Generalized (Holographic) c-Theorem and Entanglement Negativity

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Overview

Introduction

- In AdS/CFT correspondence the radial coordinate in the bulk is identified with the energy scale in the dual field theory.
- So moving in the radial direction can be thougt of RG flow in the dual field theory.
- Usually in the field theory side we deform the fixed point (UV) hamiltonian by some relevant perturbation which induces an RG flow and the flow approaches to another (IR) fixed point.
- The irreversibility of the RG flow in the field theory is encoded in the c-theorem.

- What happens in the gravity side is that in the UV we have AdS space as dual geometry and then it changes to some other geometry during the RG flow and then in the IR it again approaches to another AdS space.
- Holographic c-functions have been constructed before as radial evolution of a locally constructed function of the metric.
- It monotonically decreases along the RG flow and becomes equal to the central charge of the corresponding dual CFT at the fixed points.
- But there is another way to construct such a holographic c-function using the causal horizon (S. Banerjee, '15).
- The second law of causal horizon thermodynamics (T. Jacobson and R. Parentani, '03) plays a crucial role there.

- Brief overview of the prescription

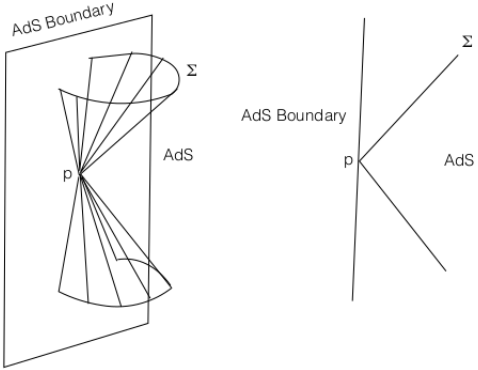


Figure: 1 Causal horizon Σ of the boundary point P in pure *AdS*

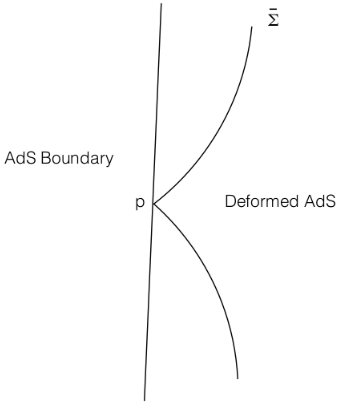


Figure: 2 Deformed causal horizon $\bar{\Sigma}$ of the boundary point p in deformed AdS.

- Now suppose instead of deforming the Hamiltonian by some relevant perturbation we take some thermal state of the CFT. The scale invariance (of this state) is clearly broken due to the finite temperature.

- Different physical quantities starts showing interesting scaling behavior as a function of (RT), $R \rightarrow$ system size and $T \rightarrow$ temperature. We call this as RG flow.

- In the bulk we have a black brane because the thermal state is dual to a black brane in the bulk.

- Our aim is to construct and study the behavior of a function which monotonically decreases along the RG from its UV value to IR value.

Calculation and Results

- We focus on four dimensional field theories only.
- The metric of the 5 dimensional black brane is

$$ds^2 = \frac{1}{z^2}\left(-(1-z^4)dt^2 + \frac{dz^2}{1-z^4} + d\vec{x}^2\right) \tag{1}$$

where we have set the radius of curvature to 1.

- Let us take a boundary point p with coordinates $z = x^\mu = 0$. The past(future) causal horizon of this point is nothing but the future(past) bulk light cone. So our job is to construct the ingoing null geodesics originating from the point p because the congruence of the ingoing null geodesics constitute the future bulk light cone.

- Introducing ingoing Eddington-Finkelstein coordinate $v = t - z^*$, $z^* = \frac{1}{2}\tan^{-1} z + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)$ the metric (1) can be written as

$$ds^2 = \frac{1}{z^2}\left[-(1-z^4)dv^2 - 2dvdz + d\vec{x}^2\right] \tag{2}$$

  we are working with the Eddington-Finkelstein coordinates because we don't want to face the coordinate singularity z=1.

- Since we will be working with null geodesics, instead of (2) we can work with the following conformally transformed metric

$$d\tilde{s}^2 = -(1-z^4)dv^2 - 2dvdz + d\vec{x}^2 \tag{3}$$

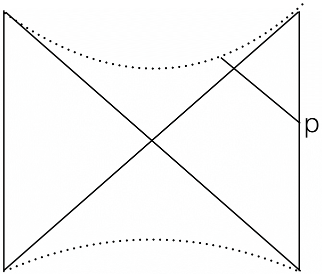


Figure: 3 Penrose diagram of the maximally extended AdS_5 black brane.

- Let λ be the affine parameter along a null geodesic in the conformally transformed metric $d\tilde{s}^2$. The null geodesics satisfy the equation

$$\tilde{g}_{AB}\frac{dx}{d\lambda}^{A}\frac{dx}{d\lambda}^{B} = 0 \tag{4}$$

- The metric is independent of v and x^i's. So we have four conserved charges

$$-(1-z^4)\frac{dv}{d\lambda} = -E + \frac{dz}{d\lambda} \tag{5}$$

$$\frac{dx^i}{d\lambda} = -p^i \tag{6}$$

E and $\vec{p}$ are the conserved charges along the null geodesic.

- Our convention is that the affine parameter λ is increasing as we move away from the boundary point $z = 0$.

- We will be working with the future bulk light-cone and with our convention for the affine parameter we have $\frac{dt}{d\lambda} \geq 0$ and hence $E \geq 0$.

- Using equations (4), (5) and (6) we get

$$\left(\frac{dz}{d\lambda}\right)^2 = E^2 - p^2(1 - z^4) \tag{7}$$

- So the null geodesics which can reach the boundary point must satisfy $E^2 - p^2 \geq 0$.

- We can parametrize the conserved charges as

$$\begin{aligned} E &= \cosh\eta && (8) \\ p^i &= \sinh\eta\ \hat{n}^i && (9) \end{aligned}$$

where $0 < \eta < \infty$ and $\hat{n}^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

- Solving these equations with boundary conditions $z(0,\eta) = v(0,\eta) = x^i(0,\eta,\hat{n}^i) = 0$ we get

$$\begin{aligned} z(\lambda,\eta) &= \sqrt{\frac{1}{\sinh\eta}\frac{1 - cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})}{1 + cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})}} \\ v(\lambda,\eta) &= \int_0^\lambda d\lambda' F(\lambda',\eta) \\ x^i(\lambda,\eta,\hat{n}^i) &= -\lambda\sinh\eta\ \hat{n}^i && (10) \end{aligned}$$

$$F(\lambda,\eta) = \sinh^2\eta \frac{(1+cn)^2\cosh\eta - \sqrt{2}\sqrt{1+cn^2}(1+cn)}{(1+cn)^2\sinh^2\eta - (1-cn)^2}$$

where cn is one of the Jacobian elliptic function.

- Equations (10) are the parametric form of a null hypersurface which is the bulk future light cone or the past causal horizon of the point $p(x^\mu = z = 0)$.
- $(\lambda, \eta, \theta, \phi)$ are the intrinsic coordinates on this null hypersurface and (η, θ, ϕ) are the comoving coordinates of a null geodesic parametrised by affine parameter λ.
- Next job is to compute the induced metric on the null hypersurface (10).

- Induced metric on the null hypersurface

$$ds^2_{ind} = \frac{1}{z^2}\left[-(1-z^4)\left(\frac{\partial v}{\partial \eta}\right)^2 - 2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta} + \lambda^2\cosh^2\eta\right] d\eta^2 + \frac{1}{z^2}\lambda^2\sinh^2\eta \, d\Omega_2^2 \tag{11}$$

(11) is the metric on $\lambda =$ constant space-like slice of the causal horizon parametrized by the coordinates $(\eta, \hat{n}^i)$.

- The volume form can be written as

$$dV_{ind} = c(\lambda,\eta) dV_H^3$$

$$c(\lambda,\eta) = \frac{\lambda^2}{z^3}\sqrt{-(1-z^4)\left(\frac{\partial v}{\partial \eta}\right)^2 - 2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta} + \lambda^2\cosh^2\eta} \tag{12}$$

- dV_H^3 is the volume form on a unit three dimensional hyperbolic space given by

$$\begin{aligned} ds^2_{H^3} &= d\eta^2 + \sinh^2\eta d\Omega_2^2 \\ dV_H^3 &= \sinh^2\eta \sin\theta d\eta d\theta d\phi \end{aligned}$$

- Now one can check that

$$\frac{\partial}{\partial\lambda} c(\lambda, \eta) \leq 0$$

- The Bekenstein-hawking entropy density associated to the volume element dV_{ind}

$$dS_{BH} = \frac{L^3}{4G_N} c(\lambda, \eta) dV_H^3$$

So our c-function is

$$c_\eta(\lambda) = \frac{L^3}{4G_N} c(\lambda, \eta) \tag{13}$$

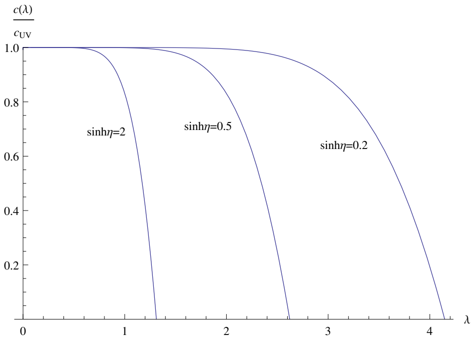


Figure: 4 We have plotted the c-function for three different values of η.

Physical Interpretation

- The boundary CFT is in a thermal state with a finite temperature. So it is effectively massive with a gap set by the temperature.
- There is a finite correlation length of the order of inverse of temperature. So in the deep IR correlation function goes to zero.
- The IR behaviour of the c-function we have constructed shows the same behavior that means it knows about the presence of this effective mass gap.
- It monotonically decreases from the central charge of UV-CFT, a_{UV} to zero at the curvature singularity.

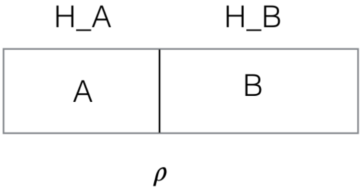
- So we can say that the causal horizon c-function faithfully quantifies the amount of pure quantum correlation or the effective number of quantum degrees of freedom that exists at different scales in the given thermal state.

- The holograpic c-function that we have constructed can't be identified with the entanglement entropy of the boundary field theory.

- First of all the space-like slices are not in general the extremal surfaces in the bulk. (Ryu and Takayanagi, '06).

- The finite temperature renormalized entanglement entropy has been calculated for a ball of radius R in R^3. (Liu and Mezei, '13)

- In the UV region $R \to 0$, $S_{REE} \to a_{UV}$. This matches with our c-function in the UV region.
- But in the IR region $R \to \infty$, $S_{REE} \neq 0$ and dominated by the thermal entropy. This does not match with our result.
- Entanglement entropy is not an entanglement measure in a mixed state. At finite temperature it becomes thermal entropy + quantum corrections.
- Thus it fails to capture the pure quantum part which should go to zero.
- Entanglement negativity - an entanglement measure in a mixed state.



- ρ is the density matrix describing the composite system $A \cup B$ with Hilbert space $H_A \bigotimes H_B$.

- Partial transpose of ρ w.r.t A $\rightarrow$

$$< i_A j_B|\rho^{T_A}|k_A l_B > = < k_A j_B|\rho|i_A l_B > \tag{14}$$

- If a density matrix is separable/unentangled, i.e ,

$$\rho = \sum_i p_i \rho_A^i \bigotimes \rho_B^i \tag{15}$$

where $p_i \geq 0$ and $\sum_i p_i = 1 \rightarrow \rho^{T_A} > 0$

- In general ρ^{T_A} is not positive semidefinite.

- Let $\{\lambda_i < 0\}$ denote the negative eigenvalues of ρ^{T_A}.
- Entanglement negativity $\rightarrow N(\rho) = \sum_i |\lambda_i|$

- Logarithmic entanglement negativity $\rightarrow$
$E(\rho) = \ln(1 + 2N(\rho))$

- If ρ is unentangled $\rightarrow \rho^{T_A} > 0 \rightarrow N(\rho) = 0 \rightarrow E(\rho) = 0$

- Logarithmic entanglement negativity at finite temperature in a two dimensional CFT was computed by Calabrese, Cardy and Tonni in 2015.

- Take an infinite line at temperature $T = \beta^{-1}$. The entanglement negativity of a single interval of length L is given by

$$E = \frac{c}{2} \ln \left[\frac{\beta}{\pi a} \sinh \left(\frac{\pi L}{\beta} \right) \right] - \frac{\pi c L}{2\beta} + f(e^{-2\pi L/\beta}) + 2 \ln c_{\frac{1}{2}} \quad (16)$$

Here a is short distance cut-off, c is the central charge of the CFT, f(x) is a universal scaling function which deends on the full operator content of the CFT such that f(1)=0 and f(0)=const.

- The renormalized negativity

$$E_R = L\frac{d}{dL}\Big|_\beta E \tag{17}$$

- UV limit : $\beta >> L$

$$E_R(UV) = \frac{c}{2}$$

- IR limit : $a << \beta << L$

$$E_R(IR) = 0 \tag{18}$$

- Question - Does it satisfy the monotonicity condition $T\frac{d}{dT}E_R \leq 0$?.

- In 4d we have to compute the logarithmic negativity for a ball of radius R in a thermal state at temperature T - UV limit is finite. Does it go to zero in the IR limit?

- If negativity becomes independent of the size of the ball in the high temperature limit, then IR limit is zero.

- Reasonable to expect - if there is a finite correlation length of order β.

- Thus we can infer that, causal horizon entropy density in the bulk $\Rightarrow$ there exist a monotonic function in the field theory, which is most likely an entanglement measure.

- There is another aspect of this problem. If we think that the space-time is built of quantum entanglement (M. Van Raamsdonk, '10) then we can interpret the c-function as an effective bulk measure of the quantum correlation between the field theory degrees of freedom at different energy scales.

- Behavior of our c-function correlates the two facts :
  Loss of quantum correlation/entanglement in the IR field theory $\Leftrightarrow$ The end of geometry which in this case is the formation of the curvature singularity behind the horizon.

- The holographic c-function is affected by things behind the horizon - Corresponding boundary c-function knows something behind the horizon.

- If entanglement negativity satisfies the monotonicity condition then this function will have some information about the interior.