Toward a complete action for heteroticstring field theory

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1. Introduction

To consider the superstring theory as a fundamental theory of the nature, the nonperturbative formulation is needed.
SuperString Field Theory is one of the promising candidates.

♠ Three kinds of formulations

- WZW-like formulation

  Berkovits, Okawa-Zwiebach, Berkovits-Okawa-Zwiebach, Matsunaga, Okawa-HK, Goto-HK
- Homotopy algebra (HA) based formulation

  Erler-Konopka-Sachs, Erler-Okawa-Takezaki
- Sen's formulation including decoupling sector Sen, Konopka-Sachs

	WZW-like	HA based	Sen's
open NS	○	○	○
R	○	○	○
heterotic NS	○	○	○
R	△	×	○
closed (NS,NS)	○	○	○
(R,R) (NS,R) (R,NS)	×	×	○

WZW-like formulation

♢ A complete action has been given for open superstring
[Okawa and HK (2015)]

♢ Partial action
$S = S^{(0)} + S^{(2)} + S^{(4)}$
has been given for heterotic string using *fermion expansion*
[Goto and HK (2016)]

Toward a complete action the purpose of this talk is to give $S^{(6)} + S^{(8)}$.

Contents of talk

2. Open superstring field theory

String fields and constraints:

$$\Phi \in \mathcal{H}^{NS}_{large} : \ |\Phi| = 0\,, (g,p) = (0,0).$$

$$\Psi_o \in \mathcal{H}^{R}_{large} : \ |\Psi_o| = 1\,, (g,p) = (1,-1/2).$$

Constraints:

$$\eta\Psi_o = 0\,, \qquad XY\Psi_o = \Psi_o\,,$$

where

$$X = -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0\,, \qquad Y = -c_0\delta'(\gamma_0)\,,$$

which satisfy

$$XYX = X, \qquad [Q, X] = 0\,.$$

We use (Erler-Okawa-Takezaki)

$$\Xi = \xi_0 + (\Theta(\beta_0)\eta\xi_0 - \xi_0)P_{-3/2} + (\xi_0\eta\Theta(\beta_0) - \xi_0)P_{-1/2}\,,$$

which satisfies $\{Q, \Xi\} = X$ on states with $p = -3/2$ subspace.

Complete action:

$$S = -\int_0^1 dt\, \langle\, A_t(t), QA_\eta(t)\,\rangle - \frac{1}{2}\,\langle\!\langle\, \Psi_{\rm o}, YQ\Psi_{\rm o}\,\rangle\!\rangle + \frac{1}{2}\langle\, \Psi_{\rm o}, F\Psi_{\rm o}\,\rangle\,,$$

where

$$A_t(t) \;=\; (\partial_t e^{\Phi(t)})e^{-\Phi(t)}\,,\; A_\eta(t) \;=\; (\eta e^{\Phi(t)}))e^{-\Phi(t)}\,,$$

with $\Phi(1) = \Phi$, $\Phi(0) = 0$ and

$$F\Psi_{\rm o} = (1 + \Xi(D_\eta - \eta))^{-1}\Psi_{\rm o}\,,$$

with

$$D_\eta A \;=\; \eta A - A_\eta A + (-1)^A A A_\eta\,.$$

Gauge transformation:

The action is invariant under

$$(\delta e^\Phi)e^{-\Phi} = Q\Lambda + D_\eta\Omega + \{F\Psi, F\Xi\,(\,\{F\Psi,\Lambda\} - \lambda\,)\}\,,$$

$$\delta\Psi = Q\lambda + X\eta\,F\Xi\,D_\eta\,(\,\{F\Psi,\Lambda\} - \lambda\,)\,,$$

with $\eta\lambda = 0$, $XY\lambda = \lambda$.

3. Heteroticstring field theory

String fields and constraints:

$$V \in \mathcal{H}^{NS}_{large} \otimes \mathcal{H}_{boson} : \ |V| = 1\,, (g,p) \ = \ (1,0)\,,$$
$$\Psi \in \mathcal{H}^{R}_{large} \otimes \mathcal{H}_{boson} : \ |\Psi| = 0\,, (g,p) \ = \ (2,-1/2).$$

Constraints: $(c_0^{\pm} = (c_0 \pm \bar{c}_0)/2$, $b_0^{\pm} = b_0 \pm \bar{b}_0)$

$$L_0^- V \ = \ b_0^- V \ = \ 0\,, \qquad L_0^- \Psi \ = \ b_0^- \Psi \ = \ 0\,,$$
$$\eta \Psi \ = \ 0\,, \qquad XY\Psi \ = \ \Psi\,,$$

where

$$X \ = \ -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0\,, \qquad Y \ = \ -2c_0^+\delta'(\gamma_0)\,,$$

satisfying

$$XYX \ = \ X, \qquad [Q, X] \ = \ 0\,.$$

We use

$$\Xi \ = \ \xi_0 + (\Theta(\beta_0)\eta\xi_0 - \xi_0)P_{-3/2} + (\xi_0\eta\Theta(\beta_0) - \xi_0)P_{-1/2}\,.$$

which satisfies $\{Q, \Xi\} = X$ on states with $p = -3/2$ subspace.

Closed string products and their dual:

Closed string products satisfying L_∞-relations:

$$
\begin{aligned}
0 \;&= Q[B_1,\cdots,B_n] + \sum_{i=1}^{n}(-1)^{(B_1+\cdots+B_{i-1})}[B_1,\cdots,QB_i,B_{i+1},\cdots,B_n] \\
&+ \sum_{\substack{\{i_l,j_k\}\\ l+k=n}} \sigma(i_l,j_k)[B_{i_1},\cdots,B_{i_l},[B_{j_1},\cdots,B_{j_k}]]\,.
\end{aligned}
$$

Dual products satisfying another L_∞-relations exchanging Q for η:

$$
\begin{aligned}
0 \;&= \eta[B_1,\cdots,B_n]^\eta + \sum_{i=1}^{n}(-1)^{(B_1+\cdots+B_{i-1})}[B_1,\cdots,\eta B_i,B_{i+1},\cdots,B_n]^\eta \\
&+ \sum_{\substack{\{i_l,j_k\}\\ l+k=n}} \sigma(i_l,j_k)[B_{i_1},\cdots,B_{i_l},[B_{j_1},\cdots,B_{j_k}]^\eta]^\eta\,.
\end{aligned}
$$

We can construct it from the original products as, for example,

$$
\begin{aligned}
[B_1, B_2]^\eta &= -[B_1, B_2], \\
[B_1, B_2, B_3]^\eta &= -\frac{1}{4}\Big(X_0[B_1, B_2, B_3] + [X_0 B_1, B_2, B_3] + [B_1, X_0 B_2, B_3] + [B_1, B_2, X_0 B_3] \\
&\quad + (-1)^{B_1}(\xi_0[B_1, [B_2, B_3]] - [\xi_0 B_1, [B_2, B_3]] \\
&\qquad + (-1)^{B_1}[B_1, [\xi_0 B_2, B_3]] + (-1)^{B_1+B_2}[B_1, [B_2, \xi_0 B_3]]) \\
&\quad + (-1)^{B_1(B_2+B_3)+B_2}(\xi_0[B_2, [B_3, B_1]] - [\xi_0 B_2, [B_3, B_1]] \\
&\qquad + (-1)^{B_2}[B_2, [\xi_0 B_3, B_1]] + (-1)^{B_2+B_3}[B_2, [B_3, \xi_0 B_1]]) \\
&\quad + (-1)^{B_3(B_1+B_2)+B_3}(\xi_0[B_3, [B_1, B_2]] - [\xi_0 B_3, [B_1, B_2]] \\
&\qquad + (-1)^{B_3}[B_3, [\xi_0 B_1, B_2]] + (-1)^{B_3+B_1}[B_3, [B_1, \xi_0 B_2]])\Big).
\end{aligned}
$$

General products can also be constructed. (Goto-Matsunaga)

Dual shifted quantities:

It is useful to introduce

$$
\begin{aligned}
[B_1, \cdots, B_n]^{\eta}_{G_\eta} &= \sum_{m=0}^{\infty} \frac{\kappa^m}{m!}[(G_\eta)^m, B_1, \cdots, B_n]^{\eta}, \\
D_\eta B &= \eta B + \sum_{m=1}^{\infty} \frac{\kappa^m}{m!}[(G_\eta)^m, B]^{\eta},
\end{aligned}
$$

where

$$
G_\eta(V) = \eta V + \frac{\kappa}{2}[V, \eta V]^{\eta} + \frac{\kappa^2}{3!}([V, (\eta V)^2]^{\eta} + [V, [V, \eta V]^{\eta}]^{\eta}) + \cdots ,
$$

is a dual pure-gauge string field satisfying

$$
\eta G_\eta + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{n!}[(G_\eta)^n]^{\eta} = 0.
$$

Shifted quantities satisfy $D_\eta^2 = 0$, and

$$
\begin{aligned}
0 &= D_\eta[B_1, \cdots, B_n]^{\eta}_{G_\eta} + \sum_{i=1}^{n}(-1)^{(B_1+\cdots+B_{i-1})}[B_1, \cdots, D_\eta B_i, B_{i+1}, \cdots, B_n]^{\eta}_{G_\eta} \\
&+ \sum_{\substack{\{i_l, j_k\} \\ l+k=n}} \sigma(i_l, j_k)[B_{i_1}, \cdots, B_{i_l}, [B_{j_1}, \cdots, B_{j_k}]^{\eta}_{G_\eta}]^{\eta}_{G_\eta}.
\end{aligned}
$$

Fermion expansion:

Expand the action and the guage tf. in powers of Ramond string fields,

$$S = \sum_{n=1}^{\infty} S^{(2n)}, \qquad B_\delta = \sum_{n=1}^{\infty} B_\delta^{(2n)}, \qquad \delta\Psi = \sum_{n=1}^{\infty} \delta\Psi^{(2n-1)}.$$

And for arbitrary variation, B_δ and $\delta\Psi$,

$$\delta S^{(2n)} = \langle\!\langle \delta\Psi, Y E^{(2n-1)} \rangle\!\rangle + \langle B_\delta, E^{(2n)} \rangle.$$

The equations of motion in this notation are given by

$$\sum_{n=0}^{\infty} E^{(2n)} = 0, \qquad \sum_{n=0}^{\infty} E^{(2n+1)} = 0.$$

Then the gauge invariance of $\mathcal{O}(\Psi^{2n})$ requires

$$0 \;=\; -\sum_{k=1}^{n}\langle\!\langle\delta\Psi^{(2n-2k+1)}, YE^{(2k-1)}\rangle\!\rangle + \sum_{k=0}^{n}\langle B_\delta^{(2n-2k)}, E^{(2k)}\rangle. \qquad (2)$$

In particular, at the lowest order $n=0$, (*i.e.* no Ramond string field), (2) reduces

$$0 \;=\; \langle B_\delta^{(0)}, E^{(0)}\rangle.$$

This is satisfied by

$$B_\delta^{(0)} \;=\; Q\Lambda + D_\eta\Omega, \qquad E^{(0)} \;=\; QG_\eta,$$

derived from the *dual* WZW-like formulation of the NS sector.

$$S^{(0)} \;=\; \int_0^1 dt\langle B_t(t), QG_\eta(t)\rangle.$$

3.1 *Up to* $\mathcal{O}(\Psi^4)$ [Goto Kunitomo]

In the previous work we solved (2) at $\mathcal{O}(\Psi^2)$ and $\mathcal{O}(\Psi^4)$.

At $\mathcal{O}(\Psi^2)$ we obtain

$$\begin{aligned} S^{(2)} &= -\frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle + \frac{\kappa}{2}\int_0^1 dt\langle B_t(t), [F(t)\Psi^2]^\eta_{G\eta} \\ &= -\frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle + \frac{1}{2}\langle\Psi, F\Psi\rangle, \end{aligned}$$

with $F\Psi = (1+\Xi(D_\eta - \eta))^{-1}\Psi$. This has the same *form* as that for opensuperstring field theory, but now

$$D_\eta\Psi = \eta\Psi + \sum_{m=1}^{\infty}\frac{\kappa^m}{m!}[(G_\eta)^m, \Psi]^\eta.$$

This is invariant at $\mathcal{O}(\Psi^2)$ under

$$\begin{aligned} B^{(2)}_\delta &= \frac{\kappa^2}{2}[F\Psi^2, \Lambda]^\eta_{G\eta} - \kappa^2[F\Psi, F\Xi[F\Psi, \Lambda]^\eta_{G\eta}]^\eta_{G\eta} - \kappa[F\Psi, F\Xi\lambda]^\eta_{G\eta}, \\ \delta_\Lambda\Psi^{(1)} &= -\kappa X\eta F\Xi D_\eta[F\Psi, \Lambda]^\eta_{G\eta} + Q\lambda - X\eta F\Xi D_\eta\lambda. \end{aligned}$$

At $\mathcal{O}(\Psi^4)$ the gauge invariance (2) for λ-gauge tf. requires

$$\frac{\kappa^2}{3!}\langle F\lambda, [F\Psi^3]^{\eta}_{G\eta}]\rangle \;=\; \langle\!\langle \delta_\lambda\Psi^{(1)}, YE^{(3)}\rangle\!\rangle + \langle\!\langle \delta_\lambda\Psi^{(3)}, YE^{(1)}\rangle\!\rangle - \langle B^{(4)}_{\delta_\lambda}, QG_\eta\rangle.$$

We can determine $E^{(3)}$, $\delta_\lambda\Psi^{(3)}$ and $B^{(4)}_{\delta_\lambda}$ to satisfy this equation using the fact that the dual string product $[\cdots]^\eta$ can be written as BRST exact form

$$[B_1,\cdots,B_n]^\eta \;=\; Q(B_1,\cdots,B_n)^{[1]} - \sum_{i=1}^{n}(-1)^{B_1+\cdots+B_{i-1}}(B_1,\cdots,QB_i,\cdots,B_n)^{[1]}, \qquad (n\geq 3).$$

Here the new products $(\cdots)^{[1]}$ is defined by

$$(B_1,\cdots,B_n)^{[1]} \;=\; \frac{1}{(n-1)(n-2)}\lambda^{\eta[1]}_n(B_1,\cdots,B_n) \equiv \rho^{[1]}_n(B_1,\cdots,B_n),$$

where $\boldsymbol{\lambda}^{[1]}_\eta(t) = \sum_{n=3}^{\infty} t^n \boldsymbol{\lambda}^{\eta[1]}_n$ is defined from the EKS's gauge product with a single picture deficit $\boldsymbol{\lambda}^{[1]}(t)$ as

$$\boldsymbol{\lambda}^{[1]}_\eta(t) \;=\; -\widehat{\boldsymbol{G}}(t)\boldsymbol{\lambda}^{[1]}(t)\widehat{\boldsymbol{G}}^{-1}(t), \qquad \widehat{\boldsymbol{G}}(t) \;=\; \overleftarrow{\mathcal{P}}\exp\left(\int_0^t dt'\boldsymbol{\lambda}^{[0]}(t')\right).$$

As a result the action and the gauge tfs. at $\mathcal{O}(\Psi^4)$ is given by

$$S^{(4)} = \frac{\kappa^2}{4!}\langle F\Psi, (F\Psi, F\Psi, F\Psi)^{[1]}_{G\eta}\rangle,$$

and

$$\begin{aligned}
B^{(4)}_\delta =& -\frac{\kappa^4}{4!}[(F\Psi^4)^{[1]}_{G\eta}, \Lambda]^\eta_{G\eta} + \frac{\kappa^4}{3!}[[F\Psi, \Xi(F\Psi^3)^{[1]}_{G\eta}]^\eta_{G\eta}, \Lambda]^\eta_{G\eta} + \frac{\kappa^4}{4!}(F\Psi^4, D_\eta\Lambda)^{[1]}_{G\eta} \\
& -\frac{\kappa^4}{3!}(F\Psi^3, F\Xi[F\Psi, D_\eta\Lambda]^\eta_{G\eta})^{[1]}_{G\eta} - \frac{\kappa^4}{3!}[F\Psi, F\Xi(F\Psi^3, D_\eta\Lambda)^{[1]}_{G\eta}]^\eta_{G\eta} \\
& -\frac{\kappa^4}{3!}[F\Xi(F\Psi^3)^{[1]}_{G\eta}, F\Psi, D_\eta\Lambda]^\eta_{G\eta} + \frac{\kappa^4}{2}[F\Psi, F\Xi(F\Psi^2, F\Xi[F\Psi, D_\eta\Lambda]^\eta_{G\eta})^{[1]}_{G\eta}]^\eta_{G\eta} \\
& +\frac{\kappa^4}{3!}[F\Psi, F\Xi[F\Xi(F\Psi^3)^{[1]}_{G\eta}]^\eta_{G\eta}]^\eta_{G\eta} + \frac{\kappa^4}{3!}[F\Xi(F\Psi^3)^{[1]}_{G\eta}, F\Xi[F\Psi, D_\eta\Lambda]^\eta_{G\eta}]^\eta_{G\eta}, \\
& -\frac{\kappa^3}{3!}(F\Psi^3, F\lambda)^{[1]}_{G\eta} + \frac{\kappa^3}{2}[F\Psi, F\Xi(F\Psi^2, F\lambda)^{[1]}_{G\eta}]^\eta_{G\eta} + \frac{\kappa^3}{3!}[F\Xi(F\Psi^3)^{[1]}_{G\eta}, F\lambda]^\eta_{G\eta}, \\
\delta\Psi^{(3)} =& X\eta F\Xi D_\eta\Big(-\frac{\kappa^3}{3!}(F\Psi^3, D_\eta\Lambda)^{[1]}_{G\eta} + \frac{\kappa^3}{2}(F\Psi^2, F\Xi[F\Psi, D_\eta\Lambda]^\eta_{G\eta})^{[1]}_{G\eta} \\
& + \frac{\kappa^3}{3!}[F\Xi(F\Psi^3)^{[1]}_{G\eta}, D_\eta\Lambda]^\eta_{G\eta} + \frac{\kappa^2}{2}(F\Psi^2, F\lambda)^{[1]}_{G\eta}\Big).
\end{aligned}$$

3.2 *Beyond* $\mathcal{O}(\Psi^4)$

Unfortunately, at the present stage, we do not yet know how can we determine a complete action. So let us continue the fermion expansion a few more steps.

Order $\mathcal{O}(\Psi^6)$:

If we concentrait on λ-gauge tf. Eq.(2) at $\mathcal{O}(\Psi^6)$ requires

$$\begin{aligned} &\frac{\kappa^4}{4!}\langle F\lambda\,,\,\Big(2(F\Psi^3\,,\,[F\Psi^2]^{\eta}_{G_\eta})^{[1]}_{G_\eta} - [F\Psi\,,\,(F\Psi^4)^{[1]}_{G_\eta}]^{\eta}_{G_\eta} \\ &\qquad\qquad + (F\Psi^2\,,\,[F\Psi^3]^{\eta}_{G_\eta})^{[1]}_{G_\eta} - [F\Psi^2\,,\,(F\Psi^3)^{[1]}_{G_\eta}]^{\eta}_{G_\eta}\Big)\rangle \\ &= \langle\!\langle QF\lambda, Y(\cdots)\rangle\!\rangle + \langle\!\langle(\cdots), YQF\Psi\rangle\!\rangle - \langle(\cdots), QG_\eta\rangle. \end{aligned} \tag{3}$$

This implies that $(\cdots)$ in the l.h.s. has to be written as

$$(\cdots) \;=\; Q(*) + D_\eta(**) + \cdots\,,$$

where $\cdots$ denotes the terms including $QF\Psi$ or QG_η.

In order to show this we introduce (generating function of) two new products with arbitrary deficit number from the EKS's product and gauge product as

$$\boldsymbol{L}_\eta^{[n]}(t) \;=\; -\widehat{\boldsymbol{G}}(t)\boldsymbol{L}^{[n]}(t)\widehat{\boldsymbol{G}}^{-1}(t), \qquad \boldsymbol{\lambda}_\eta^{[n]}(t) \;=\; -\widehat{\boldsymbol{G}}(t)\boldsymbol{\lambda}^{[n]}(t)\widehat{\boldsymbol{G}}^{-1}(t).$$

If we denote

$$[B_1, \cdots, B_n]^\eta = L_n^\eta(B_1, \cdots, B_n), \qquad \boldsymbol{L}^\eta(t) = \sum_{n=0}^{\infty} t^n L_{n+1}^\eta,$$

the dual string product is related to the new product as

$$\partial_t \boldsymbol{L}^\eta(t) \;=\; \boldsymbol{L}_\eta^{[1]}(t), \qquad [\boldsymbol{\lambda}_\eta^{[n]}(t), \boldsymbol{L}^\eta(t)] \;=\; -(n+1)\boldsymbol{L}_\eta^{[n+1]}.$$

If we further introduce generating functions

$$\boldsymbol{L}_\eta(t,s) \;=\; \sum_{n=0}^{\infty} s^n \boldsymbol{L}_\eta^{[n]}(t), \qquad \boldsymbol{\lambda}_\eta(t,s) \;=\; \sum_{n=0}^{\infty} s^n \boldsymbol{\lambda}_\eta^{[n]}(t),$$

we can show that the relation

$$\partial_t[\boldsymbol{\lambda}_\eta(t,s),\boldsymbol{L}^\eta(t)] + (1+s\partial_s)[\int_0^s ds'\partial_{s'}\boldsymbol{\lambda}_\eta(t,s'),\partial_t\boldsymbol{L}^\eta(t)]$$

$$-\,\partial_s[\int_0^s ds'\partial_{s'}\boldsymbol{\lambda}_\eta(t,s'),[\int_0^s ds'\int_0^{s'} ds''\partial_{s''}\boldsymbol{\lambda}_\eta(t,s''),\boldsymbol{L}^\eta(t)]] \;=\; -[\boldsymbol{Q},\partial_s\boldsymbol{\lambda}_\eta(t,s)].$$

or

$$(k+1)[\boldsymbol{\lambda}^{\eta[n]}_{n+k+3},\boldsymbol{L}^\eta_1] + \sum_{l=0}^{k}(k+(n+1)l+n+2)[\boldsymbol{\lambda}^{\eta[n]}_{n+k-l+2},\boldsymbol{L}^\eta_{l+2}]$$

$$-\sum_{m=1}^{n-1}\sum_{l=0}^{k}\sum_{p=0}^{l}\frac{n+1}{n-m+1}[\boldsymbol{\lambda}^{\eta[m]},[\boldsymbol{\lambda}^{\eta[n-m]}_{n-m+l-p+2},\boldsymbol{L}^\eta_{p+1}]] \;=\; -(n+1)[\boldsymbol{Q},\boldsymbol{\lambda}^{\eta[n+1]}_{n+k+3}]$$

In particular, for $n=1$ we have

$$(k+1)[\boldsymbol{\lambda}^{\eta[1]}_{k+4},\boldsymbol{L}^\eta_1] + \sum_{l=0}^{k}(k+2l+3)[\boldsymbol{\lambda}^{\eta[1]}_{k-l+3},\boldsymbol{L}^\eta_{l+2}] \;=\; -2[\boldsymbol{Q},\boldsymbol{\lambda}^{\eta[2]}_{k+4}]$$

Using $\boldsymbol{\lambda}_n^{\eta[1]} = (n-1)(n-2)\boldsymbol{\rho}_n^{[1]}$, it becomes

$$(k+1)(k+2)(k+3)\left([\rho_{k+4}^{[1]}, \boldsymbol{L}_1^{\eta}] + [\rho_{k+3}^{[1]}, \boldsymbol{L}_2^{\eta}]\right) + \sum_{l=0}^{k-1} f_k(l)[\boldsymbol{\rho}_{k-l+2}^{[1]}, \boldsymbol{L}_{l+3}^{\eta}] \ = \ -2[\boldsymbol{Q}, \boldsymbol{\lambda}_{k+4}^{\eta[2]}],$$

with $f_k(l) = (k+2l+5)(k-l+1)(k-l)$. We split $\boldsymbol{L}_1^{\eta}$, $\boldsymbol{L}_2^{\eta}$ and $\boldsymbol{L}_{l+3}^{\eta}(l \geq 0)$ because $\boldsymbol{L}_{l+3}^{\eta} = [\boldsymbol{Q}, \boldsymbol{\rho}_{l+3}^{[1]}]$. We can further find that the coeficient function $f_k(l)$ satisfies

$$f_k(l) + f_k(k-l-1) \ = \ (k+1)(k+2)(k+3),$$

and so

$$[\boldsymbol{\rho}_{k+4}^{[1]}, \boldsymbol{L}_1^{\eta}] + [\boldsymbol{\rho}_{k+3}^{[1]}, \boldsymbol{L}_2^{\eta}] + \frac{1}{2}\sum_{l=0}^{k-1}[\boldsymbol{\rho}_{k-l+2}^{[1]}, \boldsymbol{L}_{l+3}^{\eta}] \ = \ -[\boldsymbol{Q}, \tilde{\boldsymbol{\rho}}_{k+4}^{[2]}],$$

where

$$\tilde{\boldsymbol{\rho}}_{k+4}^{[2]} \ = \frac{2}{(k+1)(k+2)(k+3)}\left(\boldsymbol{\lambda}_{k+4}^{\eta[2]} + \frac{1}{8}\sum_{l=0}^{k-1}\left(f_k(l) - f_k(k-l-1)\right)[\boldsymbol{\rho}_{k-l-2}^{[1]}, \boldsymbol{\rho}_{l+3}^{[1]}]\right).$$

Then if we define shifted products as

$$\rho_{G:n}^{[p]}(B_1, \cdots, B_n) \ = \sum_{m=\max(0,p+2-n)}^{\infty} \rho_{n+m}^{[p]}((G_\eta)^m, B_1, \cdots, B_n),$$

We find

$$\sum_{m=0}^{n}[\rho^{[1]}_{G:2n+2-m}, L^{\eta}_{m+1}] \;=\; -\,[\boldsymbol{Q}, \rho^{[2]}_{G:2n+2}] + (QG_\eta)\frac{\delta}{\delta G_\eta}\rho^{[2]}_{G:2n+2}, \quad \text{on } \mathcal{H}^{2n+2},$$

$$\sum_{m=0}^{n}[\boldsymbol{\rho}^{[1]}_{G:2n+3-m}, \boldsymbol{L}^{\eta}_{G:m+1}] + \frac{1}{2}[\boldsymbol{\rho}^{[1]}_{G:n+2}, \boldsymbol{L}^{\eta}_{n+2}] \;=\; -\,[\boldsymbol{Q}, \boldsymbol{\rho}^{[2]}_{G:2n+3}] + (QG_\eta)\frac{\delta}{\delta G_\eta}\boldsymbol{\rho}^{[2]}_{G:2n+3}, \quad \text{on } \mathcal{H}^{2n+3},$$

with

$$\boldsymbol{\rho}^{[2]}_{n} \;=\; \tilde{\boldsymbol{\rho}}^{[2]}_{G:n} - \frac{1}{2}[\tilde{\boldsymbol{\rho}}^{[1]}_{G:n+1}, \tilde{\boldsymbol{\rho}}^{[1]}_{G:0}] - \frac{1}{2}\sum_{m=0}^{[\frac{n}{2}]-1}[\boldsymbol{\rho}^{[1]}_{G:n-m}, \boldsymbol{\rho}^{[1]}_{G:m+1}].$$

In particular for $n = 5$ we have

$$[\rho^{[1]}_{G:5}, L^{\eta}_{1}] + [\rho^{[1]}_{G:4}, L^{\eta}_{G:2}] + \frac{1}{2}[\rho^{[1]}_{G:3}, L^{\eta}_{G:3}] \;=\; -[\boldsymbol{Q}, \rho^{[2]}_{G:5}] + (QG_\eta)\frac{\delta}{\delta G_\eta}\rho^{[2]}_{G:5}.$$

Acting on $F\Psi^{\wedge 5}$ and note $L^{\eta}_{G:1}(F\Psi) = D_\eta F\Psi = 0$ we can show

$$\Big(2(F\Psi^3, [F\Psi^2]^{\eta}_{G\eta})^{[1]}_{G\eta} - [F\Psi, (F\Psi^4)^{[1]}_{G\eta}]^{\eta}_{G\eta}$$

$$+(F\Psi^2, [F\Psi^3]^{\eta}_{G\eta})^{[1]}_{G\eta} - [F\Psi^2, (F\Psi^3)^{[1]}_{G\eta}]^{\eta}_{G\eta}\Big)$$

$$= -\frac{1}{5}Q(F\Psi^5)^{[2]}_{G\eta} + \frac{1}{5}D_\eta(F\Psi^5)^{[1]}_{G\eta} + (F\Psi^4, QF\Psi)^{[2]}_{G\eta} + \frac{\kappa}{5}(F\Psi^5, QG_\eta)^{[2]}_{G\eta}$$

$$+\kappa((F\Psi^3)^{[1]}_{G\eta}, F\Psi^2, QG_\eta)^{[1]}_{G\eta} - \kappa(F\Psi^2, (F\Psi^3, QG_\eta)^{[1]}_{G\eta})^{[1]}_{G\eta}.$$

In this way we can solve the gauge invariance equation and find

$$E^{(5)} = -\frac{\kappa^4}{5!}X\eta F\Xi D_\eta(F\Psi^5)^{[2]}_{G\eta} + \frac{\kappa^4}{2!3!}X\eta F\Xi D_\eta(F\Psi^2, \Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta},$$

with

$$\Delta = \frac{1}{2}(F\Xi D_\eta - D_\eta F\Xi),$$

which can be derived from

$$S^{(6)} = -\frac{\kappa^4}{6!}\langle F\Psi, (F\Psi^5)^{[2]}_{G\eta}\rangle + \frac{1}{2}\frac{1}{3!3!}\langle F\Psi, (F\Psi^2, \Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta}\rangle.$$

This is invariant under the λ-gauge tf.

$$
\begin{aligned}
\delta_\lambda \Psi^{(5)} &= -\frac{\kappa^4}{4!} X\eta F\Xi D_\eta (F\Psi^4, F\lambda)^{[2]}_{G\eta} \\
&\quad + \frac{\kappa^4}{3!} X\eta F\Xi D_\eta (\Delta(F\Psi^3)^{[1]}_{G\eta}, F\Psi, F\lambda)^{[1]}_{G\eta} + \frac{\kappa^4}{2!2!} X\eta F\Xi D_\eta (F\Psi^2, \Delta(F\Psi^2, F\lambda)^{[1]}_{G\eta})^{[1]}_{G\eta}\,, \\
B^{(6)}_{\delta_\lambda} &= \frac{\kappa^5}{5!} (F\Psi^5, F\lambda)^{[2]}_{G\eta} \\
&\quad + \frac{\kappa^5}{3!2!} (D_\eta F\Xi (F\Psi^3)^{[1]}_{G\eta}, F\Psi^2, F\lambda)^{[1]}_{G\eta} + \frac{\kappa^5}{3!2!} (F\Psi^3, D_\eta F\Xi (F\Psi^2, F\lambda)^{[1]}_{G\eta})^{[1]}_{G\eta} \\
&\quad - \frac{1}{2}\frac{\kappa^5}{3!2!} [D_\eta F\Xi (F\Psi^3)^{[1]}_{G\eta}, F\Xi (F\Psi^2, F\lambda)]^{\eta}_{G\eta} - \frac{1}{2}\frac{\kappa^5}{3!2!} [F\Xi (F\Psi^3)^{[1]}_{G\eta}, D_\eta F\Xi (F\Psi^2, F\lambda)]^{\eta}_{G\eta} \\
&\quad - \frac{\kappa^5}{5!} [F\Xi (F\Psi^5)^{[2]}_{G\eta}, F\lambda]^{\eta}_{G\eta} + \frac{\kappa^4}{2!3!} [F\Xi (F\Psi^2, \Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta}, F\lambda]^{\eta}_{G\eta} \\
&\quad - \frac{\kappa^5}{4!} [F\Psi, F\Xi (F\Psi^4, F\lambda)^{[2]}_{G\eta}]^{\eta}_{G\eta} + \frac{\kappa^5}{3!} [F\Psi, F\Xi (\Delta(F\Psi^3)^{[1]}_{G\eta}, F\Psi, F\lambda)^{[1]}_{G\eta}]^{\eta}_{G\eta} \\
&\quad + \frac{\kappa^5}{2!2!} [F\Psi, F\Xi (F\Psi^2, \Delta(F\Psi^2, F\lambda)^{[1]}_{G\eta})^{[1]}_{G\eta}]^{\eta}_{G\eta}\,.
\end{aligned}
$$

From $S^{(6)}$ we can further obtain

$$E^{(6)} = D_\eta \biggl(\frac{\kappa^5}{6!}(F\Psi^6)^{[2]}_{G\eta} + \frac{\kappa^5}{3!3!}(F\Psi^3, D_\eta F\Xi(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta} - \frac{\kappa^5}{5!}[F\Psi, F\Xi(F\Psi^5)^{[2]}_{G\eta}]^{\eta}_{G\eta} + \frac{\kappa^5}{2!3!}[F\Psi, F\Xi(F\Psi^2, \Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta}]^{\eta}_{G\eta} - \frac{1}{2}\frac{\kappa^5}{3!3!}[F\Xi(F\Psi^3)^{[1]}_{G\eta}, D_\eta F\Xi(F\Psi^3)^{[1]}_{G\eta}]^{\eta}_{G\eta} \biggr) .$$

We can also obtain the Λ-gauge tf. in a similar manner.

Order $\mathcal{O}(\Psi^8)$:

Similarly, gauge invariance at $\mathcal{O}(\Psi^8)$ requires

$$0 = -\langle\!\langle \delta_\lambda \Psi^{(1)}, YE^{(7)} \rangle\!\rangle - \langle\!\langle \delta_\lambda \Psi^{(3)}, YE^{(5)} \rangle\!\rangle - \langle\!\langle \delta_\lambda \Psi^{(5)}, YE^{(3)} \rangle\!\rangle - \langle\!\langle \delta_\lambda \Psi^{(7)}, YE^{(1)} \rangle\!\rangle + \langle B^{(2)}_{\delta_\lambda}, E^{(6)} \rangle + \langle B^{(4)}_{\delta_\lambda}, E^{(4)} \rangle + \langle B^{(6)}_{\delta_\lambda}, E^{(2)} \rangle + \langle B^{(8)}_{\delta_\lambda}, E^{(0)} \rangle .$$

We find

$$
\begin{aligned}
E^{(7)} \;=\; & -X\eta F\Xi D_\eta\Bigg(-\frac{\kappa^6}{7!}(F\Psi^7)^{[3]} \\
& +\frac{\kappa^6}{3!4!}(F\Psi^4 F\Xi D_\eta(F\Psi^3)^{[1]}_{G\eta})^{[2]}_{G\eta}-\frac{\kappa^6}{2!5!}(F\Psi^2 D_\eta F\Xi(F\Psi)^{[2]}_{G\eta})^{[1]}_{G\eta} \\
& -\frac{\kappa^6}{2!2!3!}(F\Psi^2\Delta(F\Psi^2\Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta})^{[1]}_{G\eta}+\frac{1}{12}\frac{\kappa^6}{2!2!3!}(F\Psi^2(F\Psi^2(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta})^{[1]}_{G\eta} \\
& -\frac{1}{2}\frac{\kappa^6}{3!3!}(F\Psi\Delta(F\Psi^3)^{[1]}_{G\eta}D_\eta F\Xi(F\Psi^3)^{[1]}_{G\eta})-\frac{1}{24}\frac{\kappa^6}{3!3!}(F\Psi(F\Psi^3)^{[1]}_{G\eta}(F\Psi^3)^{[1]}_{G\eta})\Bigg)\,,
\end{aligned}
$$

which derived from the action

$$
\begin{aligned}
S^{(8)} \;=\; & \frac{\kappa^6}{8!}\langle F\Psi,(F\Psi^7)^{[3]}_{G\eta}\rangle \\
& +\frac{\kappa^6}{3!5!}\langle F\Psi,(F\Psi^2,D_\eta F\Xi(F\Psi^5)^{[2]}_{G\eta})^{[1]}_{G\eta}\rangle \\
& +\frac{1}{2}\frac{\kappa^6}{2!3!3!}\langle F\Psi,(F\Psi^2,\Delta(F\Psi^2,\Delta(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta})^{[1]}_{G\eta}\rangle \\
& -\frac{1}{4!}\frac{\kappa^6}{2!3!3!}\langle F\Psi,(F\Psi^2(F\Psi^2,(F\Psi^3)^{[1]}_{G\eta})^{[1]}_{G\eta})^{[1]}_{G\eta}\rangle\,.
\end{aligned}
$$

5. Summary and future prospects

- Using fermion expansion, we have constructed an action $S^{(6)}$ and $S^{(8)}$ for heterotic string field thoery.

- We have also constructed $\delta\Psi^{(5)}$, $\delta\Psi^{(7)}$, $B_\delta^{(6)}$ and, $B_\delta^{(8)}$ under which the action is invariant.

General form of $S^{(2n)}$:

We can easily extrapolate main part of $S^{(2n)}$ as

$$S^{(2n)} = \frac{\kappa^{2n-2}}{(2n!)}\langle F\Psi, (F\Psi^{2n-1})_{G\eta}^{[n-1]}\rangle + \cdots ,$$

but we don't have a method to obtain general $(\cdots)^{[n]}$.