Closed Superstring Field Theory and the Symplectic Geometry of Moduli Space

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Plan

Discuss the construction of closed superstring field theory having a non-polynomial action by exploring the hyperbolic geometry of Riemann surfaces and the symplectic geometry of the moduli space.

At the end, I would like to ask you whether this construction hints towards the possibility of a cubic closed string field theory.

Based on

arXiv:1708.04977, arXiv:1706.07366; Seyed Faroogh Moosavian, R.P.

To appear; R.P.

String Field Theory

String field theory is a refined definition of string theory, formulated in the language of QFT.

It generates the perturbative definition of the theory starting from an action. (**Witten, Zwiebach**)

SFT knows how to deal with the infrared divergences, using standard QFT techniques. (**RP, Rudra, Sen**)

Scattering amplitudes in SFT, computed using the appropriate contour prescription, is unitary. (**RP, Sen**)

Since SFT is based on an action, it may open the door towards the nonperturbative regime of string theory.

(**Schnabl; Yang, Zwiebach**)

The String Fields

The basic degrees of freedom in SFT are the string fields.

We can think of string fields as an arbitrary linear superpositions of the basis states of world-sheet CFT:

$$|\Psi\rangle = \sum_s |\Phi_s\rangle \psi_s$$

$\{|\Phi_s\rangle\}$: basis states for the Hilbert space of worldsheet CFT

ψ_s : **target space fields**

The string field $|\Psi\rangle$ must satisfy an appropriate reality condition and must be annihilated by $(b_0 - \bar{b}_0)$ and $(L_0 - \bar{L}_0)$.

The Quantum BV Master Action

Two kinds of string fields enter into the Batalin-Vilokvisky (BV) master action for closed superstring field theory:

- $|\Psi\rangle$: consists of NS states with picture number -1 and the R states with picture number $-\frac{1}{2}$.
- $|\widetilde{\Psi}\rangle$: consists of NS states with picture number -1 and the R states with picture number $-\frac{3}{2}$.

The covariant action satisfying the quantum BV master equation (**Sen**):

$$S = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Psi}|c_0^- Q_B \mathcal{G}|\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_{(n,h)=(1,0)}^{(\infty,\infty)} \frac{g_s^{n+2h}}{n!}\{\Psi^n\}_h\right]$$

On the NS states, the operator $\mathcal{G}$ acts as an identity operator, and on the R states as a PCO integrated over a cycle enclosing the puncture where the state is inserted: $\oint \frac{dz}{z}\chi(z)$

The String Vertices

Denote the h loop off-shell superstring measure for m number of NS states and n number of R states as $\Omega_{2Q}^{h,n,m}(\cdots)$.

$$\{\cdots\}_h = \int_{\mathcal{V}_{h,m,n}} \Omega_{2Q}^{h,n,m}(\cdots) \qquad Q = 3h - 3 + n + m$$

The string vertex $\mathcal{V}_{h,m,n}$ is a set of genus h Riemann surfaces with m NS and n R punctures, that does not include any degenerate Riemann surfaces. Each surface in $\mathcal{V}_{h,m,n}$ must be equipped with a choice of analytic local coordinates defined up to a constant phase around its punctures and a distribution of $2h - 2 + q_{\text{NS}}m + q_{\text{R}}n$ number of PCOs.

- The assignment of local coordinates must be independent of the labeling of the punctures
- The PCO distribution on each surface must avoid the occurrence of spurious poles.
- The PCO distribution should be invariant under the action of the mapping class group on the world-sheet.
- The PCO distribution should be invariant under the permutation of the punctures (NS and R separately).

The Geometric Identity

The BV master equation imposes a stringent consistency condition on the string vertices. (**Zwiebach, Sen**)

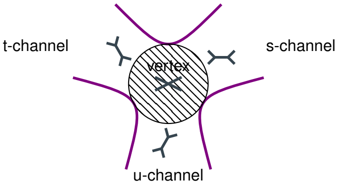
String vertex $\mathcal{V}_{h,m,n}$ together with the string Feynman diagrams F obtained by the plumbing fixture gluing of a set of string diagrams that belong to another string vertices must generate a **single cover** of the compactified moduli space $\overline{\mathcal{M}}_{h,m+n}$ (**Zwiebach**):

$$\overline{\mathcal{M}}_{h,m+n} = \mathcal{V}_{h,m,n} \bigcup F^{1}_{h,m,n} \bigcup \cdots \bigcup F^{Q}_{h,m,n}$$

$F^{Q}_{h,m,n}$: String diagram with Q gluing sites.

This is possible only if the local coordinates and the PCO distribution on the string diagrams that belong to the boundary of string string vertices match with that on the glued surface.

The Cell Decomposition of the Moduli Space



String vertices provide a cell decomposition of the moduli space and integrating the off-shell string measure over each cell is the contribution to the string amplitude from a specific Feynman diagram.

Constructing closed superstring field theory reduces to finding a suitable decomposition of the moduli spaces of closed Riemann surfaces with a choice of local coordinates and a choice of PCO distribution.

The Minimal Area Metric

For closed bosonic SFT, the complete set of string vertices can be constructed by using the local coordinates induced from the metric of least possible area under the condition that lengths of all the nontrivial closed curves on the surface be longer than or equal to 2π. (**Zwiebach**)

Unfortunately, the current understanding of the Riemann surfaces with minimal area metrics is very limited, and thus can't be used for constructing a calculable closed string field theory. (**N.Moeller; Wolf, Zwiebach**)

Therefore, in order to obtain an explicit construction of closed SFT, either study minimal area metric in great detail or find an alternate construction of string vertices.

Recently, cubic and one loop tadpole string vertices have been constructed explicitly, without using the minimal area metric. However, this construction might not be convenient for constructing the complete set of string vertices. (**Erler, Konopka, Sachs**)

Hyperbolic Geometry and Closed Superstring Field Theory

Alternate construction of the complete set of string vertices using Riemann surfaces endowed with metric having -1 constant curvature.

Every $\mathcal{R}$, hyperbolic Riemann surface with n punctures and h handles, can be obtained by the proper discontinuous action of the Fuchsian group Γ on the Poincaré upper half-plane $\mathbb{H}$:

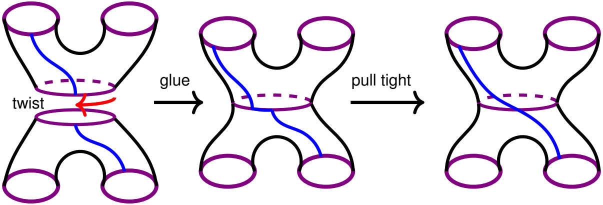
$$\mathcal{R} \simeq \mathbb{H}/\Gamma$$

The Poincaré upper half-plane is endowed with the metric:

$$ds^2 = \frac{dzd\bar{z}}{(\mathrm{Im}\, z)^2}$$

The Fuchsian group Γ is a subgroup of $PSL(2, \mathbb{R})$, the automorphism group of $\mathbb{H}$.

The Teichmüller Space of Hyperbolic Metrics



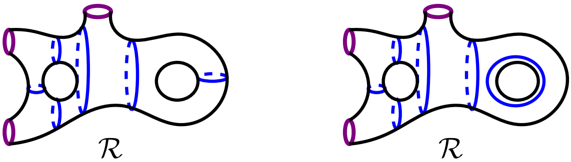
The nice feature of hyperbolic metric is that, every hyperbolic metric on a genus h Riemann surface with n punctures can be obtained by the geometric sum of $2h - 2 + n$ number of hyperbolic pairs of pants.

Each attaching site has two parameters: the geodesic length ℓ of the boundary and the twist τ performed before gluing them. The Fenchel-Nielsen parameters at $3h - 3 + n$ attaching sites

$$(\tau_j, \ell_j) \qquad 1 \leq j \leq 3h - 3 + n \qquad \tau_j \in \mathbb{R} \qquad \ell_j \in \mathbb{R}^+$$

for a fixed pants decomposition parametrizes all n punctured genus h Riemann surfaces with hyperbolic metric on it.

The Moduli Space of Hyperbolic Metric



Two Riemann surfaces with two different values for the Fenchel-Nielsen coordinates can't be related two each other by the action of infinitesimal diffeomrphisms. However, two such Riemann surfaces may be related to each other by acting with the elements in the **mapping class group** (MCG, the group of large diffeomorphisms).

Unfortunately, the action of MCG group elements on the Fenchel-Nielsen coordinates is not tractable. Therefore, a clear characterization of the moduli space in terms of the Fenchel-Nielsen coordinates is not known.

Surprisingly, Maryam Mirzakhani showed that, in spite of this difficulty, the Fenchl-Nielsen coordinates can be efficiently used for performing integrations over the moduli space.

The Naive String Vertices

Consider $\mathcal{R}^*$, genus-h hyperbolic Riemann surface with m number of NS-punctures, n number of R-punctures, and having no simple closed geodesic with length $\ell \leq c_*$, with $c_* \ll 1$.

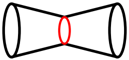
The local coordinates around the punctures are defined to be $e^{\frac{\pi^2}{c_*}} w$, where w is the natural local coordinate induced from the hyperbolic metric. The unit area disc around a puncture on a hyperbolic Riemann surface is isometric to a cusp, punctured disc with metric

$$ds_0^2 = \left(\frac{|dw|}{|w| \ln |w|} \right)^2$$

All inequivalent hyperbolic Riemann surfaces $\mathcal{R}^*$ with a consistent choice of PCO distribution form the string vertex $\mathcal{V}^0_{h,m,n}$.

Check whether the space $\mathcal{V}^0_{h,m,n}$ together with the Feynman diagrams provide a single cover of $\overline{\mathcal{M}}_{h,m+n}$.

Plumbing Fixture of Hyperbolic Riemann Surfaces



Plumbing fixture of punctured unit discs with metric ds_0^2 $\left\{ w_1 w_2 = t \,\middle|\, |w_1|, |w_2|, |t| < 1 \right\}$ provides an annulus with curvature accumulated on a curve $\Longrightarrow$ **Plumbing fixture of hyperbolic Riemann surfaces does not provide a hyperbolic Riemann surface.**

Curvature of the metric on the glued surface ds^2_{graft} has a deviation of the magnitude $\left(\frac{1}{\ln|t|}\right)^2$ from that of the hyperbolic metric.

The metric $e^{2f} ds^2_{\text{graft}}$ on a Riemann surface has constant curvature -1 provided

$$Df - e^{2f} = \mathbf{C}$$

C: Gauss curvature of the metric ds^2_{graft} $\qquad$ D: the Laplace-Beltrami operator on the surface

Hyperbolic metric on the glued surface, can be found by solving the curvature correction equation.

The Expansion for the Hyperbolic Metric

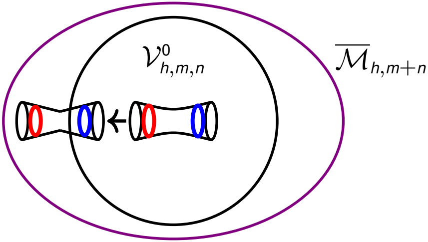
For small t the leading order term in the expansion for the hyperbolic metric on $\mathcal{R}_t$, surface with k number of plumbing collars, written in terms of the i^{th} collar geodesic length $\ell_i = -\frac{2\pi^2}{\ln |t_i|}$

$$ds^2_{\text{hyp}} = ds^2_{\text{graft}} \left\{ 1 + \sum_{i=1}^{k} \frac{\ell_i^2}{3} \left(E^{\dagger}_{i,1} + E^{\dagger}_{i,2} \right) \right\}$$

$E^{\dagger}_{i,1}$ and $E^{\dagger}_{i,2}$ are the modified Eisenstein series E associated to the pair of punctures plumbed to form the i^{th} collar, with a modification on the plumbing collar regions. (**Obitsu, Wolpert, Wolf**)

The Corrected String Vertices

The local coordinates on the surfaces lying at the boundary of the string vertices and that on the glued surfaces do not match $\Longrightarrow$ The naive string vertices provide only an approximate cell decomposition of the moduli space, which become more and more accurate as we take the parameter $c_* \longrightarrow 0$.



We can correct the string vertices to the order c_*^2 by modifying the choice of local coordinates around the punctures continuously on the surfaces that belongs to the boundary region of the string vertex in a way that compensate for the deviation of the induced metric on the glued surfaces from the hyperbolic metric.

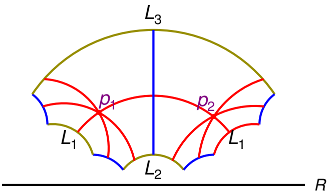
Distribution of Picture Changing Operators

The PCO distribution on a Riemann surface that belongs to a string vertex:

- Must be invariant under the action of the mapping class group.
- Must be invariant under the permutation of the punctures (NS and R separately).
- Must satisfy the geometric equation.

PCO distribution on Surface with only NS punctures

On $\mathcal{R}$, a genus h Riemann surface with only m number of NS punctures, we must insert $2h - 2 + m$ number of PCOs. There are $2h - 2 + m$ pairs of pants on $\mathcal{R} \Rightarrow$ one PCO for each pair of pants.



PCO distribution that is symmetric with respect the boundaries, and invariant under the interchange of the two hexagons

$$\widehat{\mathcal{X}}(\mathbf{P}) \equiv \frac{\widehat{\mathcal{X}}(p_1\,\{\mathbf{P}\}) + \widehat{\mathcal{X}}(p_2\,\{\mathbf{P}\})}{2} \qquad \widehat{\mathcal{X}}(z) = \mathcal{X}(z) - \partial\xi(z)dz$$

p_1, p_2: the centroids of right and left hexagons.

Identities for the Simple Closed Geodesics

Consider r multi-curves on $\mathcal{R}$ made up of simple closed geodesics:

$$\gamma_i = \sum_{v_i=1}^{k_i} \gamma_{iv_i} \qquad\qquad i = 1, \cdots, r$$

Assume that these multi-curves satisfy the identity:

$$\sum_{i=1}^{r} \sum_{h_i \in \frac{\mathrm{MCG}(\mathcal{R})}{\mathrm{Stab}(\gamma_i)}} Z_i\left(\ell_{h_i\gamma_i}\right) = 1$$

$\mathrm{Stab}(\gamma_i)$: the subgroup of $\mathrm{MCG}(\mathcal{R})$ that keeps the multi-curve γ_i invariant.

Property of Z_i: $$\lim_{\ell_{\gamma_i}\to\infty} Z_i\left(\ell_{\gamma_i}\right) = \mathcal{O}\left(e^{-\ell_{\gamma_i}}\right)$$

Two such identities are known: one is due to **McShane and Mirzakhani** and the other is due **Luo and Tan**.

The Stabilizer Group

Cutting $\mathcal{R}$ along γ_i provides the pairs of pants $\mathbf{P}_{i1}, \cdots, \mathbf{P}_{im_i}$ and the disconnected surfaces $\mathcal{R}_{i1}, \cdots, \mathcal{R}_{is_i}$

$$\bigcap_{v_i=1}^{k_i} \mathrm{Stab}(\gamma_{iv_i}) =$$

$$\mathrm{MCG}\left(\mathcal{R}_{i1}\right) \times \cdots \times \mathrm{MCG}\left(\mathcal{R}_{is_i}\right) \times \mathrm{Dehn}^*\left(\gamma_{i1}\right) \times \cdots \times \mathrm{Dehn}^*\left(\gamma_{ik_i}\right)$$

$\mathrm{Dehn}^*\left(\gamma_{iv_i}\right)$: the group generated by the half twist with respect to the curve γ_{iv_i}, if γ_{iv_i} bounds a torus with a single boundary and otherwise is generated by the full twist.

Let us also assume that the simple closed geodesics γ_{iv_i} satisfy the identity:

$$\sum_{h_{iv_i} \in \mathrm{Dehn}^*(\gamma_{iv_i})} Y\left(\ell_{h_{iv_i}\gamma_{iv_i}}, \tau_{h_{iv_i}\gamma_{iv_i}}\right) = 1$$

Combined Identity for the Simple Closed Geodesics

Combine these two identities to obtain the following identity:

$$\sum_{i=1}^{r} \sum_{g_i \in \frac{\mathrm{MCG}(\mathcal{R})}{\prod_{q_i=1}^{s_i} \mathrm{MCG}(\mathcal{R}_{iq_i})}} H_i\left(g_i\gamma_i\right) = 1$$

$$H_i\left(g_i\gamma_i\right) \equiv \frac{\left|\bigcap_{v_i=1}^{k_i} \mathrm{Stab}\left(\gamma_{iv_i}\right)\right|}{\left|\mathrm{Stab}\left(\gamma_i\right)\right|} Z_i\left(\ell_{g_i\gamma_i}\right) \prod_{v_i=1}^{k_i} Y\left(\ell_{g_i\gamma_{iv_i}}, \tau_{g_i\gamma_{iv_i}}\right)$$

The MCG Invariant PCO distribution

The elements of $\mathrm{MCG}\left(\mathcal{R}_{iq_i}\right);\ q_i = 1, \cdots, s_i$, act trivially on the pairs of pants $\mathbf{P}_{i1}, \cdots, \mathbf{P}_{im_i}$

All elements of $\mathrm{MCG}\left(\mathcal{R}\right)$ that acts non-trivially on these pairs of pants belong to the subgroup

$$\frac{\mathrm{MCG}\left(\mathcal{R}\right)}{\prod_{q_i=1}^{s_i} \mathrm{MCG}\left(\mathcal{R}_{iq_i}\right)}$$

$\Longrightarrow$ the following distribution of PCO's is invariant under the action of $\mathrm{MCG}\left(\mathcal{R}\right)$:

$$\sum_{i=1}^{r} \sum_{g_i} H_i \left(g_i \gamma_i\right) \bigwedge_{b_i=1}^{m_i} \widehat{\mathcal{X}} \left(g_i \mathbf{P}_{ib_i}\right)$$

Each term in this distribution with index i contain only m_i number of PCO's, we need $2g - 2 + m$ number of PCO's in each term .

MCG Invariant PCO distribution

Consider the identities for the surfaces $\mathcal{R}_{iq_i}$ $\quad i = 1, \cdots, r \quad q_i = 1, \cdots, s_i$ and continue this process until we obtain only pairs of pants as component surfaces.

At the last stage, we obtain the PCO distribution as a summation over all the elements in Mod$(\mathcal{R})$:

$$\mathbf{K} = \sum_{\mathbf{G}} \sum_{g \,\in\, \mathrm{MCG}(\mathcal{R})} D_{\mathbf{G}}\left(\vec{\ell}_g \circ \rho_{\mathbf{G}}, \vec{\tau}_g \circ \rho_{\mathbf{G}}\right) \wedge_{i=1}^{Q} \widehat{\mathcal{X}}\left(\mathbf{P}^i_g \circ \rho_{\mathbf{G}}\right)$$

The sum is over all possible trivalent graphs with m external legs and h loops.

$D_{\mathbf{G}}\left(\vec{\ell}\rho_{\mathbf{G}}, \vec{\tau}\rho_{\mathbf{G}}\right)$ can be found using an algorithm similar to the Feynman rules for a cubic theory.

Each term contains $2h - 2 + m$ number of PCOs, and by construction, the PCO distribution $\mathbf{K}$ is invariant under the action of all elements in $\mathrm{MCG}(\mathcal{R})$.

Symmetrize the final PCO distribution with respect to the punctures on $\mathcal{R}$.

The Improved String Vertices

It is possible to show that the PCO distribution **K** satisfy the geometric identity, up to an error. The origin of this error is same as that of the error encountered in the choice of local coordinates and can be fixed in a similar way.

PCO distribution can be generalized to the case when there are R-punctures.

To the second order in c_*, the modified string vertices $\mathcal{V}^2_{h,m,n}$ a single cover of the moduli space.

$$\mathcal{V}^2_{h,m,n} \bigcup F^{0,1}_{h,m,n} \bigcup \cdots \bigcup F^{0,Q}_{h,m,n} \overset{\mathcal{O}(c_*^2)}{=} \overline{\mathcal{M}}_{h,m+n}$$

The string vertices $\mathcal{V}^2_{h,m,n}$ provide a consistent closed superstring field theory by keeping c_* very small.

Taking c_* very small corresponds to increasing the size of the region inside the moduli space that corresponds the string vertex. For constructing a string field theory we are allowed to use string vertex having arbitrary size.

The Off-shell Superstring Measure

The superstring measure is given by

$$\Omega_p^{h,m,n}(\Phi) = \frac{\langle \mathcal{R} | \mathcal{B}_p | \Phi \rangle}{(2\pi \mathrm{i})^Q}$$

$|\Phi\rangle$: element of the Hilbert space $\mathcal{H}_{NS}^{\otimes m} \otimes \mathcal{H}_{R}^{\otimes n}$ of the world-sheet SCFT with the ghost number $n_\Phi = p + 6 - 6h$

$\langle \mathcal{R}|$: surface state associated with the surface $\mathcal{R}$

$$\mathcal{B}_p = \sum_{\substack{q=0 \\ q \leq N}}^{p} K^{(q)} \wedge B_{p-q} \qquad N = 2h - 2 + m + \frac{n}{2}$$

$K^{(q)}$: q-form component of $\mathbf{K}$

B_{p-q}: differential form constructed using the Beltrami differentials

The Effective Expression and Trivalent Graphs

We can express $\{\Psi_1, \cdots, \Psi_{m+n}\}_h$ as follows:

$$\sum_{\mathbf{G}} \int_{\ell_1^{\mathbf{G}}=c_*}^{\infty} \int_{\tau_1^{\mathbf{G}}=-\infty}^{\infty} \cdots \int_{\ell_Q^{\mathbf{G}}=c_*}^{\infty} \int_{\tau_Q^{\mathbf{G}}=-\infty}^{\infty} D_{\mathbf{G}}\left(\vec{\ell}_{\mathbf{G}}, \vec{\tau}_{\mathbf{G}}\right) \sum_{\substack{q=0 \\ q\leq N}} \frac{\langle \mathcal{R}|X^{(q)} \wedge B_{p-q}|\Phi\rangle_{\mathbf{G}}}{(2\pi \mathrm{i})^Q}$$

The sum is over all possible trivalent graphs with $m+n$ external legs and h loops.

$\left(\vec{\ell}_{\mathbf{G}}, \vec{\tau}_{\mathbf{G}}\right)$ is the Fenchel-Nielsen coordinates for the pants decomposition correspond to the trivalent graph $\mathbf{G}$.

In the absence of R-punctures, $X^{(q)}$ is the q-form component of

$$\mathbf{X} \equiv \wedge_{i=1}^{Q} \widehat{\mathcal{X}}\left(\mathbf{P}_{\mathbf{G}}^{i}\right)$$

The form of $\mathbf{X}$ requires modifications in the case with R-punctures.

The Possibility of a Cubic Closed String Field Theory?

We can express $\{\Psi_1, \cdots, \Psi_n\}_h$ in bosonic closed string field theory as follows:

$$\sum_{\mathbf{G}} \int_{\ell_1^{\mathbf{G}}=c_*}^{\infty} \int_{\tau_1^{\mathbf{G}}=-\infty}^{\infty} \cdots \int_{\ell_Q^{\mathbf{G}}=c_*}^{\infty} \int_{\tau_Q^{\mathbf{G}}=-\infty}^{\infty} D_{\mathbf{G}}\left(\vec{\ell}_{\mathbf{G}}, \vec{\tau}_{\mathbf{G}}\right) \frac{\langle \mathcal{R}|B_{2Q}|\Phi\rangle_{\mathbf{G}}}{(2\pi \mathrm{i})^Q}$$

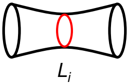
The elementary interaction strength of SFT seems to be arising from a cubic theory. This statement might be true, if the function $D_{\mathbf{G}}\left(\vec{\ell}_{\mathbf{G}}, \vec{\tau}_{\mathbf{G}}\right)$ is equal to the product of $2h-2+n$ number of identical functions.

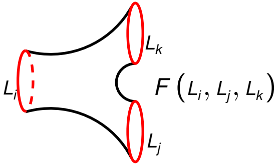
The argument of these functions must be the three boundary lengths of a pair of pants that belongs to the pants decomposition associated with the graph $\mathbf{G}$., i.e.

$$D_{\mathbf{G}}\left(\vec{\ell}_{\mathbf{G}}, \vec{\tau}_{\mathbf{G}}\right) = \prod_{i=1}^{2h-2+n} F\left(P_G^i\right)$$

The known identities may not have such property.

The Propagator and the Cubic Interaction?





Summary

Local coordinates around the punctures induced from the hyperbolic metric after a slight modification satisfy the geometric condition, imposed by the BV equation.

Identities satisfied by the simple closed geodesics on the hyperbolic Riemann surfaces can be used to obtain an MCG invariant PCO distribution that satisfy the BV equation.

Such a choice of PCO distribution leads to a simple effective expression for the quantum master action.

$\Rightarrow$**Hyperbolic Riemann surfaces can be used to construct a calculable closed superstring field theory.**

Provide efficient tools for computing the amplitudes in the conventional formulation of superstring theory.

Future Prospects

Superstring field theory in the presence of D-branes.

Implications of the developments in the Symplectic geometry of the moduli space in string theory.

Study the already known non-renormalization statements by using the SFT effective actions, and check whether there exist similar statements for massive states.

Effective action for superstring theory in AdS with large size.

Finite temperature SFT.

Tachyon condensation in closed string theory.


THANK YOU

The Mirzakhani-McShane Identity

For any genus g hyperbolic Riemann surface $\mathcal{R}_{g,n}$ with n borders $L_1, \cdots, L_n$ having lengths $l_1, \cdots, l_n$ satisfying $3g - 3 + n > 0$ (**Mirzakhani, McShane**):

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \frac{\mathcal{D}(l_1, \ell_{\gamma_1}, \ell_{\gamma_2})}{l_1} + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1i}} \frac{\mathcal{E}(l_1, l_i, \ell_\gamma)}{l_1} = 1$$

$$\mathcal{D}(x, y, z) = x - \ln\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right)$$

$$\mathcal{E}(x, y, z) = 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right)$$

$\mathcal{F}_i$: set of pairs of MCG images of the simple closed curves $\{\gamma_1, \gamma_2\}$ bounding a pair of pants with L_i.

$\mathcal{F}_{ij}$: set of MCG images of the simple closed curves γ bounding a pair of pants with L_i and L_j.

Property of Mirzakhani-MacShane Identity

$$\lim_{y,z\to\infty} \mathcal{D}(x,y,z) = \lim_{y,z\to\infty} 4e^{-\frac{y+z}{2}} \sinh(\frac{x}{2}) \to 0$$

$$\lim_{z\to\infty} \mathcal{E}(x,y,z) \to 0$$

The Luo-Tan identity for simple closed geodesics on hyperbolic Riemann surfaces with or without borders provides another decomposition of unity. (Luo, Tan)

Identity for Dehn Twist

$$\sum_{g \in \mathrm{Dehn}^*(\gamma)} \frac{\sin^2\left(2^{M_{g\gamma}} \pi \tau_{g\gamma} / \ell_{g\gamma}\right)}{\left(2^{M_{g\gamma}} \pi \tau_{g\gamma} / \ell_{g\gamma}\right)^2} = 1$$

τ_γ is the twist with respect to the simple closed geodesic γ_i

$M_\gamma = 1$ if γ bounds a torus with one boundary component otherwise $M_\gamma = 0$.

In general the twisting parameter along γ can takes value between 0 and ℓ_γ.

In the case of a simple geodesic γ separating off a one-handle, τ_γ varies with fundamental region $\left\{0 \leq \tau_\gamma \leq \frac{\ell_\gamma}{2}\right\}$.

The reason is that every Riemann surface $\mathcal{R} \in \mathcal{M}_{1,1}$ comes with an elliptic involution