Closed string field theory without the level-matching condition

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Introduction

When we parameterize a closed string world-sheet using the coordinates (τ, σ) with $0 \leq \sigma \leq 2\pi$, we do not have an appropriate gauge-fixing condition to determine the origin of σ.

Even in the light-cone gauge, the gauge redundancy is not completely fixed because of this issue, but the range of σ is compact so that we can simply path integrate over all possible choices.

The translation of σ is generated by $L_0 - \widetilde{L}_0$, and the integration over all choices requires the closed string state to be annihilated by $L_0 - \widetilde{L}_0$.

This is the origin of the level-matching condition for the closed string.

In closed string field theory, this is reflected in the constraints

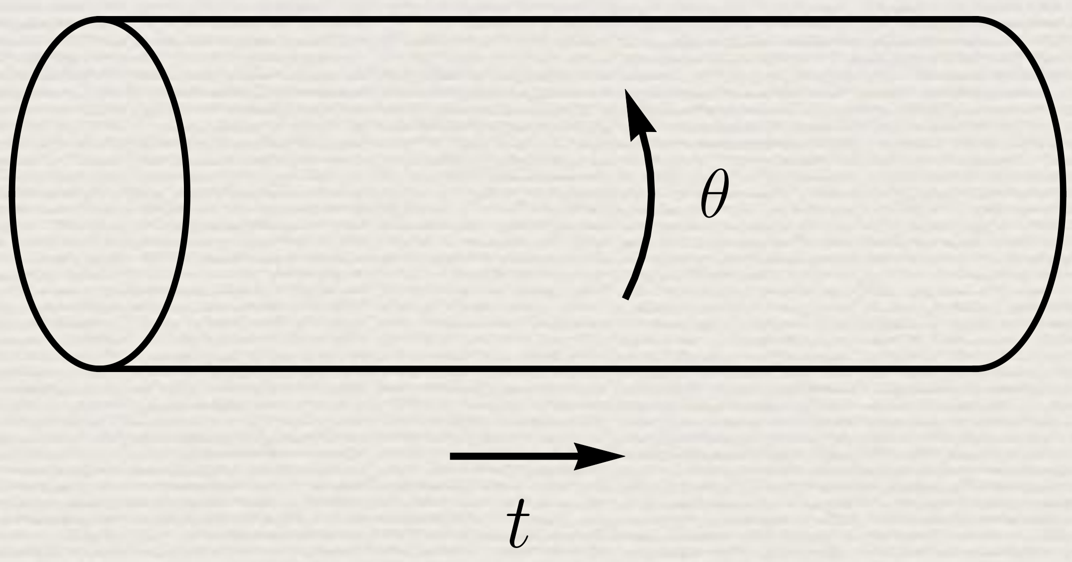
$$(L_0 - \widetilde{L}_0)\,\Psi = 0\,, \qquad (b_0 - \widetilde{b}_0)\,\Psi = 0$$

imposed on the closed string field Ψ. As can be understood from the origin of the level-matching condition, it has been difficult to formulate covariant closed string field theory without imposing these constraints on the closed string field.

In the context of the moduli space of Riemann surfaces, propagator surfaces for the closed string have two moduli, and the moduli space can be parameterized as

$$e^{-t\,(L_0+\widetilde{L}_0)+i\theta\,(L_0-\widetilde{L}_0)}\,,$$

where t and θ are the moduli.



In closed bosonic string field theory, the integration over t is implemented by the propagator:

$$\frac{b_0^+}{L_0^+} = b_0^+ \int_0^\infty dt\, e^{-t\, L_0^+} \quad \text{with} \quad L_0^+ = L_0 + \widetilde{L}_0\,, \quad b_0^+ = b_0 + \widetilde{b}_0\,.$$

On the other hand, the integration over θ yields the operator B given by

$$B = b_0^- \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta L_0^-} \quad \text{with} \quad L_0^- = L_0 - \widetilde{L}_0\,, \quad b_0^- = b_0 - \widetilde{b}_0\,.$$

The operator B can be schematically understood as $B \sim \delta(b_0^-)\,\delta(L_0^-)\,.$

The constraints on the closed string field implement the integration over this modulus in closed string field theory.

The appropriate inner product of Ψ_1 and Ψ_2 satisfying the constraints can be written as

$$\langle\, \Psi_1, c_0^- \Psi_2 \,\rangle$$

with

$$c_0^- = \frac{1}{2}\,(\, c_0 - \widetilde{c}_0 \,)\,,$$

where $\langle\, A, B \,\rangle$ is the BPZ inner product for a pair of states A and B.

The kinetic term for closed bosonic string field theory is then given by

$$S = \, -\, \frac{1}{2}\,\langle\, \Psi, c_0^- Q_B \Psi \,\rangle\,,$$

where Q_B is the BRST operator.

The operator B can also be written as

$$B = -i \int_0^{2\pi} \frac{d\theta}{2\pi} \int d\tilde{\theta} \, e^{i\theta L_0^- + i\tilde{\theta} \, b_0^-}$$

using a Grassmann-odd variable $\tilde{\theta}$.

This is the form which is natural in the context of the extended BRST transformation, which maps θ to $\tilde{\theta}$.

Witten, arXiv:1209.5461

Since $B \, c_0^- \, B = B$, the operator $B \, c_0^-$ is a projector, and the closed bosonic string field Ψ in the restricted space can be characterized as

$$B \, c_0^- \, \Psi = \Psi \, .$$

Recently, there have been important developments in the treatment of the Ramond sector in superstring field theory, which we consider is related to formulating closed string field theory without imposing the level-matching condition.

Complete actions for open superstring field theory including both the Neveu-Schwarz sector and the Ramond sector were constructed, where the string field in the Ramond sector is restricted to an appropriate subspace of the Hilbert space.

Kunitomo and Okawa, arXiv:1508.00366
Erler, Okawa and Takezaki, arXiv:1602.02582
Konopka and Sachs, arXiv:1602.02583

The restriction on the string field Ψ of picture $-1/2$ can be characterized as

$$XY\Psi = \Psi$$

with

$$X = \int d\zeta \int d\widetilde{\zeta}\, e^{\zeta G_0 - \widetilde{\zeta}\beta_0} = G_0\,\delta(\beta_0) + b_0\,\delta'(\beta_0)$$

and

$$Y = c_0 \int d\sigma\, \sigma\, e^{\sigma\gamma_0} = \, - c_0\,\delta'(\gamma_0)\,,$$

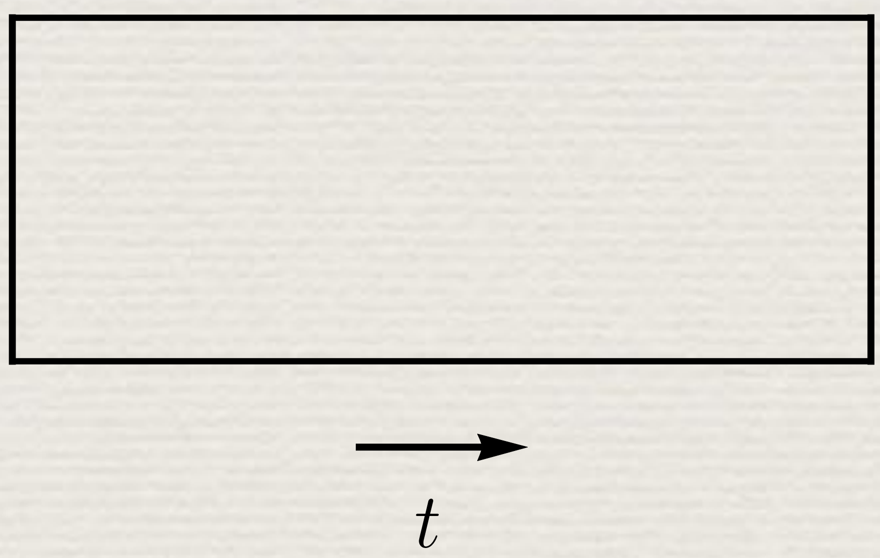
where G_0 is the zero mode of the supercurrent, ζ is Grassmann odd, and $\widetilde{\zeta}$ and σ are Grassmann even.

The integration over these Grassmann-even variables should be understood as an algebraic operation analogous to the integration over Grassmann-odd variables. The extended BRST transformation maps ζ to $\widetilde{\zeta}$.

In the context of the supermoduli space of super-Riemann surfaces, propagator strips for the Ramond sector of the open superstring have a fermionic modulus in addition to the bosonic modulus corresponding to the length t of the strip. The supermoduli space can be parameterized as

$$e^{\,tL_0+\zeta G_0}\,,$$

where ζ is the fermionic modulus.



The integration over ζ with the associated ghost insertion yields the operator X. The restriction

$$XY\Psi = \Psi$$

is therefore analogous to

$$B\, c_0^- \,\Psi = \Psi$$

for the closed bosonic string field.

The appropriate inner product of Ψ_1 and Ψ_2 in the restricted space can be written as

$$\langle\, \Psi_1, Y \Psi_2 \,\rangle\,,$$

and the kinetic term of open superstring field theory for the Ramond sector is given by

$$S = \, - \, \frac{1}{2} \, \langle\, \Psi, Y Q_B \Psi \,\rangle\,.$$

This is analogous to the kinetic term

$$S = \, - \, \frac{1}{2} \, \langle\, \Psi, c_0^- Q_B \Psi \,\rangle$$

for closed bosonic string field theory.

Another remarkable development in the treatment of the Ramond sector in superstring field theory is the construction of covariant kinetic terms by Sen, where no constraints associated with the Ramond sector are imposed but extra free fields are introduced.

Sen, arXiv:1508.05387

The construction was presented in the context of the Batalin-Vilkovisky master action for heterotic string field theory or type II superstring field theory, but the idea can be applied to the construction of a classical gauge-invariant action for open superstring field theory.

In this context, the kinetic terms are given by

$$S = \frac{1}{2} \langle \widetilde{\Psi}, Q_B X \widetilde{\Psi} \rangle + \langle \widetilde{\Psi}, Q_B \Psi \rangle \,,$$

where Ψ is a string field of picture $-1/2$ and $\widetilde{\Psi}$ is a string field of picture $-3/2$.

The interaction terms do not contain $\widetilde{\Psi}$ and the string field $\widetilde{\Psi}$ describes the extra free fields. The string field Ψ describes the interacting fields and no constraints are imposed on Ψ.

In this approach the operator X can be replaced by a different operator. For example, the zero mode of the picture-changing operator can be used, which is convenient when we describe the superconformal ghost sector in terms of ξ, η, and ϕ.

	The Ramond sector of open superstring field theory	Closed bosonic string field theory
moduli	$(\,t, \zeta\,)$	$(\,t, \theta\,)$
the theory with a restriction	Ψ satisfying $XY\Psi = \Psi$	Ψ satisfying $Bc_0^-\Psi = \Psi$
the theory with a free string field	interacting Ψ and free $\widetilde{\Psi}$	

	The Ramond sector of open superstring field theory	Closed bosonic string field theory
moduli	$(\,t, \zeta\,)$	$(\,t, \theta\,)$
the theory with a restriction	Ψ satisfying $XY\Psi = \Psi$	Ψ satisfying $Bc_0^- \Psi = \Psi$
the theory with a free string field	interacting Ψ and free $\widetilde{\Psi}$	interacting Ψ and free $\widetilde{\Psi}$

This construction indicates that we can formulate closed string field theory without imposing the level-matching condition on the closed string field if we allow extra free fields.

We claim that this is indeed possible.

The kinetic terms for the Ramond sector of open superstring field theory in the approach by Sen are

$$S = \frac{1}{2} \langle \widetilde{\Psi}, Q_B X \widetilde{\Psi} \rangle + \langle \widetilde{\Psi}, Q_B \Psi \rangle ,$$

where Ψ is a string field of picture $-1/2$ and $\widetilde{\Psi}$ is a string field of picture $-3/2$.

The interaction terms do not contain $\widetilde{\Psi}$ and the string field $\widetilde{\Psi}$ describes the extra free fields. The string field Ψ describes the interacting fields and no constraints are imposed on Ψ.

This construction indicates that we can formulate closed string field theory without imposing the level-matching condition on the closed string field if we allow extra free fields.

We claim that this is indeed possible.

Our kinetic terms for closed bosonic string field theory without the level-matching condition are given by

$$S = \frac{1}{2} \langle \widetilde{\Psi}, Q_B B \widetilde{\Psi} \rangle + \langle \widetilde{\Psi}, Q_B \Psi \rangle \,,$$

where Ψ is a string field of ghost number 2 and $\widetilde{\Psi}$ is a string field of ghost number 3.

The interaction terms do not contain $\widetilde{\Psi}$ and the string field $\widetilde{\Psi}$ describes the extra free fields. The string field Ψ describes the interacting fields and no constraints are imposed on Ψ.

Closed string field theory with the level-matching condition

The action is given by

$$
\begin{aligned}
S &= \frac{1}{2}\langle\!\langle\, \Psi, Q_B \Psi \,\rangle\!\rangle + \sum_{n=3}^{\infty} \frac{g^{n-2}}{n!}\,\langle\!\langle\, \Psi, [\![\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n-1} \,]\!] \,\rangle\!\rangle \\
&= \frac{1}{2}\langle\!\langle\, \Psi, Q_B \Psi \,\rangle\!\rangle + \frac{g}{3!}\,\langle\!\langle\, \Psi, [\![\Psi, \Psi]\!] \,\rangle\!\rangle + \frac{g^2}{4!}\,\langle\!\langle\, \Psi, [\![\Psi, \Psi, \Psi]\!] \,\rangle\!\rangle + O(g^3)\,,
\end{aligned}
$$

where g is the closed string coupling constant and

$$
\langle\!\langle\, A_1, A_2 \,\rangle\!\rangle = \langle\, A_1, c_0^- A_2 \,\rangle\,.
$$

The ghost number of the n-string product $G(\llbracket A_1, A_2, \cdots, A_n \rrbracket)$ is

$$G(\llbracket A_1, A_2, \cdots, A_n \rrbracket) = -2(n-2) - 1 + \sum_{i=1}^{n} G(A_i)\,,$$

where $G(A)$ is the ghost number of A.

The n-string product is graded-commutative,

$$\llbracket A_1, \cdots, A_{i+1}, A_i, A_{i+2}, \cdots, A_n \rrbracket = (-1)^{A_i A_{i+1}} \llbracket A_1, \cdots, A_n \rrbracket\,,$$

and the inner product $\langle\!\langle A_1, \llbracket A_2, \cdots, A_n \rrbracket \rangle\!\rangle$ has the following property:

$$\langle\!\langle A_1, \llbracket A_2, \cdots, A_n \rrbracket \rangle\!\rangle = (-1)^{A_1 A_2} \langle\!\langle A_2, \llbracket A_1, \cdots, A_n \rrbracket \rangle\!\rangle\,.$$

The action is invariant under the gauge transformation

$$\delta_\Lambda \Psi = Q_B \Lambda + \sum_{n=1}^{\infty} \frac{g^n}{n!} [\![\underbrace{\Psi, \Psi, \cdots, \Psi}_{n}, \Lambda]\!]$$

$$= Q_B \Lambda + g [\![\Psi, \Lambda]\!] + \frac{g^2}{2!} [\![\Psi, \Psi, \Lambda]\!] + O(g^3)$$

for the gauge parameter Λ satisfying

$$(L_0 - \widetilde{L}_0) \Lambda = 0 \,, \qquad (b_0 - \widetilde{b}_0) \Lambda = 0$$

if the multi-string products $[\![A_1, A_2, \cdots, A_n]\!]$ satisfy a set of relations called L_∞ relations.

The relation we need for gauge invariance at $O(g)$ is

$$Q_B \, [\![\, A_1, A_2 \,]\!] + [\![\, Q_B A_1, A_2 \,]\!] + (-1)^{A_1} [\![\, A_1, Q_B A_2 \,]\!] = 0 \, .$$

The relation we need for gauge invariance at $O(g^2)$ is

$$\begin{aligned}
& Q_B \, [\![\, A_1, A_2, A_3 \,]\!] + [\![\, Q_B A_1, A_2, A_3 \,]\!] \\
& + (-1)^{A_1} [\![\, A_1, Q_B A_2, A_3 \,]\!] + (-1)^{A_1+A_2} [\![\, A_1, A_2, Q_B A_3 \,]\!] \\
& + (-1)^{A_1} [\![\, A_1, [\![\, A_2, A_3 \,]\!] \,]\!] + (-1)^{A_2(1+A_1)} [\![\, A_2, [\![\, A_1, A_3 \,]\!] \,]\!] \\
& + (-1)^{A_3(1+A_1+A_2)} [\![\, A_3, [\![\, A_1, A_2 \,]\!] \,]\!] = 0 \, .
\end{aligned}$$

Closed string field theory without the level-matching condition

The action is given by

$$
\begin{aligned}
S &= \frac{1}{2}\langle\,\widetilde{\Psi}, Q_B B \widetilde{\Psi}\,\rangle + \langle\,\widetilde{\Psi}, Q_B \Psi\,\rangle + \sum_{n=3}^{\infty} \frac{g^{n-2}}{n!}\langle\,\Psi, [\,\underbrace{\Psi, \Psi, \cdots, \Psi}_{n-1}\,]\,\rangle \\
&= \frac{1}{2}\langle\,\widetilde{\Psi}, Q_B B \widetilde{\Psi}\,\rangle + \langle\,\widetilde{\Psi}, Q_B \Psi\,\rangle + \frac{g}{3!}\langle\,\Psi, [\,\Psi, \Psi\,]\,\rangle \\
&\quad + \frac{g^2}{4!}\langle\,\Psi, [\,\Psi, \Psi, \Psi\,]\,\rangle + O(g^3)\,.
\end{aligned}
$$

The ghost number of $[\,A_1, A_2, \cdots, A_n\,]$ is

$$G(\,[\,A_1, A_2, \cdots, A_n\,]\,) = \,-\,2(n-2) + \sum_{i=1}^{n} G(A_i)\,.$$

The n-string product is graded-commutative,

$$[\,A_1, \cdots, A_{i+1}, A_i, A_{i+2}, \cdots, A_n\,] = (-1)^{A_i A_{i+1}}[\,A_1, \cdots, A_n\,]\,,$$

and the inner product $\langle\, A_1, [\,A_2, \cdots, A_n\,]\rangle$ has the following property:

$$\langle\, A_1, [\,A_2, \cdots, A_n\,]\,\rangle = (-1)^{A_1 A_2}\langle\, A_2, [\,A_1, \cdots, A_n\,]\,\rangle\,.$$

The action is invariant under the gauge transformations given by

$$
\begin{aligned}
\delta_\Lambda \Psi &= Q_B \Lambda + \sum_{n=1}^{\infty} \frac{g^n}{n!} \, B \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n}, \Lambda \,] \\
&= Q_B \Lambda + g \, B \, [\, \Psi, \Lambda \,] + \frac{g^2}{2!} \, B \, [\, \Psi, \Psi, \Lambda \,] + O(g^3) \,, \\
\delta_\Lambda \widetilde{\Psi} &= \; - \sum_{n=1}^{\infty} \frac{g^n}{n!} \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n}, \Lambda \,] \\
&= \; - g \, [\, \Psi, \Lambda \,] - \frac{g^2}{2!} \, [\, \Psi, \Psi, \Lambda \,] + O(g^3) \,, \\
\delta_{\widetilde{\Lambda}} \Psi &= 0 \,, \\
\delta_{\widetilde{\Lambda}} \widetilde{\Psi} &= Q_B \widetilde{\Lambda} \,,
\end{aligned}
$$

where Λ and $\widetilde{\Lambda}$ are gauge parameters, if the multi-string products satisfy a set of relations analogous to the L_∞ relations.

The relation we need for gauge invariance at $O(g)$ is

$$Q_B\,[\,A_1, A_2\,] - [\,Q_B A_1, A_2\,] - (-1)^{A_1}[\,A_1, Q_B A_2\,] = 0\,.$$

The relation we need for gauge invariance at $O(g^2)$ is

$$\begin{aligned}
&Q_B\,[\,A_1, A_2, A_3\,] - [\,Q_B A_1, A_2, A_3\,] \\
&- (-1)^{A_1}[\,A_1, Q_B A_2, A_3\,] - (-1)^{A_1+A_2}[\,A_1, A_2, Q_B A_3\,] \\
&- (-1)^{A_1}[\,A_1, B\,[\,A_2, A_3\,]\,] - (-1)^{A_2(1+A_1)}[\,A_2, B\,[\,A_1, A_3\,]\,] \\
&- (-1)^{A_3(1+A_1+A_2)}[\,A_3, B\,[\,A_1, A_2\,]\,] = 0\,.
\end{aligned}$$

Note that the L_∞ relation

$$
\begin{aligned}
&Q_B \, [\![\, A_1, A_2, A_3 \,]\!] + [\![\, Q_B A_1, A_2, A_3 \,]\!] \\
&+ (-1)^{A_1} [\![\, A_1, Q_B A_2, A_3 \,]\!] + (-1)^{A_1+A_2} [\![\, A_1, A_2, Q_B A_3 \,]\!] \\
&+ (-1)^{A_1} [\![\, A_1, [\![\, A_2, A_3 \,]\!] \,]\!] + (-1)^{A_2(1+A_1)} [\![\, A_2, [\![\, A_1, A_3 \,]\!] \,]\!] \\
&+ (-1)^{A_3(1+A_1+A_2)} [\![\, A_3, [\![\, A_1, A_2 \,]\!] \,]\!] = 0
\end{aligned}
$$

follows from

$$
\begin{aligned}
&Q_B \, [\, A_1, A_2, A_3 \,] - [\, Q_B A_1, A_2, A_3 \,] \\
&- (-1)^{A_1} [\, A_1, Q_B A_2, A_3 \,] - (-1)^{A_1+A_2} [\, A_1, A_2, Q_B A_3 \,] \\
&- (-1)^{A_1} [\, A_1, B \, [\, A_2, A_3 \,] \,] - (-1)^{A_2(1+A_1)} [\, A_2, B \, [\, A_1, A_3 \,] \,] \\
&- (-1)^{A_3(1+A_1+A_2)} [\, A_3, B \, [\, A_1, A_2 \,] \,] = 0
\end{aligned}
$$

under the identification

$$
[\![\, A_1, A_2, \cdots, A_n \,]\!] = B \, [\, A_1, A_2, \cdots, A_n \,] \, .
$$

The equations of motion are then

$$
\begin{aligned}
Q_B B \widetilde{\Psi} + Q_B \Psi = 0 \,, \\
Q_B \widetilde{\Psi} + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n} \,] = 0 \,.
\end{aligned}
$$

We can eliminate $\widetilde{\Psi}$ by multiplying the second equation by B and adding the resulting equation to the first equation. We then obtain

$$
Q_B \Psi + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} \, B \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n} \,] = 0 \,.
$$

This takes the same form as the equation of motion in the theory with the level-matching condirion

$$Q_B\Psi + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} \, [\![\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n} \,]\!] = 0$$

under the identification

$$[\![\, A_1, A_2, \cdots, A_n \,]\!] = B \, [\, A_1, A_2, \cdots, A_n \,] \, .$$

We can show the perturbative equivalence of the two equations of motion with respect to g based on the fact that the cohomology of Q_B for the space with the constraints is the same as the cohomology of Q_B for the space without the constraints.

Once we have a solution Ψ to

$$Q_B \Psi + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} B\,[\,\underbrace{\Psi, \Psi, \cdots, \Psi}_{n}\,] = 0\,,$$

the equation

$$Q_B \widetilde{\Psi} + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!}\,[\,\underbrace{\Psi, \Psi, \cdots, \Psi}_{n}\,] = 0$$

can be regarded as an equation for $\widetilde{\Psi}$.

We can show that this equation can be solved without imposing any conditions on Ψ.

General solutions can be written as

$$\widetilde{\Psi} = \widetilde{\Psi}_* + \Delta\widetilde{\Psi}\,,$$

where $\widetilde{\Psi}_*$ and $\Delta\widetilde{\Psi}$ satisfy, respectively,

$$\begin{aligned} Q_B \widetilde{\Psi}_* &= \, -\sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n} \,]\,, \\ Q_B \Delta\widetilde{\Psi} &= 0\,. \end{aligned}$$

The fluctuation $\Delta\widetilde{\Psi}$ around $\widetilde{\Psi}_*$ obeys the free equation of motion, and it describes the extra free fields.

To summarize, the theory without the level-matching condition is perturbatively equivalent to the theory with the level-matching condition up to extra free fields described by $\Delta\widetilde{\Psi}$.

Expansion in terms of component fields

We consider closed strings in a flat spacetime of 26 dimensions.

When the eigenvalues of L_0 and $\widetilde{L}_0$ for a state carrying spacetime momentum k_μ are $-1+\ell+\alpha' k^2/4$ and $-1+\tilde{\ell}+\alpha' k^2/4$, respectively, we say that the level of the state is $(\,\ell,\,\tilde{\ell}\,)$.

The string field Ψ and the gauge parameter Λ can be expanded with respect to the level as

$$\Psi = \sum_{\ell,\,\widetilde{\ell}=0}^{\infty} \Psi_{(\,\ell\,,\,\tilde{\ell}\,)}\,, \qquad \Lambda = \sum_{\ell,\,\widetilde{\ell}=0}^{\infty} \Lambda_{(\,\ell\,,\,\tilde{\ell}\,)}\,,$$

where $\Psi_{(\,\ell\,,\,\widetilde{\ell}\,)}$, and $\Lambda_{(\,\ell\,,\,\widetilde{\ell}\,)}$ consist of states of the level $(\,\ell,\,\tilde{\ell}\,)$.

We are interested in $\Psi_{(1,1)}$ because the graviton is described by this string field.

Let us further decompose $\Psi_{(1,1)}$ and $\Lambda_{(1,1)}$ based on the world-sheet parity as follows:

$$\Psi_{(1,1)} = \Psi^{\text{odd}}_{(1,1)} + \Psi^{\text{even}}_{(1,1)}\,, \qquad \Lambda_{(1,1)} = \Lambda^{\text{odd}}_{(1,1)} + \Lambda^{\text{even}}_{(1,1)}\,.$$

The string field $\Psi^{\text{odd}}_{(1,1)}$ carrying ghost number 2 is expanded as

$$\begin{aligned}\Psi^{\text{odd}}_{(1,1)} = \int \frac{d^{26}k}{(2\pi)^{26}} \Big[& B(k)\, c_0 \widetilde{c}_0 \,|\, 0; k\,\rangle + \frac{1}{2} D(k) \left(c_{-1}c_1 - \widetilde{c}_{-1}\widetilde{c}_1\right) |\, 0; k\,\rangle \\ & + A_\mu(k) \left(\alpha^\mu_{-1} c_0^- c_1 + \widetilde{\alpha}^\mu_{-1} c_0^- \widetilde{c}_1\right) |\, 0; k\,\rangle \\ & + E_\mu(k) \left(\alpha^\mu_{-1} c_0^+ c_1 - \widetilde{\alpha}^\mu_{-1} c_0^+ \widetilde{c}_1\right) |\, 0; k\,\rangle \\ & + \frac{1}{4} G_{\mu\nu}(k) \left(\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1} + \alpha^\nu_{-1}\widetilde{\alpha}^\mu_{-1}\right) c_1 \widetilde{c}_1 \,|\, 0; k\,\rangle \Big]\,,\end{aligned}$$

where $B(k)$, $D(k)$, $A_\mu(k)$, $E_\mu(k)$, and $G_{\mu\nu}(k)$ are five component fields.

In this expansion, $c_0\widetilde{c}_0 \,|\, 0; k \,\rangle$ and $\big(\alpha^\mu_{-1} c_0^- c_1 + \widetilde{\alpha}^\mu_{-1} c_0^- \widetilde{c}_1\big) \,|\, 0; k \,\rangle$ are not annihilated by b_0^-. Therefore, the corresponding component fields $B(k)$ and $A_\mu(k)$ are absent in the theory with the level-matching condition.

On the other hand, the component fields $D(k)$, $E_\mu(k)$, and $G_{\mu\nu}(k)$ exist in closed string field theory with the level-matching condition and the graviton and the dilaton are described by these fields.

The equation

$$Q_B \Psi + \sum_{n=2}^{\infty} \frac{g^{n-1}}{n!} \, B \, [\, \underbrace{\Psi, \Psi, \cdots, \Psi}_{n} \,] = 0 \,,$$

is expanded at this level as follows:

$$
\begin{aligned}
\frac{\alpha' k^2}{4} D(k) - 2B(k) - \sqrt{\frac{\alpha'}{2}} k^\mu E_\mu(k) &= \mathcal{J}_D(k)\,, \\
\frac{1}{2}\sqrt{\frac{\alpha'}{2}} k^\mu D(k) - A^\mu(k) + E^\mu(k) + \frac{1}{2}\sqrt{\frac{\alpha'}{2}} k_\nu G^{\mu\nu}(k) &= \mathcal{J}_E^\mu(k)\,, \\
\frac{1}{4}\sqrt{\frac{\alpha'}{2}} k^\mu E^\nu(k) + \frac{1}{4}\sqrt{\frac{\alpha'}{2}} k^\nu E^\mu(k) + \frac{\alpha' k^2}{16} G^{\mu\nu}(k) &= \mathcal{J}_G^{\mu\nu}(k)\,, \\
2B(k) - \sqrt{\frac{\alpha'}{2}} k_\mu A^\mu(k) &= 0\,, \\
\sqrt{\frac{\alpha'}{2}} k^\mu B(k) - \frac{\alpha' k^2}{4} A^\mu(k) &= 0\,, \\
k^\mu A^\nu(k) - k^\nu A^\mu(k) &= 0\,,
\end{aligned}
$$

where the source terms $\mathcal{J}_D(k)$, $\mathcal{J}_E^\mu(k)$, and $\mathcal{J}_G^{\mu\nu}(k)$ are from the interaction terms.

The gauge parameter $\Lambda^{\rm odd}_{(1,1)}$ of ghost number 1 is expanded as

$$\Lambda^{\rm odd}_{(1,1)} = \int \frac{d^{26}k}{(2\pi)^{26}} \left[\chi(k)\, c_0^- \,|\, 0; k \,\rangle - \frac{1}{2}\, \xi_\mu(k) \left(\alpha^\mu_{-1} c_1 - \widetilde{\alpha}^\mu_{-1} \widetilde{c}_1 \right) |\, 0; k \,\rangle \right],$$

where $\chi(k)$ and $\xi_\mu(k)$ are two component fields.

The field $\xi_\mu(k)$ also exists in the theory with the level-matching condition, and it corresponds to the gauge parameter for the general coordinate transformation.

The field $\chi(k)$ is a new component field which appears in the theory without the level-matching condition.

The gauge transformation relevant for $\Psi^{\text{odd}}_{(1,1)}$ is $\delta_\Lambda \Psi^{\text{odd}}_{(1,1)} = Q_B \Lambda^{\text{odd}}_{(1,1)}$, and it is expanded in terms of component fields as

$$
\begin{aligned}
\delta_\Lambda B(k) &= -\frac{\alpha' k^2}{4}\chi(k)\,, \\
\delta_\Lambda D(k) &= 2\chi(k) + \sqrt{\frac{\alpha'}{2}} k^\mu \xi_\mu(k)\,, \\
\delta_\Lambda A_\mu(k) &= -\sqrt{\frac{\alpha'}{2}} k_\mu \chi(k)\,, \\
\delta_\Lambda E_\mu(k) &= -\frac{\alpha' k^2}{4}\xi_\mu(k)\,, \\
\delta_\Lambda G_{\mu\nu}(k) &= \sqrt{\frac{\alpha'}{2}} k_\mu \xi_\nu(k) + \sqrt{\frac{\alpha'}{2}} k_\nu \xi_\mu(k)\,.
\end{aligned}
$$

The corresponding equations of motion in the theory with the level-matching condition are given by

$$\begin{aligned}
\frac{\alpha' k^2}{4} D(k) - \sqrt{\frac{\alpha'}{2}} k^\mu E_\mu(k) &= \mathcal{J}_D(k)\,, \\
\frac{1}{2}\sqrt{\frac{\alpha'}{2}} k^\mu D(k) + E^\mu(k) + \frac{1}{2}\sqrt{\frac{\alpha'}{2}} k_\nu G^{\mu\nu}(k) &= \mathcal{J}_E^\mu(k)\,, \\
\frac{1}{4}\sqrt{\frac{\alpha'}{2}} k^\mu E^\nu(k) + \frac{1}{4}\sqrt{\frac{\alpha'}{2}} k^\nu E^\mu(k) + \frac{\alpha' k^2}{16} G^{\mu\nu}(k) &= \mathcal{J}_G^{\mu\nu}(k)\,,
\end{aligned}$$

which are obtained from the equations in the theory without the level-matching condition by simply setting $B(k) = 0$ and $A^\mu(k) = 0$.

It is easy to see the equivalence of the equations in the theory without the level-matching condition to the equations in the theory with the level-matching condition up to gauge transformations.

The equation

$$k^\mu A^\nu(k) - k^\nu A^\mu(k) = 0$$

means that the field strength of $A^\mu(k)$ vanishes. Therefore, we can bring any solution to the form where

$$A^\mu(k) = 0$$

by the gauge transformation using the gauge parameter $\chi(k)$. In this gauge, the component field $B(k)$ also vanishes:

$$B(k) = 0\,.$$

This establishes the equivalence under the gauge transformation.

Note that we can have solutions which cannot be brought to the form where $A^\mu(k) = 0$ by the gauge transformation if we compactify the target space on a torus. In this case the theory without the level-matching condition can be inequivalent to the theory with the level-matching condition nonperturbatively.

We expect that there will be a lot of such nonperturbative solutions for generic backgrounds. It will be also possible that there are similar nonperturbative solutions in superstring field theory without any restrictions for the Ramond sector and consequently the theory can be inequivalent to superstring field theory with the restricion for the Ramond sector. It would be interesting to explore more about such nonperturbative differences.

Conclusions and discussion

We constructed closed bosonic string field theory without imposing the constraints

$$(L_0 - \widetilde{L}_0)\, \Psi = 0\,, \qquad (b_0 - \widetilde{b}_0)\, \Psi = 0$$

on the closed string field Ψ. This is the first implementation of general covariance in the context of string theory without using the level-matching condition.

Even in closed string field theory with the level-matching condition, the general covariance is not implemented by simply replacing ordinary derivatives with covariant derivatives, but in closed string field theory without the level-matching condition its implementation is more exotic. This is why the extra free fields from $\widetilde{\Psi}$ do not couple to gravity despite the fact that they have kinetic terms.

In the formulations of open superstring field theory based on the restriction for the Ramond sector the operator X plays a distinctive role, and it seems difficult to replace it with a different operator.

On the other hand, the operator X does not have a special meaning in the approach by Sen and in fact the zero mode of the picture-changing operator was used instead of X.

In closed bosonic string field theory with the level-matching condition, the operator B plays a distinctive role, and it has been difficult to replace it with a different operator.

On the other hand, the operator B does not have a special meaning in closed bosonic string field theory without the level-matching condition, and we expect that it is possible to replace it with a different operator. While we have not understood the reason clearly, somehow use of extra free string fields seems to make string field theory more flexible.

This flexibility might play a role when we try to extract closed strings from open strings in the context of the AdS/CFT correspondence. For example, the world-sheet of closed strings is constructed from the world-sheet of open strings via unconventional gluing in the hexagon approach, and the representation of closed strings without using the level-matching condition might be useful.

The approach by Sen to the covariant treatment of the Ramond sector using extra free fields has opened a new direction of research in string field theory, and we believe that we have revealed that the approach has a counterpart which is related to the level-matching condition on closed string fields.

Some aspects of the new approach is still mysterious, and we hope that our results will help demystify this interesting new direction of research in string field theory.