On universal solutions in OSFT

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Plan

- 30 years of level truncation in OSFT
  20 years of Sen’s conjectures
- SU(1,1) symmetry of the ghost vertex in Siegel gauge
- Scan of universal solutions in level truncation

30 years of level truncation in OSFT

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THE STATIC TACHYON POTENTIAL IN THE OPEN BOSONIC STRING THEORY

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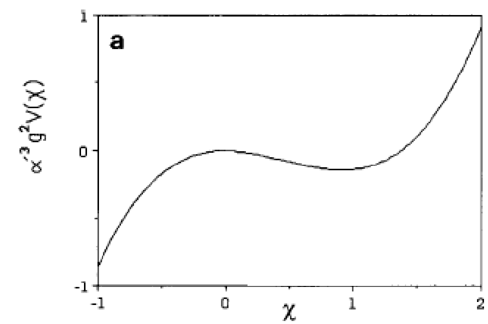
In an effort to understand whether the open bosonic string theory has a stable vacuum, the four-point contribution to the static tachyon potential is computed. This off-shell calculation is carried out using covariant string field theory.

1. Introduction

String theory [1,2] is a proposal for a consistent theory of gravity [3], that can contain anomaly-free gauge groups large enough to include SU(3)×SU(2)×U(1) [4,5]. It is at present the unique theory having the possibility of combining the four fundamental interactions in a unified quantum theory.

in this direction and may be helpful in analyzing more realistic string theories.

The bosonic string theory is an ideal arena for such investigations. Since it contains a tachyon, the perturbative vacuum is unstable. Our goal is to try to determine, at tree level, whether a stable vacuum exists. Unfortunately, we are unable to incorporate all the tree-level effects. Instead, we include tree-level diagrams up to four external legs.



First level 0 calculation

$$V_{\text{static}} = \frac{1}{\alpha'^3 g^2}\left(-\tfrac{1}{2}\chi^2 + \frac{1}{3!}\,\frac{3^{9/2}}{2^6}\,\chi^3 + \frac{\bar{\lambda}}{4!}\,\chi^4 + \ldots\right)$$

First level ∞ calculation

20 years of Sen's conjectures

- The difference in the action between the unstable vacuum and the perturbatively stable vacuum should be $E = V\,T_{25}$, where V is the volume of space-time and T_{25} is the tension of the D25-brane.
- Lower-dimensional Dp-branes should be realized as soliton configurations of the tachyon and other string fields.
- The perturbatively stable vacuum should correspond to the closed string vacuum. In particular, there should be no physical open string excitations around this vacuum.

Ashoke Sen '98 - '99 (this version taken from Ellwood, Taylor '01)

20 years of Sen's conjectures

Sen's conjectures are now considered proven analytically in OSFT:

1. In bosonic theory: MS '05
   In superstring: Erler '13

2. In bosonic theory: Erler, Maccaferri '14

3. In bosonic theory: MS, Ellwood '06
   In superstring: Erler '13

20 years of Sen's conjectures

- First evidence, however, has been provided by a numerical approach – level truncation

0	-0.684616		
2	-0.959377		
4	-0.987822		Sen, Zwieabach 1999
6	-0.995177		
8	-0.997930		
10	-0.999182		Moeller, Taylor 2000
12	-0.999822		
14	-0.999826		
16	-1.000375		
18	-1.000494		Gaitotto, Rastelli 2002
20	-1.000563		Kishimoto, Takahashi 2009
22	-1.000602		
24	-1.000623		
26	-1.000631		Kishimoto 2011
28	-1.000632		
30	-1.000627		

← $-\frac{2^{12}}{3^{10}}\pi^2$

← Actually they used (L,2L) scheme, so their numbers are a bit different

Lots of other references for superstring and/or lump solutions etc.

20 years of Sen's conjectures

- The computation gets messy pretty quickly, already at level 2 where the string field can takes the form

$$|T\rangle = tc_1|0\rangle + uc_{-1}|0\rangle + v \cdot \frac{1}{\sqrt{13}} L_{-2}c_1|0\rangle$$

one has to find stationary points of

Sen, Zwiebach 1999

$$\begin{aligned} f^{(4)}(T) \;=\; & 2\pi^2\Big(-\frac{1}{2}t^2 + \frac{3^3\sqrt{3}}{2^6}\,t^3 \\ & -\frac{1}{2}u^2 \;+\; \frac{1}{2}v^2 \;+\; \frac{11\cdot 3\sqrt{3}}{2^6}\,t^2\,u - \frac{5\cdot 3\sqrt{39}}{2^6}\,t^2\,v \\ & +\frac{19}{2^6\sqrt{3}}\,t\,u^2 + \frac{7\cdot 83}{2^6\cdot 3\sqrt{3}}\,t\,v^2 - \frac{11\cdot 5\sqrt{13}}{2^5\cdot 3\sqrt{3}}\,t\,u\,v\Big)\,. \end{aligned}$$

Tricks for level truncation

Critical ingredients for a useful computerized level truncation

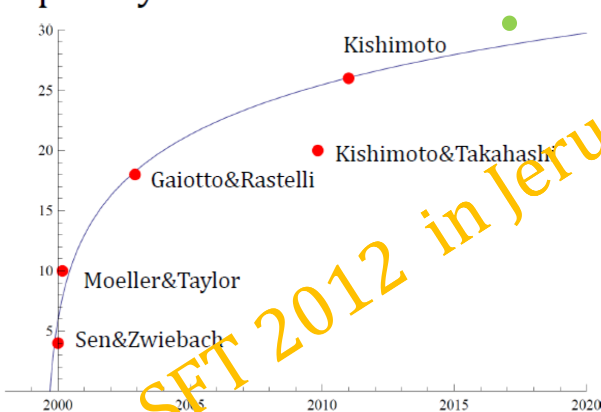
1. Convenient basis of states
   universality, twist condition, gauge condition, SU(1,1) condition,…
2. Conservation laws for vertex computation
3. Finding a good starting point for Newton’s method
4. Algorithmic tricks (parallelism)
5. Having good observables
6. Fits to infinite level

Tricks for level truncation

- Employing these tricks and better computer power one moves on a logarithmic curve in time

Future of level truncation

- Hopefully soon level 30 should be reached



Required tools:
-universal basis
- conservation laws
- C++
-SU(1,1) singlet basis
-Parallelism
- ???

N.B.: Level 30 is interesting, as we should see the oscillation for the tachyon vacuum energy predicted by Gaiotto and Rastelli.

SU(1,1) symmetry of Witten vertex

- Zwiebach (2000), following an observation of Hata and Shinohara, has shown that the SU(1,1) generators

$$J_3 = \frac{1}{2}\sum_{n=1}^{\infty}\left(c_{-n}b_n - b_{-n}c_n\right), \qquad J_+ = \sum_{n=1}^{\infty} n c_{-n}c_n, \qquad J_- = \sum_{n=1}^{\infty}\frac{1}{n}b_{-n}b_n,$$

preserve the Witten's vertex in Siegel gauge $b_0\Psi = 0$

$$_{123}\langle V|\left(J^{(1)}_{\pm,3} + J^{(2)}_{\pm,3} + J^{(3)}_{\pm,3}\right) = 0$$

For free action the symmetry was known by Siegel and Zwiebach already in 1986

SU(1,1) symmetry of Witten vertex

- Under this symmetry $(n\, c_{-n}, b_{-n})$ transforms as a doublet, so it mixes ghost numbers
- It allows however to restrict the string field states to the singlet sector. In consequence the coefficients of $(3b_{-1}c_{-3})|\Omega\rangle$ and $(b_{-3}c_{-1})|\Omega\rangle$ are equal for such solutions, and there are more relations like this at higher levels

SU(1,1) symmetry of Witten vertex

- Convenient basis is obtained by forming twisted descendants using c=-2 twisted Virasoro generators

$$L_n'^{gh} = L_n^{gh} + n j_n^{gh} + \delta_{n,0} = \sum_{m=-\infty}^{\infty} (n-m) : b_m c_{n-m} :$$

GRSZ 2001

on the twisted Virasoro primaries

$$|j,m\rangle \equiv N_{j,m} (J_-)^{j-m} |j,j\rangle$$

where

$$|j,j\rangle \equiv c_{-2j} \ldots c_{-1} c_1 |0\rangle$$

$$N_{j,m} = \prod_{k=m+1}^{j} \left(j(j+1) - k(k-1)\right)^{-\frac{1}{2}}$$

SU(1,1) symmetry of Witten vertex

- Generic string field in the universal sector in Siegel gauge can be written as

$$\Psi = \sum_{K,L,j,m} t_{K,L,j,m} L^{m}_{-K} L'^{gh}_{-L} |j, m\rangle$$

K, L are multi-indices

For classical solutions we are interested in *m=0*, since ghost number *g=2m+1.*

SU(1,1) symmetry of Witten vertex

- Completeness of the presentation (verification of the multiplicities) can be checked by computing the character for Siegel gauge

$$
\begin{aligned}
\mathrm{ch}^{\mathrm{Siegel}}(q,y) &= \frac{1}{1+y}\mathrm{ch}^{\mathrm{gh}}(q,y) \\
&= \mathrm{ch}^{\mathrm{gen}}(q)\sum_{g=-\infty}^{\infty}\sum_{s=|g-1|}^{\infty}(-1)^{s+g-1}y^g q^{\frac{s^2+s}{2}} \\
&= \mathrm{ch}^{\mathrm{gen}}(q)\sum_{g=-\infty}^{\infty}\sum_{s=|g-1| \bmod 2}^{\infty} y^g\left(q^{\frac{s^2+s}{2}} - q^{\frac{(s+1)^2+s+1}{2}}\right)
\end{aligned}
$$

and rewriting it as

$$
\begin{aligned}
\mathrm{ch}^{\mathrm{Siegel}}(q,y) &= \sum_{g=-\infty}^{\infty}\sum_{j=|g-1|/2 \bmod 1}^{\infty} y^g \mathrm{ch}_j(q) \\
&= \sum_{j=0 \bmod 1/2}^{\infty}\sum_{m=-j}^{j} y^{2m+1}\mathrm{ch}_j(q)
\end{aligned}
$$

where

$$\mathrm{ch}_j(q) = \mathrm{ch}^{\mathrm{gen}}(q)q^{h_j}\left(1-q^{2j+1}\right)$$

$$h_j = j(2j+1)$$

N.B. The c=-2 theory possesses null states for ghost numbers 0,-3,-6,…

SU(1,1) symmetry of Witten vertex

- The SU(1,1) symmetry of the vertex

$$ {}_{123}\langle V| \left(J^{(1)}_{\pm,3} + J^{(2)}_{\pm,3} + J^{(3)}_{\pm,3} \right) = 0 $$

can be used to derive the Wigner-Eckart theorem

$$ \langle V_3| L'^{gh}_{-I_1} |j_1, m_1\rangle L'^{gh}_{-I_2} |j_2, m_2\rangle L'^{gh}_{-I_3} |j_3, m_3\rangle = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} C(j_1, j_2, j_3, I_1, I_2, I_3) $$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the standard SU(2) 3-j symbols

and $C(j_1, j_2, j_3, I_1, I_2, I_3)$ is *m*-independent reduced vertex

SU(1,1) symmetry of Witten vertex

- Cyclic property nicely manifest

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

- Also the selection rules for which spins can combine together at the vertex trivially follow SU(2)
- Less expected properties stemming from

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

imply that at *m=0* the sum of all three spins must be an even integer

SU(1,1) symmetry of Witten vertex

- Let us introduce $J_{1,2} = \frac{1}{2}\left(J_+ \pm J_-\right)$

  (this is where SU(1,1) differs from SU(2))

- Exponentiating the generators we find two which preserve the gh.n.=1 (m=0) subspace

$$\begin{aligned} e^{i\pi J_1}|j,m\rangle &= (-1)^j|j,-m\rangle \\ e^{\pi J_2}|j,m\rangle &= (-1)^{j+m}|j,-m\rangle \end{aligned}$$

  These finite transformations change the sign of odd-spins and provide independent explanation why $\sum j_i \in 2\mathbb{Z}$

Conservation laws for vertex computations

- For non-singlet string fields standard Virasoro basis is the most convenient and one can use the standard conservation laws
- For singlet string fields it is useful to be able to compute the vertices directly. It can be done by combining *L* and *J* conservation laws

$$\begin{aligned}\langle V_3|L'^{gh(2)}_{-m} &= \langle V_3| \left(\sum_n \alpha_n^{m(1)} L'^{gh(1)}_n + \sum_n \alpha_n^{m(2)} L'^{gh(2)}_n + \sum_n \alpha_n^{m(3)} L'^{gh(3)}_n + \alpha^{m(c)} c\right) \\ &-\langle V_3| \left(\sum_n \left(m\beta_n^{m(1)} + n\alpha_n^{m(1)}\right) j_n^{gh(1)} + \sum_n \left(m\beta_n^{m(2)} + n\alpha_n^{m(2)}\right) j_n^{gh(2)}\right. \\ &\left. + \sum_n \left(m\beta_n^{m(3)} + n\alpha_n^{m(3)}\right) j_n^{gh(3)} + m\beta^{m(q)} q\right).\end{aligned}$$

and observing that we need only few extra states $j^{gh}_{-k} L'^{gh}_{-M} c_1|0\rangle$

Starting points for Newton's method

- For tachyon solution one simple uses the solution from a previous level and improves it by Newton's method for the next level. The very first starting point is thus at level 0.

- For more exotic solutions one has to start at higher level where there are more solutions. A convenient way is to use linear homotopy method. The trick is to continuously deform our system of equations to something we can solve completely e.g.

$$
\begin{aligned}
t_1(1-t_1) &= 0 \\
t_2(1-t_2) &= 0 \\
&\dots
\end{aligned}
$$

Parallelism

- These days computers tend to have more cores often with lower performance (for energy consumption reasons). So one has to parallelize the computation.
- For vertices the parallelization is a bit undeterministic (but it works!), since we have a recursive algorithm, but we do not know when a given core finishes its task
- Parallelization can be also employed for matrix manipulations in the Newton’s method

Observables for universal solutions

- Energy computed from the action *E=V+1*

- The only independent Ellwood invariant

$$E_0 = -4\pi i \langle E[c\bar{c}\partial X^0 \bar{\partial} X^0] | \Psi \rangle + 1$$

- Out-of-Siegel-gauge equations (we take just the first one)

$$\Delta_S = \left| \langle 0 | c_{-1} j_2^{gh} \, | Q\Psi + \Psi * \Psi \rangle \right| = | \langle 0 | c_{-1} c_0 b_2 \, | Q\Psi + \Psi * \Psi \rangle |$$

- Ratios $$R_n = (-1)^n \frac{54}{65} \frac{\langle \Psi | c_0 L_{2n}^m | \Psi \rangle}{\langle \Psi | c_0 L_0 | \Psi \rangle}$$

Siegel gauge universal solutions

Our results:

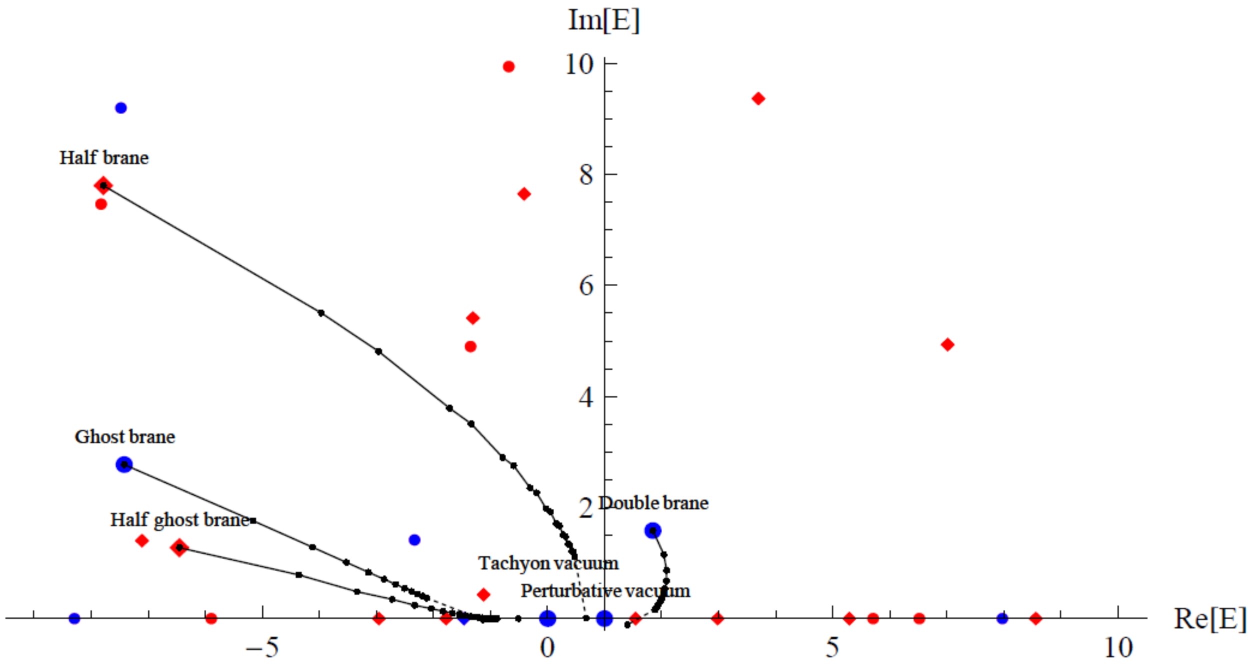


Figure 1: Twist even solutions at level 6 and twist non-even solutions at level 5 represented as dots or diamonds respectively in the energy complex plane. SU(1,1) singlet solutions are represented by blue color, non-singlet solutions by red. By black line and dots we show the improvement of the interesting solutions to higher levels, the dashed part of the line is infinite level extrapolation.

Siegel gauge universal solutions

- Properties of the surviving solutions

Solution	$\text{Energy}^{L=\infty}$	$E_0^{L=\infty}$	Δ_S^∞	Reality	Twist even	SU(1,1) singlet
tachyon vacuum	-8×10^{-6}	0.0004	-7×10^{-6}	yes	yes	yes
single brane	1	1	0	yes	yes	yes
"ghost brane"	$-1.13+0.024i$	$-1.01+0.11i$	0.08	no	yes	yes
"double brane"	$1.40+0.11i$	$1.23+0.04i$	0.20	possibly*	yes	yes
"half ghost brane"	-0.51	-0.66	0.17	pseudoreal**	no	no
"half brane"	$0.68-0.01i$	$0.54+0.1i$	0.23	no	no	no

* as $L\to\infty$

** for $L\geq 22$

Siegel gauge universal solutions

Twist even SU(1,1) singlets

Solution	level	Energy	E_0	Δ_S	Im/Re
perturbative vacuum		1	1	0	0
tachyon vacuum	30	-0.000627118	0.0120671	0.00090829	0
	∞	-8×10^{-6}	0.0004	-7×10^{-6}	0
"double brane"	28	$1.8832-0.161337i$	$1.32953+0.178426i$	0.535827	0.51589
	∞	$1.40+0.11i$	$1.23+0.04i$	0.20	-0.05
"ghost brane"	28	$-2.11732-0.371832i$	$-1.19063+0.165908i$	0.267977	0.398931
	∞	$-1.13+0.024i$	$-1.01+0.11i$	0.08	0.33
No. 9	24	-0.35331	3.35886	1.97365	0
	∞	0.4	2.9	0.5	0
No. 10	24	-4.89552	2.46007	7.37201	0
	∞	$-6.$	2.3	10.	0
No. 14	28	$-0.61986-2.07194i$	$-0.304394-0.125328i$	0.51849	0.567639
	∞	$-0.059-0.28i$	$-0.18-0.08i$	0.16	0.4
No. 16	24	$19.1573+6.2523i$	$1.39521+0.659265i$	2.64483	1.40716
No. 49	24	$-6.50268-8.39148i$	$2.15961+0.077867i$	1.79055	0.750635
No. 51	24	$1.67067-4.96206i$	$-0.091597-0.341405i$	2.03866	1.31641
No. 55	24	$-13.18-1.514i$	$0.538205+0.192177i$	1.35519	0.222453
No. 65	24	$-16.6534-5.7377i$	$0.541071-0.782319i$	1.40638	0.756182
No. 77	24	$-5.74905-4.10849i$	$0.508235+0.516736i$	0.438803	0.844295
	∞	$-5.-1.8i$	$0.4+0.1i$	0.2	0.8
No. 81	24	$-7.96846-3.62476i$	$0.95657-0.64505i$	1.63462	0.783216
	∞	$-5.+1.i$	$0.8-0.6i$	1.9	0.8
No. 91	24	$-3.86278+0.78003i$	$-0.75477-0.028228i$	1.49166	0.314904
	∞	$-1.-1.i$	$-0.5+0.0i$	0.9	0.4
No. 93	24	$-6.60883-8.49812i$	$1.83907-0.047786i$	1.58305	0.532952
No. 95	24	$-9.83474-8.05476i$	$0.24612+0.462712i$	1.02673	0.812354
	∞	$0.-8.i$	$0.4+0.5i$	-0.7	0.6

Twist even non-singlets

No. 231	22	$6.27071-30.8278i$	$-1.35262+1.22029i$	6.83759	0.496046

Non-even non-singlets

"half ghost brane"	26	-0.88489	-0.427091	0.105198	0.600394
	∞	-0.51	-0.66	0.17	0.31
"half brane"	24	$0.47454-1.11238i$	$0.488976+0.107413i$	0.565206	1.20649
	∞	$0.68-0.010i$	$0.54+0.10i$	0.23	1.3
No. 264	22	-10.5493	1.10974	7.58984	0.229845
	∞	-11	0.9	11	-0.3

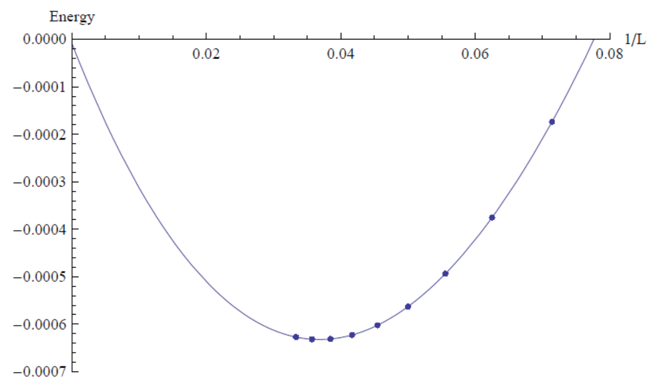
List of all solutions
With $|E| < 50$

They are not that many!

Tachyon vacuum

- Tachyon vacuum up to level 30

Level	Energy	E_0	Δ_S
2	0.0406234	0.110138	0.0333299
4	0.0121782	0.0680476	0.0145013
6	0.00482288	0.0489211	0.00841347
8	0.00206982	0.0388252	0.00564143
10	0.000817542	0.0318852	0.00412431
12	0.000177737	0.0274405	0.00319231
14	−0.00017373	0.0238285	0.00257255
16	−0.000375452	0.0213232	0.00213597
18	−0.000493711	0.0190955	0.00181467
20	−0.000562955	0.0174832	0.00156995
22	−0.000602262	0.0159666	0.00137834
24	−0.000622749	0.0148397	0.00122487
26	−0.000631156	0.0137381	0.0010996
28	−0.000631707	0.0129049	0.00099569
30	−0.000627118	0.0120671	0.00090829
∞	-8×10^{-6}	0.0004	-7×10^{-6}
σ	2×10^{-6}	0.0013	2×10^{-6}



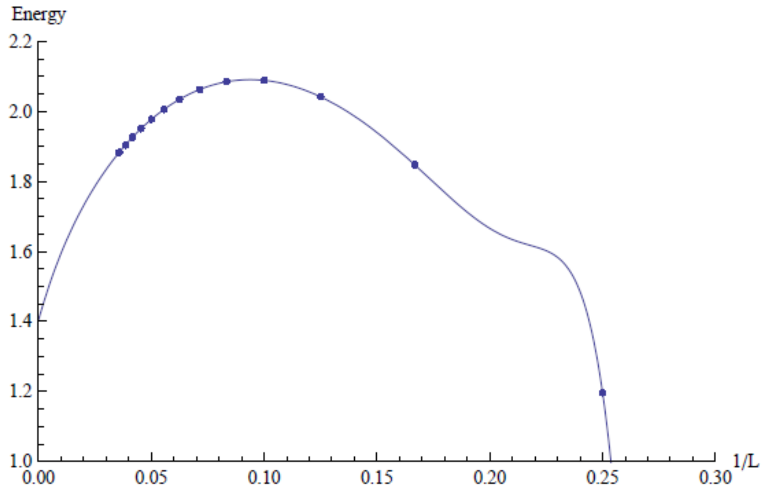
Level	R_1	R_2	R_3	R_4
2	1.12793	0	0	0
4	1.06964	1.07986	0	0
6	1.04647	1.0519	1.05352	0
8	1.03459	1.03755	1.04077	1.03698
10	1.02744	1.0293	1.03137	1.03308
12	1.02269	1.02398	1.02537	1.0268
14	1.01931	1.02026	1.02126	1.02232
16	1.01679	1.01752	1.01828	1.01908
18	1.01485	1.01542	1.01602	1.01664
20	1.0133	1.01377	1.01425	1.01474
22	1.01204	1.01243	1.01282	1.01323
24	1.01099	1.01132	1.01165	1.01199
26	1.01011	1.01039	1.01067	1.01096
28	1.00936	1.0096	1.00985	1.01009
30	1.00871	1.00892	1.00913	1.00935
∞	0.999972	0.999974	0.99998	1.000
σ	0.000009	0.000006	0.00005	0.002

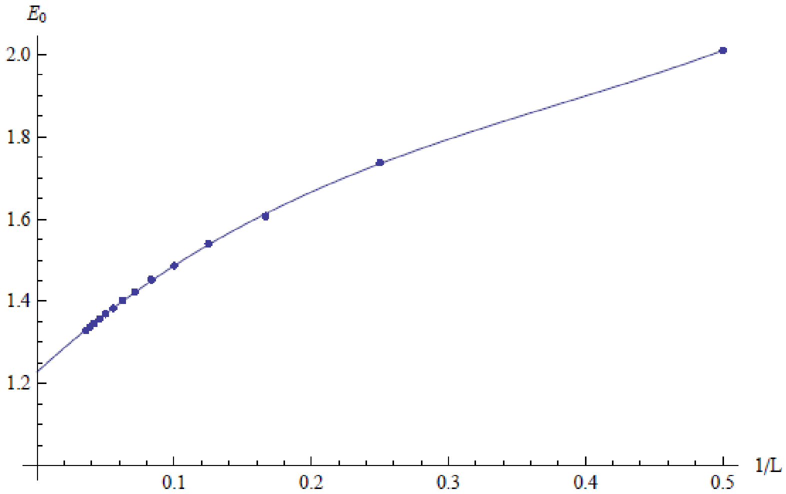
“Double brane”

Level	Energy	E_0	Δ_S	Im/Re
2	$-1.42791 - 3.40442i$	$2.00934 - 0.054534i$	2.9861	2.47302
4	$1.19625 - 2.25966i$	$1.73651 + 0.117637i$	1.65103	5.23177
6	$1.84813 - 1.58507i$	$1.60634 + 0.195442i$	1.2563	2.20828
8	$2.04207 - 1.14971i$	$1.53973 + 0.217911i$	1.05199	1.51157
10	$2.08908 - 0.866428i$	$1.48598 + 0.228010i$	0.921766	1.20095
12	$2.08515 - 0.674602i$	$1.45210 + 0.227059i$	0.83043	1.01867
14	$2.06302 - 0.53887i$	$1.42232 + 0.224184i$	0.762358	0.895048
16	$2.03499 - 0.439057i$	$1.40194 + 0.218266i$	0.709389	0.804018
18	$2.00593 - 0.363272i$	$1.38304 + 0.212332i$	0.666812	0.733269
20	$1.9778 - 0.304197i$	$1.36942 + 0.205378i$	0.631713	0.675682
22	$1.95135 - 0.257139i$	$1.35632 + 0.198765i$	0.602187	0.627204
24	$1.92679 - 0.218971i$	$1.34654 + 0.191784i$	0.576934	0.585387
26	$1.90411 - 0.187545i$	$1.33691 + 0.185169i$	0.555036	0.548697
28	$1.8832 - 0.161337i$	$1.32953 + 0.178426i$	0.535827	0.51589
∞	$1.40 + 0.11i$	$1.23 + 0.04i$	0.20	-0.05
σ	$0.02 + 0.02i$	$0.02 + 0.05i$	0.01	0.16

Out-of-Siegel-gauge equations seem to be problematic !
But it is not far from being asymptotically real.

“Double brane”





At level 10 it looks like a double brane, but then it wonders away…

Interestingly the quadratic identities are asymptotically violated at 10-20% level

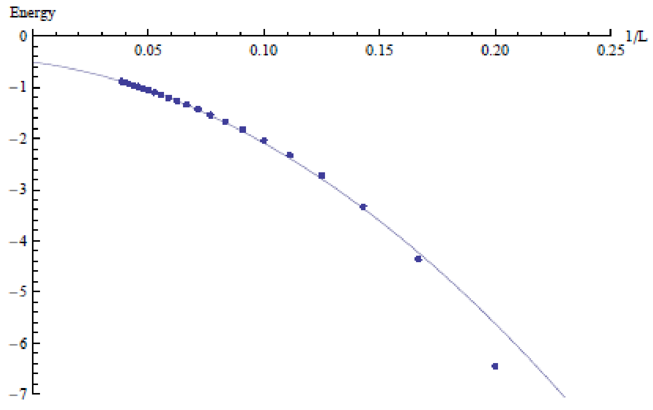
“Ghost brane”

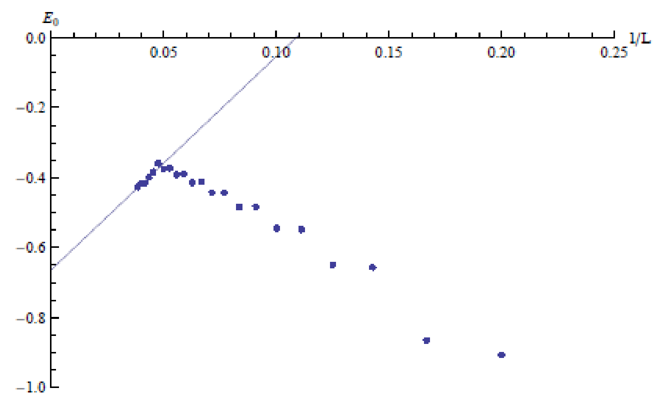
Level	Energy	E_0	Δ_S	Im/Re
4	$-15.534 - 6.15021i$	$-2.67655 + 0.349878i$	2.06533	0.410863
6	$-7.41971 - 2.77358i$	$-1.94969 + 0.275957i$	1.02147	0.418622
8	$-5.16142 - 1.76196i$	$-1.68072 + 0.250813i$	0.71436	0.432211
10	$-4.12161 - 1.28659i$	$-1.52959 + 0.229158i$	0.56818	0.434482
12	$-3.52505 - 1.0123i$	$-1.43859 + 0.216265i$	0.482212	0.432647
14	$-3.13749 - 0.834158i$	$-1.37324 + 0.204977i$	0.425217	0.429121
16	$-2.86477 - 0.709192i$	$-1.32713 + 0.19685i$	0.384407	0.424871
18	$-2.66193 - 0.616662i$	$-1.29055 + 0.189495i$	0.353578	0.42024
20	$-2.50477 - 0.545352i$	$-1.26259 + 0.183675i$	0.329357	0.415469
22	$-2.37916 - 0.488679i$	$-1.23917 + 0.178292i$	0.309749	0.410702
24	$-2.27628 - 0.442528i$	$-1.22039 + 0.173791i$	0.293495	0.406007
26	$-2.19032 - 0.404195i$	$-1.20411 + 0.169568i$	0.279762	0.402137
28	$-2.11732 - 0.371832i$	$-1.19063 + 0.165908i$	0.267977	0.398931
∞	$-1.13 + 0.024i$	$-1.01 + 0.11i$	0.08	0.33
σ	$0.03 + 0.003i$	$0.04 + 0.02i$	0.01	0.03

Solution inherently complex, but it has smallest Δ_S ,,
difference between the two energies, and also the R’s are close to 1!

“Half ghost brane”

Level	Energy	E_0	Δ_S	Im/Re
4	$-12.316 - 3.03642i$	$-1.67202 + 0.546917i$	1.2313	0.865144
5	$-6.45268 - 1.27696i$	$-0.906424 + 0.418332i$	0.620185	1.16737
6	$-4.35705 - 0.78598i$	$-0.865572 + 0.35625i$	0.415511	0.954718
7	$-3.33673 - 0.484238i$	$-0.656885 + 0.311362i$	0.312715	1.06523
8	$-2.72264 - 0.346946i$	$-0.649858 + 0.280771i$	0.254214	0.941507
9	$-2.32595 - 0.239544i$	$-0.548369 + 0.252446i$	0.214425	1.00849
10	$-2.03976 - 0.180921i$	$-0.544639 + 0.229285i$	0.187496	0.919346
11	$-1.83136 - 0.130289i$	$-0.483438 + 0.20816i$	0.166609	0.966213
12	$-1.6664 - 0.0999362i$	$-0.483896 + 0.190449i$	0.151204	0.894168
13	$-1.53812 - 0.0724385i$	$-0.442433 + 0.172804i$	0.138347	0.930032
14	$-1.43071 - 0.0550496i$	$-0.442547 + 0.157289i$	0.128364	0.86812
15	$-1.34368 - 0.0389625i$	$-0.412389 + 0.141337i$	0.119637	0.896007
16	$-1.26801 - 0.028504i$	$-0.413658 + 0.127156i$	0.112624	0.838677
17	$-1.20493 - 0.0188315i$	$-0.390587 + 0.111523i$	0.106296	0.860183
18	$-1.14861 - 0.0125338i$	$-0.391441 + 0.0971558i$	0.101085	0.804464
19	$-1.10069 - 0.00689461i$	$-0.373155 + 0.0800877i$	0.0962737	0.819172
20	$-1.05703 - 0.00341677i$	$-0.374456 + 0.0632733i$	0.0922374	0.760415
21	$-1.01932 - 0.00075642i$	$-0.359549 + 0.0384798i$	0.0884464	0.760498
22	-0.984567	-0.383157	0.0927055	0.65173
23	-0.955695	-0.399688	0.0997222	0.662377
24	-0.929072	-0.415166	0.102253	0.618092
25	-0.906413	-0.417504	0.104245	0.637956
26	-0.884894	-0.427091	0.105198	0.600394
∞	-0.51	-0.66	0.17	0.31

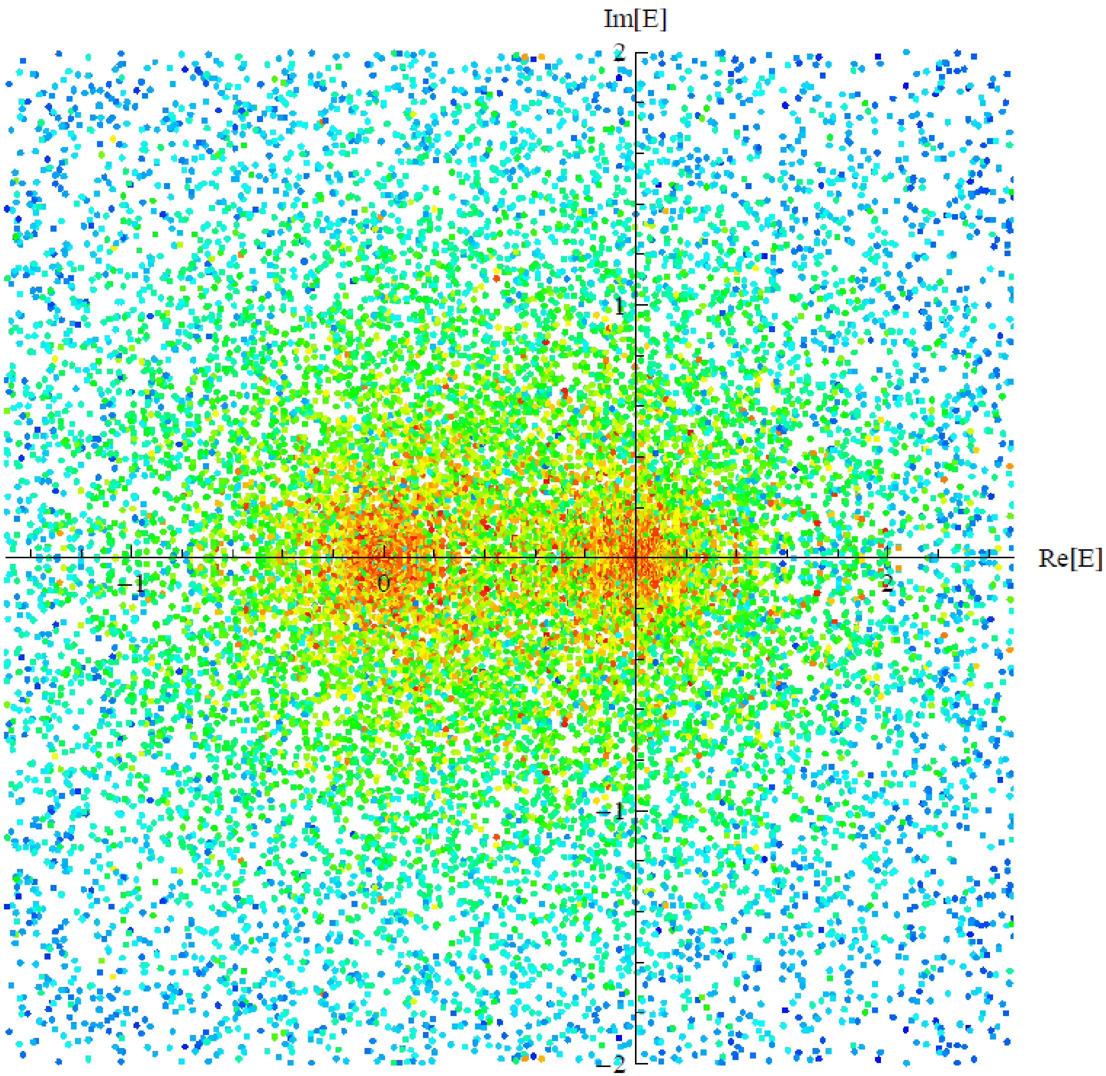




Solution becomes pseudo-real starting at L=22.

Some fits are difficult!

Relaxing the Siegel gauge?



- Without gauge we have enormous proliferation of solutions (figure shows level 6)
- No good notion of good solution except for closeness of the two energies (denoted by color)
- Newton's method erratic without gauge fixing

Conclusions

- Level truncation is still fun after 30 years
- Provides interesting information and clues about possible solutions (see talk by Vošmera).
- Are there proper exotic solutions (possibly complex) ? If yes, what is their interpretation?
- One can go beyond twist symmetry, reality, singlet condition etc., but not past gauge fixing (cf. talk by Kudrna)