^-$ acts on wedge states with insertions. Consider first a string field $\mathcal{O}$ defined by an insertion of an operator $\mathcal{O}(x)$ of scaling dimension h on the real axis inside an infinitely thin strip.



Placing the energy momentum contours appropriate to $\frac{1}{2}\mathcal{L}^-$ on the left and right boundaries of the strip, it is easy to see that they can be joined into a single contour integral

$$\oint_0 \frac{dz}{2\pi i} z T(z)$$

surrounding $\mathcal{O}(x)$. Plugging in the T-$\mathcal{O}$ OPE, the contour integral singles out the double pole, and we obtain $h\mathcal{O}(x)$. We therefore have

$$\tfrac{1}{2}\mathcal{L}^- \mathcal{O} = h\mathcal{O}$$

Next consider the action of $\frac{1}{2}\mathcal{L}^-$ on K. Note that the operator insertion defining K has scaling dimension 1; if $S_\lambda(z) = \lambda z$ is the scale transformation by a constant λ, we have

$$S_\lambda \circ \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z) = \lambda \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z)$$

Indeed, we find

$$\tfrac{1}{2}\mathcal{L}^- K = K$$

[Excercise 6: Prove this]

Thus the action of $\frac{1}{2}\mathcal{L}^-$ on an infinitely thin strip containing an operator insertion of scaling dimension h simply gives the scaling dimension as an overall factor. Combined with the derivation property, we can use this to compute the action of $\frac{1}{2}\mathcal{L}^-$ on more general states. For example, on a wedge state we have

$$\tfrac{1}{2}\mathcal{L}^- \Omega^\alpha = -\alpha K \Omega^\alpha$$

It is also useful to consider, for an arbitrary state A

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}^-(\sqrt{\Omega}A\sqrt{\Omega}) &= (\tfrac{1}{2}\mathcal{L}^-\sqrt{\Omega})A\sqrt{\Omega} + \sqrt{\Omega}(\tfrac{1}{2}\mathcal{L}^-A)\sqrt{\Omega} + \sqrt{\Omega}A(\tfrac{1}{2}\mathcal{L}^-\sqrt{\Omega}) \\ &= -\tfrac{1}{2}K\sqrt{\Omega}A\sqrt{\Omega} + \sqrt{\Omega}(\tfrac{1}{2}\mathcal{L}^-A)\sqrt{\Omega} + \sqrt{\Omega}A\sqrt{\Omega}(-\tfrac{1}{2}K)\end{aligned}$$

Bringing the first and last terms to the other side gives Schnabl's $\mathcal{L}_0$. This gives a convenient expression of the relationship between $\mathcal{L}_0$ and $\frac{1}{2}\mathcal{L}^-$:

$$\mathcal{L}_0(\sqrt{\Omega}A\sqrt{\Omega}) = \sqrt{\Omega}(\tfrac{1}{2}\mathcal{L}^-A)\sqrt{\Omega}$$

<u>KBc subalgebra</u> Next we introduce a subalgebra of wedge states with insertions that is sufficient to give analytic solutions for the endpoint of tachyon condensation — the tachyon vacuum. It is natural to guess that this subalgebra should include the zero momentum tachyon state

$$c_1|0\rangle = \frac{\pi}{2}\sqrt{\Omega}c\sqrt{\Omega}$$

Since this is the most important fluctuation field on the D-brane which acquires expectation value after tachyon condensation. Therefore we can consider a subalgebra given by products of fields K and c

$$K,\ c$$

where c is defined as an infinitessimal strip with a boundary insertion of the c-ghost. However, it turns out that this subalgebra is not rich enough to describe interesting tachyon vacuum solutions — we need in addition fields with negative ghost number. One way to motivate this is gauge fixing; Since we are not interested in constructing the entire gauge orbit of tachyon vacuum solutions, it makes sense to look for a solution in a particular gauge. In the numerical construction of solutions in level truncation, the most common gauge choice is Siegel gauge:

$$b_0 \Psi = 0$$

where b_0 is the zero mode of the b-ghost. This is inconvenient in analytic calculations, since b_0 and L_0 take us outside the algebra of wedge states. A natural alterative is

$$\mathcal{B}_0 \Psi = 0$$

where $\mathcal{B}_0$ is the zero mode of the b-ghost in the sliver coordinate frame— it is the same as $\mathcal{L}_0$ after replacing T by b. This is called "Schnabl gauge." We have the property

$$\mathcal{B}_0(\sqrt{\Omega}A\sqrt{\Omega}) = \sqrt{\Omega}(\tfrac{1}{2}\mathcal{B}^- A)\sqrt{\Omega}$$

where $\frac{1}{2}\mathcal{B}^-$ is a derivation of the star product — it is the same as $\frac{1}{2}\mathcal{L}^-$ after replacing T by b. We have the property

$$\tfrac{1}{2}\mathcal{B}^- K = B$$

where the string field B is the same as K after replacing T by b — it is given by a vertical contour integral

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\, b(z)$$

inside an infinitely thin strip. Therefore, Schnabl gauge leads us to consider a subalgebra of states given by multiplication of the string fields K, B, and c:

K = Grassmann even; ghost # 0

B = Grassmann odd; ghost # -1

c = Grassmann odd; ghost # 1

We have the properties

$$\tfrac{1}{2}\mathcal{L}^- K = K \qquad \tfrac{1}{2}\mathcal{B}^- K = B$$

$$\tfrac{1}{2}\mathcal{L}^- B = B \qquad \tfrac{1}{2}\mathcal{B}^- B = 0$$

$$\tfrac{1}{2}\mathcal{L}^- c = -c \qquad \tfrac{1}{2}\mathcal{B}^- c = 0$$

For example, using these relations we can check that the zero-momentum tachyon state is in Schnabl gauge:

$$\begin{aligned}\mathcal{B}_0\, c_1|0\rangle &= \mathcal{B}_0\left(\tfrac{\pi}{2}\sqrt{\Omega}\,c\sqrt{\Omega}\right)\\ &= \tfrac{\pi}{2}\sqrt{\Omega}\left(\tfrac{1}{2}\mathcal{B}^- c\right)\sqrt{\Omega}\\ &= 0\end{aligned}$$

Alternatively, we can check this by noting

$$\mathcal{B}_0 = b_0 + \tfrac{2}{3}b_2 - \tfrac{2}{15}b_4 + \dots$$

The positively moded b oscillators pass through c_1 and annihilate the $SL(2,\mathbb{R})$ vacuum.

The fields K, B, c satisfy a number of important relations with respect to eachother and with respect to the BRST operator:

$$[K, B] = 0 \qquad B^2 = c^2 = 0 \qquad [B, c] = 1$$

$$QK = 0 \qquad QB = K \qquad Qc = cKc$$

We use $[\,,\,]$ to denote the commutator graded w.r.t Grassmann parity. The important thing to note is that the K, B, c subalgebra is closed under the action of the BRST operator. Therefore it is consistent to look for a solution to $Q\Psi + \Psi^2 = 0$ in this subalgebra. Most of these relations can be easily verified by the appropriate contour deformations inside correlation functions on the cylinder. The computation of Qc however merits a brief comment. First, the computation of Qc gives a string field we can write as $c\partial c$, defined by a boundary insertion of the operator $c\partial c(x)$ inside an infititly thin strip. The operator ∂c however, is different from c and it looks like we leave the KBc subalgebra. However, we note that

$$[K, c] = \partial c$$

In terms of correlation functions, the commutator with K produces a contour of the energy-momentum tensor around the c-insertion, which picks out the pole contribution of the T-c OPE, giving ∂c. Therefore we have

$$Qc = c\partial c = c[K, c] = cKc - c^2K = cKc$$

Generally, it is always true that $[K, \cdot\,]$ computes the worldsheet derivative of the operator insertion defining a string field.

Now that we have a simple algebraic setup, it is hard to resist playing around a bit to see if we can find some solutions. One thing you might notice is that if you multiply Qc by K, you find the identity

$$Q(cK) = (cK)^2$$

This means that

$$\Psi = -cK$$

is a solution to the open SFT equations of motion. Unfortunately, this solution is somewhat of a dud — it has no physical meaning or significance, as far as we know. Generally, it is a nontrivial matter understanding when a solution is really "there," or when it is an artifact of some kind of singularity. Rather than dwell on this, we note that adding c results in something more interesting:

$$\Psi = c(1-K)$$

I turns out that this is a solution for the tachyon vacuum. This statement should be understood with some qualification, since the solution is singular: it is defined by a pair of operator insertions on an infinitly thin strip, and behaves similarly to the identity string field. The thing we would like to do given a tachyon vacuum solution is prove Sen's conjectures. For example, the action evaluated on the solution should give the brane tension:

$$\frac{S[\Psi]}{\text{Vol}} = \frac{1}{2\pi^2}$$

Since Ψ is an infinitely thin strip with operator insertions, computing star products and traces of Ψ produces a correlation function on a cylinder with vanishing circumference. There is no surface, and no unambiguous way to define the correlator, so it seems that the action evaluated on Ψ is undefined. Nevertheless, I say that Ψ is a tachyon vacuum solution due to its genetic relationship to other tachyon vacuum solutions which are well-defined, and, more interestingly, due to the fact that the solution supports no open string excitations. The linearized fluctuations of the background defined by Ψ are given by the cohomology of the shifted kinetic operator

$$Q_\Psi = Q + [c(1-K), \cdot\,]$$

We claim that the cohomology of Q_Ψ is empty, and therefore any linearized fluctuation of the solution is pure gauge. The absence of cohomology follows from the following computation:

$$Q_\Psi B = QB + [c(1-K), B] = K + [c, B](1-K) = K + 1 - K = 1$$

Given a fluctuation φ around Ψ satisfying the linearized EOM:

$$Q_\Psi \varphi = 0$$

we can write

$$\varphi = 1\cdot\varphi = (Q_\Psi B)\varphi = Q_\Psi(B\varphi)$$

Therefore φ is a trivial element of the cohomology — all linearized fluctuations are pure gauge.

<u>Lecture 3</u> In this lecture we present some of the most important analytic solutions in Witten's open bosonic SFT: Schnabl's solution for the tachyon vacuum; Schnabl gauge solutions for marginal deformations; the simple tachyon vacuum; and the KOS (Kiermaier-Okawa-Soler) solution for marginal deformations. Among the important solutions we will not discuss is the Fuchs-Kroyter-Potting-Kiermaier-Okawa solutions for marginal deformations (often called the Kiermaier-Okawa solution for short, perhaps unfairly, since its essential structure was first described by Fuchs, Kroyter and Potting). The Kiermaier-Okawa solutions have a rather beautiful structure,

but are conceptually rather different from the other solutions, and, for one reason or another, have not played a central role in recent developments.

Schnabl's solution The first fully regular analytic solution in open bosonic SFT was Schnabl's solution for the tachyon vacuum. We will give a "derivation" of this solution which is rather different from Schnabl's original approach, but is more direct from the perspective of our development. We look for solutions among states in the KBc subalgebra satisfying the Schnabl gauge condition:

$$\mathcal{B}_0 \Psi = 0$$

A fairly general class of such states takes the form

$$\Psi = \sqrt{\Omega}\, cBF(K)c\sqrt{\Omega}$$

With a little more work we can actually write the completely general ansatz for KBc states in Schnabl gauge, but the other possible combinations of K,B,c do not appear in the solution, so we will make our lives a little simpler by ignoring them. First let's check the Schnabl gauge condition; acting $\mathcal{B}_0$ on the rhs gives

$$\mathcal{B}_0(\sqrt{\Omega}\,cBF(K)c\sqrt{\Omega}) = \sqrt{\Omega}\,\tfrac{1}{2}\mathcal{B}^-(cBF(K)c)\sqrt{\Omega}$$

Now recall that $\frac{1}{2}\mathcal{B}^-$ acts as a derivation, and annihilates B and c. The only possible contribution appears when $\frac{1}{2}\mathcal{B}^-$ acts on F(K), giving

$$\tfrac{1}{2}\mathcal{B}^- F(K) = BF'(K)$$

However, this contribution vanishes due to $B^2 = 0$. Next we plug this ansatz into the EOM to fix the form of F(K):

$$Q\Psi = -\sqrt{\Omega}\,cKBcFc\sqrt{\Omega} + \sqrt{\Omega}\,cBFcKc\sqrt{\Omega}$$

$$\Psi^2 = \sqrt{\Omega}\,cBFc\Omega Fc\sqrt{\Omega} - \sqrt{\Omega}\,cB\Omega FcFc\sqrt{\Omega}$$

These expressions look rather different, but if you stare at it a little bit you will find that the equations of motion are equivalent to the following functional equation for F(K):

$$F(K_1)K_2 + F(K_1)F(K_2)e^{-K_2} - K_1F(K_2) - e^{-K_1}F(K_1)F(K_2) = 0$$

Since there are two variables K_1, K_2 and only one undetermined function F(K), this looks overconstrained; still there is a solution. After some algebra we can rewrite this as

$$\frac{F(K_1)}{K_1 + e^{-K_1}F(K_1)} = \frac{F(K_2)}{K_2 + e^{-K_2}F(K_2)}$$

Since the lhs is only a function of K_1, and the right hand side only a function of K_2, the only way this can be consistent is if both sides are equal to a constant, which we call λ:

$$\frac{F(K)}{K+\Omega F(K)} = \lambda$$

This implies

$$F(K) = \frac{\lambda K}{1-\lambda\Omega}$$

and

$$\Psi = \lambda\sqrt{\Omega}\, c\, \frac{KB}{1-\lambda\Omega}\, c\sqrt{\Omega}$$

We have a 1-parameter family of KBc solutions in Schnabl gauge. Note that if $\lambda = 0$ we obtain the trivial solution

$$\Psi = 0$$

This is the perturbative vacuum — the configuration where all fluctuation fields vanish, and the D-brane defining the SFT is undisturbed. If λ is small we can expand the solution perturbatively in λ:

$$\Psi = \lambda\sqrt{\Omega}\, cKBc\sqrt{\Omega} + \mathcal{O}(\lambda^2)$$

The leading order contribution is BRST exact:

$$\sqrt{\Omega}\, cKBc\sqrt{\Omega} = Q(\sqrt{\Omega}\, Bc\sqrt{\Omega})$$

This means that, for sufficiently small λ the solution represents a deformation of the perturbative vacuum by a trivial element of the BRST cohomology; physically, this represents no deformation at all: for small enough λ, the solution is pure gauge. It is a little puzzling to find pure gauge solutions since we have supposedly fixed the gauge — Schnabl gauge. Apparently, the Schnabl gauge condition does not completely fix the gauge.

However, what we wanted to find is a solution for the tachyon vacuum. One way to know if we have a tachyon vacuum is if the linearized excitations of the solution are gauge trivial — the background supports no open strings. This follows from the existence of a string field A satisfying

$$Q_\Psi A = 1$$

The field A is called a "homotopy operator." We can try to construct A for the KBc Schnabl gauge solutions. A ~~must have~~ has ghost number -1, and assuming it can be expressed in the KBc subalgebra, it must take the form

$$A = B\,G(K)$$

Now compute

$$Q_\Psi A = Q(BG(K)) + \lambda\sqrt{\Omega}\, c\frac{KB}{1-\lambda\Omega}\, cB\sqrt{\Omega}\, G(K) + \lambda\, BG(K)\sqrt{\Omega}\, c\frac{KB}{1-\lambda\Omega}\, c\sqrt{\Omega}$$

$$= KG(K) + \lambda\sqrt{\Omega}\, cB\frac{KG(K)}{1-\lambda\Omega}\sqrt{\Omega} + \lambda\sqrt{\Omega}\frac{KG(K)}{1-\lambda\Omega}Bc\sqrt{\Omega}$$

$$= G(K)\left(K + \lambda\frac{K\Omega}{1-\lambda\Omega}\right) + \lambda\sqrt{\Omega}\left[\frac{KG(K)}{1-\lambda\Omega}, Bc\right]\sqrt{\Omega}$$

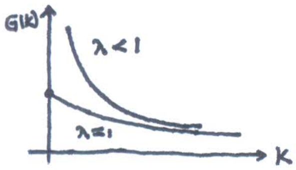
This can only be equal to 1 if

$$\frac{K\,G(K)}{1-\lambda\Omega} = 1$$

or

$$G(K) = \frac{1-\lambda\Omega}{K}$$

Have we just proven that the solutions represent the tachyon vacuum after all? Not yet: we still have to see that the homotopy operator makes sense as a string field. For this recall our previous discussion about the wedge algebra: acceptable states in the wedge algebra must be bounded, continuous functions of $K \geq 0$. The



function $G(K)$ has a pole at $K=0$ for all values of λ <u>except</u> $\lambda = 1$; where

$$\| G(K) \| = 1$$

To see that this pole should be taken seriously, consider the homotopy operator at $\lambda = 0$:

$$A = \frac{B}{K}$$

The existence of this state would imply that the original D-brane supports no open string excitations — which is obviously not true. We can nevertheless attempt to construct this state, for example by defining $1/K$ through the Schwinger parameterization as an integral over all wedge states:

$$\frac{B}{K} = B\int_0^\infty d\alpha\, \Omega^\alpha$$

We may worry about the upper limit in the integration since Ω^α approaches the sliver state; but actually the integral is finite since $B\Omega^\alpha$ vanishes in the Fock space as

$$B\Omega^\alpha \sim \mathcal{O}\left(\frac{1}{\alpha^3}\right)$$

Excercise 2: Prove this by finding the operator ϕ of lowest scaling dimension such that $\langle \phi | B\Omega^\alpha \rangle \neq 0$, and determine the behavior of this correlator for large α.

The difficulty does not come from ~~the fact that the state diverges~~ a divergence of the state, but from the fact that it does not actually define a homotopy operator:

$$Q\left(\frac{B}{K}\right) = \int_0^\infty d\alpha\, K\Omega^\alpha = -\int_0^\infty d\alpha \frac{d}{d\alpha}\Omega^\alpha = 1 - \Omega^\infty$$

The presence of the sliver state negates the construction. Its presence is related to the fact that the identity operator is nontrivial in the BRST cohomology. For other $\lambda \neq 0$ the homotopy operator may be formally written

$$A = B\frac{1-\lambda\Omega}{K} = B\left(\int_0^1 d\alpha\,\Omega^\alpha + (1-\lambda)\int_1^\infty d\alpha\,\Omega^\alpha\right)$$

The second term, with integration out to the sliver state, is the problematic contribution and is absent precisely when $\lambda = 1$. Thus at $\lambda=1$ we have a well defined homotopy operator and linearized excitations are trivial. The corresponding solution

$$\Psi = \sqrt{\Omega}\, c \frac{KB}{1-\Omega} c \sqrt{\Omega}$$

is Schnabl's solution for the tachyon vacuum.

An important aspect of Schnabl's solution is understanding how to define the string field $\frac{K}{1-\Omega}$ concretely. One possible approach is to define it by a power series in K. Recalling the generating function for Bernoulli numbers, this gives

$$\Psi = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \sqrt{\Omega}\, c B K^n c \sqrt{\Omega}$$

$$= \sqrt{\Omega}\, c \sqrt{\Omega} - \frac{1}{2} \sqrt{\Omega}\, c K B c \sqrt{\Omega} + \frac{1}{12} \sqrt{\Omega}\, c K^2 B c \sqrt{\Omega} + \cdots$$

Each term in this series is an eigenstate of $\mathcal{L}_0$:

$$\mathcal{L}_0 \left(\sqrt{\Omega}\, c K^n B c \sqrt{\Omega} \right) = (n-1) \sqrt{\Omega}\, c K^n B c \sqrt{\Omega}$$

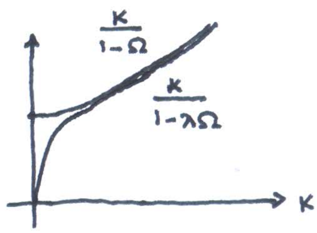
This defines the so-called $\mathcal{L}_0$ level expansion. It is somewhat analogous to the expansion of the string field into a basis of Fock states with definite L_0 eigenvalue — the ordinary level expansion — except formulated in the sliver frame. The "level" of a state in this expansion is defined to be its $\mathcal{L}_0$ eigenvalue. The lowest level state in the $\mathcal{L}_0$ expansion of Schnabl's solution is the zero-momentum tachyon $\frac{2}{\pi} c_1|0\rangle$, at level -1. It is also interesting to see the $\mathcal{L}_0$ level expansion of KBc solutions in Schnabl gauge when $\lambda \neq 1$

$$\Psi = \frac{1}{1-\lambda} \sqrt{\Omega}\, c K B c \sqrt{\Omega} - \frac{\lambda}{(1-\lambda)^2} \sqrt{\Omega}\, c K^2 B c \sqrt{\Omega} + \cdots$$

The zero-momentum tachyon state is now absent from the expansion, and the leading state, at level 0, is trivial in the BRST cohomology. More interestingly, the $\mathcal{L}_0$ expansion is divergent in the limit $\lambda \to 1$; this is further evidence that the solutions for $\lambda = 1$ and $\lambda \neq 1$ are really distinct. A related observation is that the states $\frac{K}{1-\lambda\Omega}$ do not form a Cauchy sequence in the $\lambda \to 1$ limit w.r.t the wedge algebra norm, and in particular

$$\lim_{\lambda \to 1} \left\| \frac{K}{1-\Omega} - \frac{K}{1-\lambda\Omega} \right\| = 1$$

due to the fact that $\frac{K}{1-\lambda\Omega}$ vanishes at $K=0$ for all $\lambda \neq 1$ but $\frac{K}{1-\Omega}$ is 1 at $K=0$.



Note that the $\mathcal{L}_0$ level expansion represents Schnabl's solution as a unit strip containing an infinite number of energy-momentum contour insertion. Such a representation can be useful for some purposes, but is not always appropriate. Another approach is to represent $\frac{K}{1-\Omega}$ as a geometric series in Ω. More precisely, we have the following identity:

$$\frac{K}{1-\Omega} = \sum_{n=0}^{N} K\Omega^n + \frac{K}{1-\Omega}\,\Omega^{N+1}$$

In the last term we may expand using Bernoulli numbers:

$$\frac{K}{1-\Omega} = \sum_{n=0}^{N} K\Omega^n + \sum_{\ell=0}^{\infty} \frac{(-)^\ell B_\ell}{\ell!} K^\ell \Omega^{N+1}$$

Note that in the limit of large N, Ω^{N+1} approaches the sliver state and higher powers of K acting on Ω^{N+1} are suppressed. For the calculations that concern us the higher powers of K can be ignored, and we can replace the Bernoulli sum with its leading term $\frac{(-1)^0 B_0}{0!} K^0 = 1$. Therefore

$$\frac{K}{1-\Omega} = \lim_{N\to\infty}\left[\sum_{n=0}^{N} K\Omega^n + \Omega^{N+1}\right]$$

and Schnabl's solution is expressed

$$\Psi = \lim_{N\to\infty}\left[\sum_{n=0}^{N}\sqrt{\Omega}\,cKB\Omega^n c\sqrt{\Omega} + \sqrt{\Omega}\,cB\Omega^{N+1}c\sqrt{\Omega}\right]$$

The last term in this expression is called the "phantom term," and is a source of a lot of puzzlement about Schnabl's solution. On the one hand, as follows from the computation of Excercise 7, the phantom term vanishes as a state in the Fock space:

$$\langle\phi|\sqrt{\Omega}\,cB\Omega^{N+1}c\sqrt{\Omega}\rangle = \mathcal{O}\left(\frac{1}{N^3}\right) \qquad |\phi\rangle = \text{Fock space state.}$$

On the other hand, the phantom term is essentially what is responsible for the physics of the solution — it is the origin of the zero momentum tachyon contribution in the $\mathcal{L}_0$ level expansion. Note that for $\lambda \neq 1$ the KBc Schnabl gauge solution can be written in a similar form

$$\Psi = \sum_{n=0}^{\infty} \lambda^{n+1}\sqrt{\Omega}\,cKB\Omega^n c\sqrt{\Omega}$$

This works for $|\lambda| < 1$, since otherwise the geometric series is divergent. However, since the phantom term vanishes in the Fock space, the limit $\lambda \to 1$ is perfectly smooth in the Fock space. Thus it seems that Schnabl's solution is "close" to being pure gauge.

It was realized by Okawa that Schnabl gauge KBc solutions can be expressed explicitly as a finite gauge transformation of the perturbative vacuum

$$\Psi = (1-\lambda\sqrt{\Omega}Bc)Q\frac{1}{1-\lambda\sqrt{\Omega}Bc\sqrt{\Omega}}$$

and this converges to Schnabl's solution in the Fock space in the limit $\lambda \to 1$.

[Excercise 8: Prove this formula]

The problem with the pure gauge expression as $\lambda \to 1$ is that the inverse gauge parameter

$$\frac{1}{1-\lambda\sqrt{\Omega}\,Bc\sqrt{\Omega}} = 1 + \lambda\sqrt{\Omega}\,\frac{1}{1-\lambda\Omega}\,Bc\sqrt{\Omega}$$

develops a pole at $K=0$. Therefore, the finite gauge transformation relating Schnabl's solution to the perturbative vacuum is singular. There is a close relationship between this singularity and the presence of a phantom term in the solution, which we will describe later

Now that we have an analytic solution for the tachyon vacuum, we would like to evaluate the action to reproduce the D-brane tension. The computation, as originally given by Schnabl, is unfortunately a little too complicated to present here. Instead, we will give a computation of the Ellwood invariant.

$$\mathrm{Tr}_V[\Psi] = \lim_{N\to\infty}\left[\sum_{n=0}^{N}\mathrm{Tr}_V[\sqrt{\Omega}\,cKB\Omega^n c\sqrt{\Omega}] + \mathrm{Tr}_V[\sqrt{\Omega}\,cB\Omega^{N+1}c\sqrt{\Omega}]\right]$$

Let us first deal with the terms in the sum. They also appear in the KBc Schnabl gauge solutions when $\lambda \neq 1$:

$$\sum_{n=0}^{\infty}\lambda^{n+1}\,\mathrm{Tr}_V[\sqrt{\Omega}\,cKB\Omega^n c\sqrt{\Omega}]$$

However, we know that the Schnabl gauge KBc solutions are pure gauge for $\lambda\neq 1$, and the Ellwood invariant must therefore vanish. Therefore, the above expression vanishes order-by-order in λ, which implies

$$\mathrm{Tr}_V[\sqrt{\Omega}\,cKB\Omega^n c\sqrt{\Omega}] = 0$$

Therefore the only nonvanishing contribution to the Ellwood invariant of Schnabl's solution must come from the phantom term

$$\mathrm{Tr}_V[\Psi] = \lim_{N\to\infty}\mathrm{Tr}_V[\sqrt{\Omega}\,cB\Omega^{N+1}c\sqrt{\Omega}]$$

$$= \lim_{N\to\infty}\mathrm{Tr}_V[c\Omega cB\Omega^{N+1}]$$

To simplify the ghost correlator, we note the following identities:

$$\mathrm{Tr}_V[c\Omega cB\Omega^{N+1}] + \mathrm{Tr}_V[c\Omega Bc\Omega^{N+1}] - \mathrm{Tr}_V[c\Omega^{N+2}] = 0$$

$$\mathrm{Tr}_V[\tfrac{1}{2}\mathcal{B}^-(c\Omega c\Omega^{N+1})] = \mathrm{Tr}_V[cB\Omega c\Omega^{N+1}] - \mathrm{Tr}_V[c\Omega cB\Omega^{N+1}](N+1) = 0$$

Subtracting the second equation from the first gives

$$(N+2)\,\mathrm{Tr}_V[c\Omega c\Omega^{N+1}] - \mathrm{Tr}_V[c\Omega^{N+2}] = 0$$

Therefore

$$\mathrm{Tr}_V[\Psi] = \lim_{N\to\infty}\frac{1}{N+2}\,\mathrm{Tr}_V[c\Omega^{N+2}]$$

Using $\frac{1}{2}\mathcal{L}^-$ invariance of the trace,

$$\mathrm{Tr}_V[\Psi] = \lim_{N\to\infty}\frac{1}{N+2}\,\mathrm{Tr}_V\left[\left(\frac{1}{N+2}\right)^{\frac{1}{2}\mathcal{L}^-}(c\Omega^{N+2})\right] = \lim_{N\to\infty}\frac{N+2}{N+2}\,\mathrm{Tr}_V[c\Omega] = \mathrm{Tr}_V[c\Omega]$$

The final trace can be written as a correlation function on a cylinder of circumference 1, which can be mapped to the unit disk using

$$f(z) = e^{2\pi i z}$$

This gives

$$\begin{aligned}
\mathrm{Tr}_V[\Psi] &= \langle c\bar{c} V^m(i\infty)\, c(0) \rangle_{C_1} \\
&= \frac{1}{2\pi i} \langle c\bar{c}\, V^m(0)\, c(1) \rangle_{\text{disk}} \\
&= -\frac{1}{2\pi i} \langle V^m(0) \rangle^{\text{matter}}_{\text{disk}} \qquad = -A_0(V)
\end{aligned}$$

where in the last step we used the fact that the ghost correlator evaluates to -1. This is precisely the difference in the closed string tadpole amplitude between the tachyon vacuum (where the tadpole vanishes) and the perturbative vacuum.

Schnabl gauge Marginal solutions We now describe analytic solutions for marginal deformations; these correspond to deformations of the reference D-brane given by moving along flat directions in the string field potential. At linearized order, such solutions are represented by a nontrivial element of the BRST cohomology, which we will assume takes the form

$$\Psi = cV(0)|0\rangle + \text{nonlinear corrections}$$

where $V(z)$ is a boundary matter primary operator of weight 1. If we introduce a string field V as an infinitessimally thin strip with boundary insertion of $V(z)$, we can write

$$\Psi = \sqrt{\Omega}\, cV \sqrt{\Omega} \;+\; \text{nonlinear corrections}$$

Some important properties of V are

$$Q(cV) = 0 \qquad \tfrac{1}{2}\mathcal{L}^- V = V \qquad \tfrac{1}{2}\mathcal{B}^- V = 0$$

The second property says that V has scaling dimension 1, and the third property follows because $V(x)$ is a matter operator. Often we multiply V by a constant λ, corresponding to the expectation value of the field generated by the vertex operator $V(x)$. We will absorb this constant into the normalization of V, to avoid writing too many λs in formulas. The solution can be expanded perturbatively

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \ldots$$

where Ψ_n contains n insertions of V; they represent the nonlinear corrections that account for the fact that the field generated by cV has finite expectation value. Matching terms that contain the same number of V's the EOM imply:

$$Q\Psi_1 = 0$$

$$Q\Psi_2 + \Psi_1^2 = 0$$

$$Q\Psi_3 + \Psi_1\Psi_2 + \Psi_2\Psi_1 = 0$$

$$\vdots$$

There may be an obstruction to solution of these equations if the quadratic terms containing lower order corrections are not BRST exact. The physical interpretation of this obstruction is that the potential for the field cV is not exactly flat; a finite expectation value for the field is not at a stationary point of the potential. If the construction fails to find a solution for Ψ_n, that means that the potential goes roughly as λ^{n+1} for small λ. If the obstruction is absent, then the deformation generated by cV is called "exactly marginal." We assume that cV generates an exactly marginal deformation in what follows.

We look for an analytic solution for the marginal deformation in Schnabl gauge:

$$\mathcal{B}_0\Psi_n = 0$$

Acting $\mathcal{B}_0$ on the EOM and using $[Q,\mathcal{B}_0] = \mathcal{L}_0$, we obtain a recursive set of equations for the corrections of the form

$$\mathcal{L}_0\Psi_n + \mathcal{B}_0(\text{lower order corrections}) = 0$$

If the second term in this equation does not produce states in the kernel of $\mathcal{L}_0$, we can invert $\mathcal{L}_0$ to obtain an explicit formula for Ψ_n. Let us work this out for the second order correction:

$$\Psi_2 = -\frac{\mathcal{B}_0}{\mathcal{L}_0}\Psi_1^2$$

Substituting $\Psi_1 = \sqrt{\Omega}\,cV\sqrt{\Omega}$ and representing the inverse of $\mathcal{L}_0$ using the Schwinger parameterization, this is straightforward to compute:

$$\Psi_2 = -\frac{\mathcal{B}_0}{\mathcal{L}_0}\sqrt{\Omega}\,cV\,\Omega\,cV\sqrt{\Omega}$$

$$= \frac{-1}{\mathcal{L}_0}\sqrt{\Omega}\,cVB\Omega\,cV\sqrt{\Omega}$$

$$= -\int_0^\infty dt\, e^{-t\mathcal{L}_0}\left(\sqrt{\Omega}\,cVB\Omega\,cV\sqrt{\Omega}\right)$$

$$= -\int_0^\infty dt\, \sqrt{\Omega}\, e^{-t\frac{1}{2}\mathcal{L}^-}(cVB\Omega\,cV)\sqrt{\Omega}$$

$$= -\int_0^\infty dt\, e^{-t}\sqrt{\Omega}\,cVB\,\Omega^{e^{-t}}cV\sqrt{\Omega}$$

Substitute $\alpha = e^{-t}$:

$$\Psi_2 = -\int_0^1 d\alpha\,\sqrt{\Omega}\,cVB\Omega^{\alpha}cV\sqrt{\Omega}$$

You might recognize this integral from the homotopy operator for Schnabl's solution.

We have

$$\Psi_2 = -\sqrt{\Omega}\, cVB\frac{1-\Omega}{K}cV\sqrt{\Omega}$$

Let us mention a puzzle. Usually marginal operators, being dimension 1 primaries have a double pole in their OPE proportional to the identity operator:

$$V(x)V(0) = \frac{1}{x^2} + \cdots$$

However, the strip seperating the two V insertions in Ψ_2 can be arbitrarily thin, and the OPE of Vs will produce a divergence in Ψ_2. The interpretation of this is fairly clear: apparently, $\mathcal{B}_0\Psi_1^2$ contains states in the kernel of $\mathcal{L}_0$. The puzzle is that normally this would indicate that the deformation is not exactly marginal. In Siegel gauge, if $b_0\Psi_1^2$ has a state in the kernel of L_0, Ψ_1^2 is not BRST exact and Ψ_2 doesn't exist. However, we know that there are plenty of exactly marginal operators with double pole OPEs — it is the generic expectation. The resolution to this puzzle is that $\mathcal{L}_0$ — unlike L_0 — has a state in its kernel which is trivial in the BRST cohomology

$$\sqrt{\Omega}\, cKBc\sqrt{\Omega} = Q(\sqrt{\Omega}\, Bc\sqrt{\Omega})$$

Therefore, it is possible that we can have an obstruction to solution in Schnabl gauge which does not indicate an obstruction to solution in other gauges. However, since we want a solution in Schnabl gauge we assume that all poles in the OPE of two Vs are absent

$$V(x)V(0) = \text{regular as } x\to 0$$

To avoid problems at higher orders, we will in fact assume that all powers of the string field V are finite. This assumption is not as constraining as it might seem at first. One example of V with regular OPE is the rolling tachyon deformation:

$$V(x) = e^{X^0}(x) \qquad e^{X^0}(x)\, e^{X^0}(0) = |x|^2 :e^{2X^0}(0): + \cdots$$

This represents a time-dependent solution where the reference D-brane decays starting from an infinitessimal, homogeneous tachyon fluctuation in the infinite past. For ~~margin time ide~~ time independent marginal deformations there is a trick for obtaining a solution in Schnabl gauge: Given V with a double pole (and only a double pole) in the OPE with itself with unit coefficient, we may consider a modified marginal operator

$$V'(x) = V(x) + i\partial X^0(x)$$

This has regular self OPE since the double pole of ∂X^0-∂X^0 cancels that of V-V, and V-∂X^0 is regular since V is independent of the X^0 BCFT. Technically, this deformation turns on the field corresponding to $V(x)$

in addition to a timelike gauge field A_0 on the D-brane. However, in physical configurations the timelike direction on the D-brane worldvolume is noncompact, and a constant, timelike gauge field A_0 can be removed by gauge transformation. Therefore the deformation generated by V' is physically indistinguishable from that of V.

In any case, to proceed in Schnabl gauge we must assume marginal operators with regular self OPE. Then Ψ_2 is well-defined and we can proceed to higher order. We will simply quote the result:

$$\Psi_{n+1} = \sqrt{\Omega}\, cV\left(B\frac{1-\Omega}{K}cV\right)^n\sqrt{\Omega}\,(-)^{n+1}$$

Adding all corrections gives the full solution:

$$\Psi = \sqrt{\Omega}\, cV\frac{1}{1+B\frac{1-\Omega}{K}cV}\sqrt{\Omega}$$

$$= \sqrt{\Omega}\, cV\frac{B}{1+\frac{1-\Omega}{K}V}c\sqrt{\Omega}$$

In the last step we simplified Bc insertions using $[B,c]=1$.

[Exercise 8: Prove that this solution satisfies the EOM.]

With the Schnabl gauge marginal solution constructed, we can try to compute some observables. The action is not very interesting to compute since a marginal solution moves the string field along a flat direction in the potential; the value of the action does not change. More interesting is the Ellwood invariant

$$\mathrm{Tr}_{\mathcal{V}}[\Psi] = A_*(\mathcal{V}) - A_0(\mathcal{V})$$

Let us explain the expected result. The solution shifts the background from BCFT_0 to a marginally deformed background BCFT_*. The closed string tadpole in the new background is computed by a matter 1-point function on the disk in BCFT_*:

$$A_*(\mathcal{V}) = \frac{1}{2\pi i}\langle \mathcal{V}^m(0,0)\rangle^{\text{matter}}_{\text{disk, BCFT}_*}$$

The question is how this correlator is related to that of BCFT_0. The matter correlator on the disk in BCFT_* can be computed as a path integral

$$\langle \ldots \rangle^{\text{matter}}_{\text{disk, BCFT}_*} = \int_{|z|<1}[dX(z,\bar{z})]\, e^{-S_{\text{disk}} - \int_0^{2\pi} d\theta\, V(\theta)}(\ldots)$$

where S_{disk} is the Polyakov action on the unit disk and the additional term $\int_0^{2\pi} d\theta\, V(\theta)$ is a boundary interaction representing the shift in boundary condition implemented by the marginal operator V. This boundary interaction can be viewed either as part of the worldsheet action, or as an operator insertion inside the path integral of BCFT_0. Therefore we expect

$$\langle \dots \rangle^{\text{matter}}_{\text{disk, BCFT}_*} = \left\langle \exp\left[-\int_0^{2\pi} d\theta\, V(\theta)\right] \dots \right\rangle^{\text{matter}}_{\text{disk, BCFT}_0}$$

One subtlety is that the exponential operator typically needs renormalization to make sense in BCFT_0. However, for Schnabl gauge marginal deformations all OPEs of V are finite, so renormalization is not necessary. Therefore the expected result for the Ellwood invariant is

$$\mathrm{Tr}_{\mathcal{V}}[\Psi] = \frac{1}{2\pi i}\left\langle \left(\exp\left[-\int_0^{2\pi} d\theta\, V(\theta)\right] - 1\right) \mathcal{V}^m(0,0)\right\rangle^{\text{matter}}_{\text{disk, BCFT}_0}$$

This result was proven in a paper by Kishimoto, though the paper looks complicated and the notation cumbersome; I have not really read it, but I have always felt that a more elegant demonstration is possible using the streamlined algebraic formalism we have been developing. It would be great if someone here could give a nice proof, and explain their calculation to me (Excercise).

Simple solution It is interesting to look for other tachyon vacuum solutions in the KBc subalgebra. With some analysis, it is possible to prove that the general tachyon vacuum takes the form

$$\Psi = T\frac{KB}{1-F(K)}T + Q(BT)$$

where T is the "zero momentum tachyon" state characterizing the solution. For Schnabl's solution, T is litterally the zero-momentum tachyon:

$$T = \sqrt{\Omega}\, c\sqrt{\Omega} = \frac{2}{\pi} c_1|0\rangle \qquad \text{(Schnabl's solution)}$$

and for more general solutions it may take the form

$$T = \int_0^\infty d\alpha \int_0^\infty d\beta\, f(\alpha,\beta)\, \Omega^\alpha c\, \Omega^\beta$$

$F(K)$ is the "vacuum state" of the solution, and is related to T through

$$[B, T] = F(K)$$

For Schnabl's solution, this is litterally the $SL(2,\mathbb{R})$ vacuum state:

$$F(K) = \Omega \qquad \text{(Schnabl's solution)}$$

To get a well-defined tachyon vacuum, it seems necessary that the following states are bounded, continuous functions of $K \geq 0$:

$$\frac{KF(K)}{1-F(K)} \qquad \frac{1-F(K)}{K}$$

The first should be a good state since it appears from computing $B\Psi B$; The second should be a good state since

$$A = B\frac{1-F(K)}{K}$$

defines the homotopy operator which implies the absence of open string states.

The structure of Schnabl's solution is fairly representative of the generic tachyon vacuum in the KBc subalgebra. Given this general structure, it is interesting to ask whether there is a simplest possible realization. The tachyon vacuum defines two distinguished states in the wedge algebra: the "vacuum" state $F(K)$ and the state $\frac{1-F(K)}{K}$ which appears in the homotopy operator. We can define a solution by requiring them to be equal:

$$F(K) = \frac{1-F(K)}{K}$$

This implies

$$F(K) = \frac{1}{1+K}$$

A convenient choice of the zero momentum tachyon state is

$$T = c\frac{1}{1+K}$$

The solution is then

$$\Psi = c(1+K)Bc\frac{1}{1+K}$$

I call this the "simple" tachyon vacuum solution. We can define $\frac{1}{1+K}$ via the Schwinger parameterization

$$\frac{1}{1+K} = \int_0^\infty d\alpha\, e^{-\alpha}\,\Omega^\alpha$$

The explicit definition of the solution does not require a phantom term. However, the solution is not quite as well-behaved as Schnabl's solution in the Fock space expansion. One can compute the energy of the solution by dropping Fock space component fields with masses above a fixed integer L, plugging into the action, and taking the limit $L\to\infty$. For Schnabl's solution, this procedure appears to converge on the correct answer, whereas for the simple solution it does not. Instead, the sum of contributions from each mass level gives a divergent series. but the series can be resummed to give the correct answer. The origin of this behavior is that the simple solution involves a continuous superposition of wedge states all the way down to the identity string field; the identity string field is just at the border between normalizable and non-normalizable Gaussian functionals, and is poorly behaved as a state in the Fock space expansion.

One great advantage of the simple solution, however, is that the analytic calculation of the energy is very easy. Assuming that the string field Ψ satisfies the EOM, the action simplifies to

$$S = -\frac{1}{2}\mathrm{Tr}[\Psi Q\Psi] - \frac{1}{3}\mathrm{Tr}[\Psi^3] = \left(-\frac{1}{2}+\frac{1}{3}\right)\mathrm{Tr}[\Psi Q\Psi] = -\frac{1}{6}\mathrm{Tr}[\Psi Q\Psi]$$

If we normalize the spacetime volume to unity, this should give the energy (or tension) of the D-brane before tachyon condensation. Normalizing the spacetime volume to 1 implies that the vacuum matter correlator in the UHP is equal to 1, so the totall correlator with 3 c-ghost insertions is given by

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{UHP} = (z_1-z_2)(z_2-z_3)(z_1-z_3)$$

We write the simple solution as

$$\Psi = c\frac{1}{1+K} + Q\left(Bc\frac{1}{1+K}\right)$$

Plugging into the action, the BRST exact term drops out since $Q^2=0$ and BRST invariance of the trace. Then

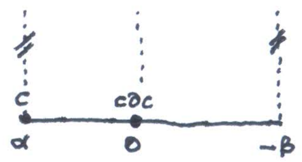
$$S = -\frac{1}{6}\mathrm{Tr}[\Psi Q\Psi]$$

$$= -\frac{1}{6}\mathrm{Tr}\left[c\frac{1}{1+K}c\partial c\frac{1}{1+K}\right]$$

$$= -\frac{1}{6}\int_0^\infty d\alpha\int_0^\infty d\beta\ \mathrm{Tr}[c\Omega^\alpha c\partial c\Omega^\beta]\,e^{-\alpha-\beta}$$

The trace can be computed as a correlator on a cylinder of circumference $\alpha+\beta$, which can be mapped to the UHP using

$$\tan\frac{\pi z}{\alpha+\beta}$$



Fixing the origin on $c\partial c$, this gives

$$\mathrm{Tr}[c\Omega^\alpha c\partial c\Omega^\beta] = \langle c(\alpha)c\partial c(0)\rangle_{C_{\alpha+\beta}}$$

$$= \left(\frac{\pi}{\alpha+\beta}\sec^2\frac{\pi\alpha}{\alpha+\beta}\right)^{-1}\left(\frac{\pi}{\alpha+\beta}\sec^2 0\right)^{-1}\left\langle c\left(\tan\frac{\pi\alpha}{\alpha+\beta}\right)c\partial c(0)\right\rangle_{UHP}$$

Using

$$\langle c(z_1)c\partial c(z_2)\rangle_{UHP} = -(z_1-z_2)^2$$

gives

$$\mathrm{Tr}[c\Omega^\alpha c\partial c\Omega^\beta] = \left(\frac{\alpha+\beta}{\pi}\right)^2\cos^2\frac{\pi\alpha}{\alpha+\beta}\cdot\left(-\tan^2\frac{\pi\alpha}{\alpha+\beta}\right)$$

$$= -\left(\frac{\alpha+\beta}{\pi}\right)^2\sin^2\frac{\pi\alpha}{\alpha+\beta}$$

This produces the integral

$$S = \frac{1}{6}\int_0^\infty d\alpha\int_0^\infty d\beta\left(\frac{\alpha+\beta}{\pi}\right)^2 e^{-(\alpha+\beta)}\sin^2\frac{\pi\alpha}{\alpha+\beta}$$

The form of the integral suggests a substitution

$$L = \alpha+\beta \in [0,\infty] \qquad d\alpha\, d\beta = \frac{1}{\pi}L\,dL\,d\theta$$
$$\theta = \frac{\pi\alpha}{\alpha+\beta} \in [0,\pi]$$

giving

$$S = \frac{1}{6\pi}\int_0^\infty dL\int_0^\pi d\theta\ \frac{L^3}{\pi^2}e^{-L}\sin^2\theta$$

$$= \frac{1}{6\pi^3}\left(\int_0^\infty dL\, L^3 e^{-L}\right)\left(\int_0^\pi d\theta\,\sin^2\theta\right)$$

$$= \frac{1}{6\pi^3} \cdot 3! \cdot \frac{\pi}{2}$$

$$= \frac{1}{2\pi^2}$$

This confirms Sen's conjecture.

<u>KOS solution</u> A similar simplification of the Schnabl-gauge marginal solutions was given by Kiermaier, Okawa, and Soler, and turns out to have far-reaching consequences. We will present the solution with some refinements following work by C. Maccaferri and myself.

The simple solution is related to the Schnabl gauge tachyon vacuum by replacing Ω with $\frac{1}{1+K}$. A similar replacement is also possible for the Schnabl gauge marginal solution, giving

$$\Psi = cV \frac{B}{1+\frac{1}{1+K}V} c \frac{1}{1+K}$$

In this solution it is also necessary to assume that the Vs can be multiplied without singularity. This unassuming expression can be turned into something profound with a few simple algebraic manipulations whose significance and necessity are not easy to see at first. It goes as follows:

$$\Psi = cV \frac{B}{\frac{1}{1+K}(1+K+V)} c \frac{1}{1+K}$$

$$= cV \frac{B}{1+K+V}(1+K) c \frac{1}{1+K}$$

$$= c\big(1+K+V-(1+K)\big)\frac{B}{1+K+V}(1+K)c\frac{1}{1+K}$$

$$= \underbrace{c(1+K)Bc\frac{1}{1+K}}_{\Psi_{tv}} - \underbrace{c(1+K)\frac{B}{1+K+V}(1+K)c\frac{1}{1+K}}_{\Psi_*}$$

The first term is the simple tachyon vacuum. Since Ψ is a solution, the second term is a solution to the equations of motion around the tachyon vacuum:

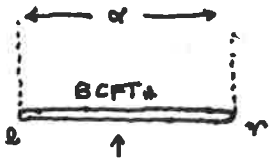
$$Q_{\Psi_{tv}}\Psi_* + \Psi_*^2 = 0$$

In a sense, the first term destroys the reference D-brane, and the second term creates a marginally deformed D-brane out of the tachyon vacuum.

To make sense of the solution in this form, it is necessary to give a concrete definition to the state $\frac{1}{1+K+V}$. Using the Schwinger parameterization we can write

$$\frac{1}{1+K+V} = \int_0^\infty d\alpha\, e^{-\alpha} e^{-\alpha(K+V)}$$

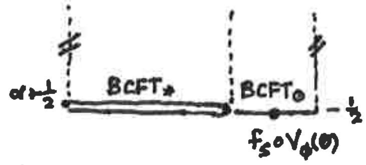
The string field in the integrand looks somewhat like a wedge state, except that the combination $K+V$ appears in the exponential. We claim that this is a wedge state containing an exponential insertion of line integrals of V on the boundary:

$$e^{-\alpha(K+V)} =$$ 

$$\exp\left[-\int_r^l dx\, V(x)\right]$$

The cumulative effect of this operator insertion is to deform the boundary condition from the reference D-brane $BCFT_0$ into a new D-brane $BCFT_*$, related to $BCFT_0$ by marginal deformation through the marginal operator $V(x)$. Let us prove this relation. Suppose that Ω_*^α is a wedge state defined with the boundary conditions of $BCFT_*$. Contract with a test state and compute

$$\frac{d}{d\alpha}\langle\phi|\Omega_*^\alpha\rangle = \frac{d}{d\alpha}\mathrm{Tr}[\Omega_*^\alpha\,\phi]$$

$$= \frac{d}{d\alpha}\left\langle \exp\left[-\int_{1/2}^{\alpha+1/2} dx\, V(x)\right] f_S\circ V_\phi(0)\right\rangle_{C_{\alpha+1}}$$ 

The ~~integran~~ derivative first acts on the exponential, bringing down an insertion of $V(\alpha+\frac{1}{2})$; it also acts on the cylinder circumference. In the coordinate system we have chosen, this shifts the leftmost boundary of the strip (leaving the operator insertions unchanged). This shift can be implemented by a vertical contour insertion of the energy-momentum tensor intersecting the real axis at $\alpha+\frac{1}{2}$. Therefore

$$\frac{d}{d\alpha}\langle\phi|\Omega_*^\alpha\rangle = \left\langle\left(-V(\alpha+\tfrac{1}{2}) - \int_{-i\infty+\alpha+\frac{1}{2}}^{i\infty+\alpha+\frac{1}{2}}\frac{dz}{2\pi i}T(z)\right)\exp\left[-\int_{1/2}^{\alpha+1/2}dx\, V(x)\right] f_S\circ V_\phi(0)\right\rangle_{C_{\alpha+1}}$$

which implies

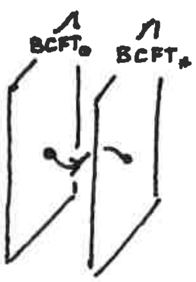
$$\frac{d}{d\alpha}\Omega_*^\alpha = -(K+V)\,\Omega_*^\alpha$$

and

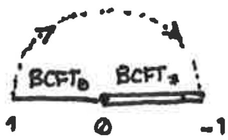
$$\Omega_*^\alpha = e^{-\alpha(K+V)}$$

as claimed.

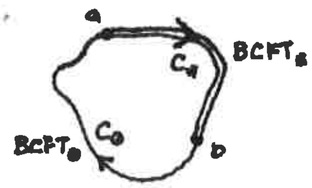
It will be helpful to adopt a slightly different language for describing the boundary interaction. Consider an open string connecting a D-brane $BCFT_0$ to another D-brane $BCFT_*$. From the point of view of radial quantization



such an open string can be associated to ~~a or~~ the semicircle bounding a half disk with $BCFT_0$ boundary conditions on the positive real axis and $BCFT_*$ boundary conditions on the negative real axis.



It is natural to think of this state as being created by a vertex operator, which somehow changes the boundary condition from $BCFT_0$ to $BCFT_*$. This is called a boundary condition changing operator (bcc operator), which we denote as $\sigma(0)$. There is also a bcc operator which shifts the boundary condition from BCFT_* to BCFT_0, which we write as $\bar{\sigma}(0)$. Consider a surface Σ with two boundary components C_0 and C_*, carrying respectively BCFT_0 and BCFT_* boundary conditions; moving clockwise, let a and b be the ~~two~~ points on the boundary at the junction of C_0 and C_* and at the junction of C_* and C_0, respectively. The bcc operators are related to the boundary interaction through



$$\bar{\sigma}(b)\sigma(a) = \exp\left[-\int_{C_*} dz\, V(z)\right]$$

We can learn a few things from this identification. If b approaches a from a ~~counter~~clockwise direction, the whole boundary of Σ carries an exponential insertion of line integrals of V. From the point of view of an open string in BCFT_*, this is simply a trivial insertion of the identity operator

$$\lim_{b\to a} \bar{\sigma}(b)\sigma(a) = \exp\left[-\int_{\partial\Sigma} dz\, V(z)\right] = 1_{\mathrm{BCFT}_*}$$

If a approaches b from a ~~counter~~clockwise direction, the boundary interaction shrinks to nothing and we obtain

$$\lim_{a\to b} \sigma(a)\bar{\sigma}(b) = 1_{\mathrm{BCFT}_0}$$

Note that these properties hold only because the OPEs of $V(z)$ are finite. If they were not finite, the boundary interaction would need to be defined in some renormalization scheme; and in general it will not be possible to renormalize so that the limits $a\to b$ and $b\to a$ remain finite. In a sense, σ and $\bar{\sigma}$ develop singularities in their OPE. It is also clear that the boundary interaction transforms in a trivial way under conformal transformations of Σ:

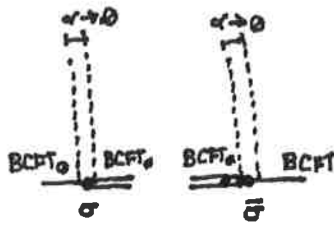
$$f\circ \exp\left[-\int_{C_*} dz\, V(z)\right] = \exp\left[-\int_{f\circ C_*} dz\, V(z)\right]$$

This implies

$$f\circ\left[\bar{\sigma}(b)\sigma(a)\right] = \bar{\sigma}(f(b))\,\sigma(f(a))$$

Therefore $\sigma, \bar{\sigma}$ are primary operators of weight 0. Again, this is true because the OPEs of V are nonsingular. If renormalization was necessary, the renormalization schemes in different conformal frames would be nontrivially related, which would have the effect of implying nontrivial transformation properties of $\sigma, \bar{\sigma}$. An important advantage of the bcc operator point of view is that it is universal: for any BCFT_0 and BCFT_*, regardless of whether they are ~~are~~ related by marginal deformation, there are bcc operators $\sigma, \bar\sigma$ relating them. This is because for any pair of D-branes, the spectrum of excitations always includes open strings which connect one D-brane to another. However, it is not generally true that $\sigma, \bar\sigma$ are dimension 0 primaries which multiply to give 1; it is also not generally true that a pair of bcc insertions can be represented by insertion of a boundary interaction between the locations of σ and $\bar\sigma$ — at least not a boundary interaction that can be explicitly defined by renormalization of operators in $BCFT_0$. These additional properties follow from the fact $BCFT_0$ and $BCFT_*$, in our case, are related by a nonsingular marginal deformation.

With this motivation, it is natural to describe the solution in terms of $\sigma, \bar\sigma$, rather than V. We introduce two string fields σ and $\bar\sigma$, defined by infinitessimally thin strips containing boundary insertions of $\sigma(x)$, $\bar\sigma(x)$. The string fields have the properties



$$\bar\sigma\sigma = \sigma\bar\sigma = 1 \qquad [B,\sigma] = [c,\sigma] = 0$$
$$[B,\bar\sigma] = [c,\bar\sigma] = 0$$

Since $\sigma, \bar\sigma$ represent insertions of dimension 0 matter primaries,

$$\tfrac{1}{2}\mathcal{L}^-\sigma = \tfrac{1}{2}\mathcal{L}^-\bar\sigma = 0 \qquad Q\sigma = c\partial\sigma \qquad Q\bar\sigma = c\partial\bar\sigma$$

where ∂ is equivalent to a commutator with K. This gives two ways of representing the wedge states of $BCFT_*$:

$$\Omega_*^\alpha = e^{-\alpha(K+V)} = \sigma\Omega^\alpha\bar\sigma$$

[Excercise 9: Show that V and $\sigma, \bar\sigma$ are related by $V = \sigma\partial\bar\sigma$. Use this to prove the above relation] The KOS marginal solution can then be expressed

$$\Psi = \Psi_{tv} + \Psi_* = c(1+K)Bc\frac{1}{1+K} - c(1+K)\sigma\frac{B}{1+K}\bar\sigma(1+K)c\frac{1}{1+K}$$

We are not done yet. It will be interesting and useful to introduce a trivial factor of the simple tachyon vacuum between σ and $\bar\sigma$ with the replacement:

$$\frac{B}{1+K} = \frac{B}{1+K}c(1+K)Bc\frac{B}{1+K}$$

The second term in the solution can then be expressed

$$\Psi_* = cB(1+K)\sigma\frac{1}{1+K}\left(-c(1+K)Bc\frac{1}{1+K}\right)\bar\sigma(1+K)Bc\frac{1}{1+K}$$

Now we make the following observations. Ψ_* is a state in $BCFT_0$, which describes the creation of the D-brane $BCFT_*$ out of the tachyon vacuum.

The factor in parentheses is a state in $BCFT_*$; since it is minus the simple tachyon vacuum, it describes the creation of the D-brane $BCFT_*$ out of the tachyon vacuum. That is the states

$$\Psi_* \in BCFT_0 \qquad -c(1+K)Bc\frac{1}{1+K} \in BCFT_*$$

physically mean the same thing; they are just expressed using the degrees of freedom of different D-branes. This suggests that the factors containing σ and $\bar\sigma$ on either side of the simple tachyon vacuum serve as a kind of dictionary between the fluctuation fields of $BCFT_0$ and $BCFT_*$. We write these factors as Σ and $\bar\Sigma$, so that Ψ_* takes the form

$$\Psi_* = \Sigma(-\Psi_{tv})\bar\Sigma$$

The fact that Ψ_* and $-\Psi_{tv}$ are solutions to the equations of motion around the tachyon vacuum in the respective BCFTs leads to the following properties:

$$Q_{\Psi_{tv}}\Sigma = Q_{\Psi_{tv}}\bar\Sigma = 0 \qquad \bar\Sigma\Sigma = 1$$

One can show that $\Sigma, \bar\Sigma$ can be defined as

$$\Sigma = Q_{\Psi_{tv}}\left(\sigma\frac{B}{1+K}\right) = cB(1+K)\sigma\frac{1}{1+K} + \sigma Bc$$

$$\bar\Sigma = Q_{\Psi_{tv}}\left(\bar\sigma\frac{B}{1+K}\right) = \bar\sigma(1+K)Bc\frac{1}{1+K} + (1+K)\bar\sigma cB\frac{1}{1+K}$$

Note that only the first terms above respectively appear in Ψ_* ; the second terms drop out due to $c^2=0$. In this form it is manifest that Σ and $\bar\Sigma$ are annihilated by $Q_{\Psi_{tv}}$; we may also compute

$$\begin{aligned}
\bar\Sigma\Sigma &= Q_{\Psi_{tv}}\left(\bar\sigma\frac{B}{1+K}\right)Q_{tv}\left(\sigma\frac{B}{1+K}\right) \\
&= Q_{\Psi_{tv}}\left(\bar\sigma\frac{B}{1+K}Q_{\Psi_{tv}}\left(\sigma\frac{B}{1+K}\right)\right) \\
&= Q_{\Psi_{tv}}\left(-\bar\sigma\underbrace{Q_{\Psi_{tv}}\left(\frac{B}{1+K}\sigma\frac{B}{1+K}\right)}_{0} + \bar\sigma\underbrace{Q_{\Psi_{tv}}\left(\frac{B}{1+K}\right)}_{1}\sigma\frac{B}{1+K}\right) \\
&= Q_{\Psi_{tv}}\left(\underbrace{\bar\sigma\sigma}_{1}\frac{B}{1+K}\right) \\
&= 1
\end{aligned}$$

In summary, we have found that the KOS solution can be expressed as

$$\Psi = \Psi_{tv} - \Sigma\Psi_{tv}\bar\Sigma \qquad \Psi_{tv} = c(1+K)Bc\frac{1}{1+K}$$

$$\Sigma = Q_{\Psi_{tv}}\left(\sigma\frac{B}{1+K}\right)$$

$$\bar\Sigma = Q_{\Psi_{tv}}\left(\bar\sigma\frac{B}{1+K}\right)$$

To reiterate, the first term destroys the D-brane of $BCFT_0$; the second term creates the D-brane of $BCFT_*$ out of the tachyon vacuum of $BCFT_*$, and reexpresses it in terms of ~~the~~ $BCFT_0$ degrees of freedom.

The thing that impresses about this solution is that it encapsulates a general structure which could apply to any reference $BCFT_0$ and any target $BCFT_*$. The difficult part is implementing the condition

$$\bar{\Sigma}\Sigma = 1$$

which is directly related to a condition on the bcc operators

$$\bar{\sigma}\sigma = 1$$

This condition is satisfied for nonsingular marginal deformations, but is rather unnatural for bcc operators connecting general backgrounds. One resolution is related to an earlier comment that marginal deformations can be made nonsingular by turning on a timelike gauge potential. This trick can be generalized to any time independent background as follows. Suppose we have bcc operators $\sigma', \bar{\sigma}'$ which are primaries of weight h with OPE:

$$\bar{\sigma}'(s)\sigma'(0) = \frac{1}{s^{2h}} + \dots$$

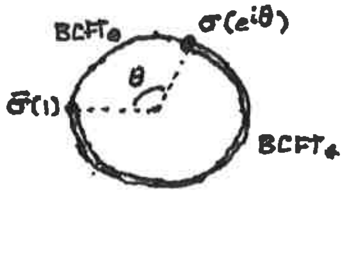
We may construct primaries of weight 0 and satisfying $\bar{\sigma}(x)\sigma(0) = 1 + \dots$ by tensoring with a timelike, plane wave vertex operator

$$\sigma(0) = \sigma' e^{i\sqrt{h}X^0}(0)$$

$$\bar{\sigma}(0) = \bar{\sigma}' e^{-i\sqrt{h}X^0}(0)$$

The planewave vertex operators are bcc operators which turn on a timelike gauge potential on the D-brane of $BCFT_*$. However, a timelike gauge potential is physically trivial. This gives an analytic solution in open bosonic SFT describing any time-independent D-brane configuration.

One subtlety with this solution is that star products of $\sigma, \bar{\sigma}$ are generally not associative. To see why, consider a ~~2-point~~ matter 2-point function of the bcc operators on the unit disk



$$\langle \bar{\sigma}(1)\sigma(e^{i\theta})\rangle^{\text{matter}}_{\text{disk}}$$

For angles between 0 and θ the boundary of the disk carries $BCFT_0$ boundary conditions, and outside that range it carries $BCFT_*$ boundary conditions. Since $\sigma, \bar{\sigma}$ are dimension 0 primaries the correlator is independent of θ, we may evaluate the correlator by taking the limit $\theta \to 0^+$ ~~and connecting~~ where $\bar{\sigma}$ annihilates σ:

$$\langle \bar{\sigma}(1)\sigma(e^{i\theta})\rangle^{\text{matter}}_{\text{disk}} = \langle 1 \rangle^{\text{matter}}_{\text{disk}, BCFT_*} \equiv g_*$$

g_* is called the disk partition function (of $BCFT_*$): it is essentially the norm of the $SL(2,\mathbb{R})$ vacuum in $BCFT_*$. This is proportional to the volume and energy of the D-brane system described by $BCFT_*$. Now we can also take the limit $\theta \to 2\pi^-$, where the boundary condition on the disk is $BCFT_0$. This produces the disk partition function g_0 of $BCFT_0$. This leads to a paradox: for general $BCFT_0$ and $BCFT_*$, $g_0 \neq g_*$. They happen to be ~~the same if~~ equal if $BCFT_0$ and $BCFT_*$ are related by nonsingular marginal deformation, but if they describe systems with different energy, ~~this will not hold~~ they will not be equal. The resolution to this paradox is that the σ-$\bar{\sigma}$ OPE depends on whether $BCFT_0$ or $BCFT_*$ boundary conditions are being squeezed between the bcc operators. By choice of normalization we have

$$\lim_{x\to 0^+} \bar{\sigma}(x)\sigma(0) = 1$$

but ~~this~~ in the opposite order this requires

$$\lim_{x\to 0^+} \sigma(x)\bar{\sigma}(0) = \frac{g_*}{g_0}$$

The string fields $\sigma, \bar{\sigma}$ will multiply as

$$\bar{\sigma}\sigma = 1 \qquad \sigma\bar{\sigma} = \frac{g_*}{g_0}$$

but this is inconsistent with associativity

$$(\bar{\sigma}\sigma)\bar{\sigma} \neq \bar{\sigma}(\sigma\bar{\sigma})$$

The important point is that the product $\sigma\bar{\sigma}$ which causes problems with associativity does not appear in the solution or when evaluating the equations of motion. While it is true that products of generic states in the $K,B,c,\sigma,\bar{\sigma}$ subalgebra may be ambiguous if $g_0 \neq g_*$, these ambiguities do not appear to apply to the solution itself.

Perhaps the most remarkable feature of the KOS solution is that it provides a simple proof of background independence in open bosonic SFT. Let S_0 denote the action around $BCFT_0$, and S_* denote the action around $BCFT_*$. We want to use the KOS solution to prove that the actions are related by field redefinition. First we seperate the string field of $BCFT_0$ into the KOS solution plus a fluctuation:

$$\Psi + \varphi_0 \in BCFT_0$$

The action is reexpresed

$$S_0[\Psi + \varphi_0] = S_0[\Psi] - \frac{1}{2}\mathrm{Tr}[\varphi_0 Q_\Psi \varphi_0] - \frac{1}{3}\mathrm{Tr}[\varphi_0^3]$$

Since this expression and S_* are both cubic actions for fluctuations around $BCFT_*$, we know that φ_0 should be linearly related to the string field $\varphi_* \in BCFT_*$ in the action S_*. It is natural to guess:

$$\varphi_0 = \Sigma \varphi_* \bar{\Sigma}$$

We can immediately see that the cubic terms in the actions agree due to $\bar{\Sigma}\Sigma = 1$. To identify the kinetic terms we have to deal with the shifted kinetic operator Q_Ψ. To do this, it is useful to introduce the operator

$$Q_{\Psi_1\Psi_2} X \equiv QX + \Psi_1 X - (-)^X X\Psi_2$$

If Ψ_1, Ψ_2 are solutions to the equations of motion

$$Q^2_{\Psi_1\Psi_2} = 0$$

We also have a version of the Leibniz rule

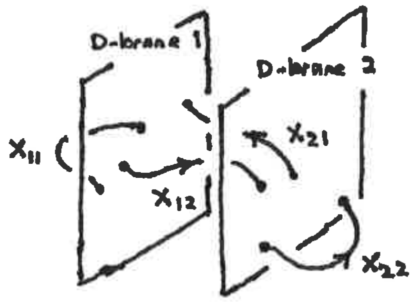
$$Q_{\Psi_1\Psi_3}(XY) = (Q_{\Psi_1\Psi_2}X)Y + (-)^X X(Q_{\Psi_2\Psi_3}Y)$$

where Ψ_2 on the right hand side is any solution to the equations of motion; it does not appear on the left hand side. The significance of this operator is that it naturally appears in the SFT formulated on a pair of D-branes. On a pair of D-branes the string field naturally carries 2×2 Chan-Paton factors, and can be arranged into a 2×2 matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

Note that a-priori it is not necessary to assume that the matrix entries are states in the same BCFT; the two D-branes which comprise the system need not be identical. Thus X_{11} is a state in the BCFT of the first D-brane, X_{22} is a state in the BCFT of the second and X_{21}, X_{12} are stretched string states connecting the two BCFTs.

If we condense the first D-brane to a solution Ψ_1 and the second D-brane to a solution Ψ_2, the solution on the combined system is



$$\Psi = \begin{pmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix}$$

and the kinetic operator expanded around Ψ is

$$Q_\Psi X = \begin{pmatrix} Q_{\Psi_1} X_{11} & Q_{\Psi_1\Psi_2} X_{12} \\ Q_{\Psi_2\Psi_1} X_{21} & Q_{\Psi_2} X_{22} \end{pmatrix}$$

Therefore $Q_{\Psi_1\Psi_2}$ is the shifted kinetic operator for a stretched string connecting D-brane 1 condensed to a solution Ψ_1 and D-brane 2 condensed to a solution Ψ_2. We claim that $\Sigma, \bar{\Sigma}$ satisfy

$$Q_{\Psi 0}\Sigma = 0 \qquad Q_{0\Psi}\bar{\Sigma} = 0$$

This can be seen as follows:

$$\begin{aligned} Q_{\Psi 0}\Sigma &= Q\Sigma + \Psi\Sigma \\ &= Q\Sigma + (\Psi_{tv} - \Sigma\Psi_{tv}\bar{\Sigma})\Sigma \\ &= Q\Sigma + \Psi_{tv}\Sigma - \Sigma\Psi_{tv} \\ &= Q_{\Psi_{tv}}\Sigma \\ &= 0 \end{aligned}$$

with a similar computation for $Q_{0\Psi}\bar{\Sigma}$. The interpretation is that Σ is killed by the kinetic operator for a stretched string connecting $BCFT_0$ condensed to the KOS solution Ψ and $BCFT_*$ at the perturbative vacuum 0. Note that $BCFT_0$ condensed to Ψ and $BCFT_*$ physically represent the same background; therefore the cohomology of $Q_{\Psi 0}$ should be the same as the cohomology of Q in $BCFT_*$. From this point of view, $\Sigma, \bar{\Sigma}$ are clearly representatives of the cohomology class of the identity operator in $BCFT_*$. Returning to background independence, we use $\varphi_0 = \Sigma\varphi_*\bar{\Sigma}$ to compute

$$\begin{aligned} Q_\Psi \varphi_0 &= Q_\Psi(\Sigma\varphi_*\bar{\Sigma}) \\ &= (Q_{\Psi 0}\Sigma)\varphi_*\bar{\Sigma} + \Sigma(Q\varphi_*)\bar{\Sigma} + \Sigma\varphi_*(Q_{0\Psi}\bar{\Sigma}) \\ &= \Sigma(Q\varphi_*)\bar{\Sigma} \end{aligned}$$

Plugging into the kinetic term and using $\bar{\Sigma}\Sigma = 1$ we have shown

$$S_0[\Psi + \varphi_0] = S_0[\Psi] + S_*[\varphi_*]$$

which establishes background independence.

Given this general solution, it is possible to study ~~specific solutions~~ realizations for specific backgrounds in some detail. One important example are lump solutions describing lower dimensional D-branes. We may for example consider a Dp-brane with one spacelike world volume coordinate X^1 compactified on a circle of radius R. Inhomogeneous tachyon condensation on this circle can produce a lump solution describing a D(p-1)-brane. Since on the Dp-brane X^1 satisfies Neumann boundary conditions and on the D(p-1)-brane it satisfies Dirichlet boundary conditions, we can construct the ~~soluti~~ lump solution using bcc operators $\sigma_{ND}, \bar{\sigma}_{ND}$ which change the boundary condition from Dirichlet to Neumann. These are known as Neumann-Dirichlet twist operators, and the lowest possible conformal weight for such operators is $\frac{1}{16}$. Unlike nonsingular marginal deformations, it is not known how to represent $\sigma_{ND}, \bar{\sigma}_{ND}$ ~~construct such operators from~~ as suitably renormalized composite operators built from $X^1(z,\bar{z})$. Nevertheless, quite a lot is known about their correlation functions. To construct the solution we must turn on a timelike gauge potential on the D(p-1) brane so that the OPEs of the bcc operators are regular:

$$\sigma(0) = \sigma_{ND}\, e^{i/4X^0}(0) \qquad \bar{\sigma}(0) = \bar{\sigma}_{ND}\, e^{-i/4X^0}(0)$$

When $R<1$ this lump solution has higher energy than the original Dp-brane. From the T-dual interpretation $R \to \frac{1}{R}$, the solution actually represents the formation of a higher dimensional D-brane in terms of the fluctuations of a D-brane with one lower dimension.

Another interesting class of solutions represent multiple D-brane systems. A curious feature of backgrounds in open string theory is that they can be superimposed to create new backgrounds: Given a D-brane represented by $BCFT_1$ and another D-brane represented by $BCFT_2$, by adding Chan-Paton factors we can obtain a background where both D-branes are present. It is almost as though the D-branes do not interact; in ordinary field theories, simply adding soliton solutions together does not give a multi-soliton since the field equations are non-linear. Using the KOS solution, we can try to create a multiple D-brane system by adding solutions around the tachyon vacuum creating $BCFT_1$ and $BCFT_2$

$$\Psi = \Psi_{tv} - \Sigma_1 \Psi_{tv} \bar{\Sigma}_1 - \Sigma_2 \Psi_{tv} \bar{\Sigma}_2$$

In fact, this is a solution provided that the bcc operators in Σ_1, Σ_2 have been chosen so that

$$\bar{\Sigma}_1 \Sigma_2 = 0 \qquad \bar{\Sigma}_2 \Sigma_1 = 0$$

The composite solution can be written in the form of the original solution

$$\Psi = \Psi_{tv} - \Sigma \Psi_{tv} \bar{\Sigma}$$

provided that $\Sigma, \bar{\Sigma}$ are interpreted as column and row vectors which "create" Chan-Paton factors:

$$\Sigma = (\Sigma_1 \quad \Sigma_2) \qquad \bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_1 \\ \bar{\Sigma}_2 \end{pmatrix}$$

In particular

$$\bar{\Sigma}\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the identity matrix acting on Chan-Paton factors of the composite system.

The KOS solution has brought our understanding of background independence in open bosonic SFT to a new level. The solution is very simple, but there are reasons to be skeptical: Like the simple tachyon vacuum, the structure of the solution is highly dependent on special but possibly singular properties of the identity string field. The presence of associativity anomalies prevents a useful generalization to open superstring field theory; the timelike gauge potential breaks manifest Lorentz invariance, and prevents a construction of time-dependent backgrounds. We hope to see new developments which address some of these problems in the future.