LINEAR INDEPENDENCE OF SPECIAL VALUES OF JACOBI-THETA CONSTANTS

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ABSTRACT. Let $m \geq 2$ be any integer and let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1 (in other words, β is a Pisot-Vijayaraghavan number). Let $a_1, a_2, \ldots, a_m$ be distinct positive integers. In this article we prove that the following infinite sums

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{\beta^{a_1 n^2}}, \quad \sum_{n=1}^{\infty} \frac{1}{\beta^{a_2 n^2}}, \ldots, \sum_{n=1}^{\infty} \frac{1}{\beta^{a_m n^2}}$$

are $\mathbb{Q}(\beta)$-linearly independent. As a consequence, we prove the linear independence of special values of Jacobi-theta-constants.

1. Introduction

For a complex number τ which lies in the upper complex half plane $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, the Jacobi-theta-constants (see for details [8], Chapter 10) are defined as

$$\theta_2(\tau) = 2\sum_{n=1}^{\infty} q^{(n+1/2)^2}, \ \theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \text{ and } \quad \theta_4(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}$$

where $q = e^{i\pi\tau}$. These constants $\theta_2(\tau)$, $\theta_3(\tau)$ and $\theta_4(\tau)$ are periodic function with period 2.

Elsner, Luca and Tachiya [4] proved that the values of the Jacobi-theta constants $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any distinct integers m and n, under some conditions on τ. On the other hand, in 2018, Elsner and Tachiya [3] also proved that for any three distinct integers ℓ, m and n, the constants $\theta_3(\ell\tau)$, $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$. These results motivate us to ask the following questions.

Question 1. Let $m \geq 2$ be an integer and let $a_1, \ldots, a_m$ be distinct positive integers. Let $k \in \{2, 3, 4\}$ be a given integer. Then, under what conditions on τ, the m-theta values

$$\theta_k(a_1\tau), \ldots, \theta_k(a_m\tau)$$

are linearly independent over $\overline{\mathbb{Q}}$ with 1?

Question 2. Let $m \geq 2$ be an integer and let $a_1, \ldots, a_m$ be distinct positive integers. Let $k \neq \ell \in \{2, 3, 4\}$ be given integers. Then under what conditions on τ, the set of m-theta values

$$\{\theta_k(a_i\tau), \theta_\ell(a_j\tau) : 1 \leq i \neq j \leq m\}$$

is linearly independent over $\overline{\mathbb{Q}}$ with 1?

In this direction, in 2019, the second author proved the following result.

Theorem 1.1. [[7], *Theorem 1*] *Let $m, b \geq 2$ be integers and $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$. Let $\tau_0 = \dfrac{i \log b}{\pi}$. Then the numbers*

$$1, \quad \theta_3(a_1\tau_0), \quad \theta_3(a_2\tau_0), \ldots, \theta_3(a_m\tau_0)$$

are $\mathbb{Q}$-linearly independent.

Recently, in 2020, C. Elsner and second author [5] proved that for distinct positive integers $a_1, \ldots, a_m$, the functions $\theta_3(a_1\tau), \ldots, \theta_3(a_m\tau)$ are linearly independent over $\mathbb{C}(\tau)$ and they also studied $\overline{\mathbb{Q}}$-linear independence results as values of these function in the case $m = 3$ and for certain class of τ. This latter result answered the above Question 1 in the case $m = 3, k = 3$ and under some restriction on a_i's.

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The main result of this article is to strengthening Theorem 1.1. In particular, we remove the restrictive condition $\sqrt{a_i/a_j} \notin \mathbb{Q}$ on a_i's and we extend the result for an algebraic integer β instead of an integer $b \geq 2$ in Theorem 1.1. More precisely, we prove the following result.

Theorem 1.2. *Let $m \geq 2$ be any integer and $f_1, f_2, \ldots, f_m : \mathbb{N} \to \mathbb{Z}\backslash\{0\}$ be functions of polynomial growth. Let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1, and let $a_1, a_2, \ldots, a_m$ be distinct positive integers. Then the m infinite sums*

$$1, \quad \sum_{n=1}^{\infty} \frac{f_1(n)}{\beta^{a_1 n^2}}, \quad \sum_{n=1}^{\infty} \frac{f_2(n)}{\beta^{a_2 n^2}}, \ldots, \sum_{n=1}^{\infty} \frac{f_m(n)}{\beta^{a_m n^2}} \tag{1.1}$$

are $\mathbb{Q}(\beta)$-linearly independent with 1.

The number β in Theorem 1.2 is called *Pisot-Vijayaraghavan number* or *Pisot number*. For example all the positive integer greater than 1 are Pisot numbers. For any natural number $d \geq 1$, there are algebraic integers of degree d which are Pisot-Vijayaraghavan numbers. As an immediate consequence of Theorem 1.2, and by the periodicity of $\theta_3(\tau)$ and $\theta_4(\tau)$, we have some interesting corollaries.

If we take $f_i(n) = 1$ for $i = 1, \ldots, m$ in Theorem 1.2, we have the first corollary as follows.

Corollary 1.1. *Let $\beta > 1$ and a_i's be as in Theorem 1.2. Set $\tau_n = \frac{i \log \beta}{\pi} + 2n$. Then for every integer $n \geq 1$, the m-Jacobi theta values*

$$\theta_3(a_1\tau_n), \quad \theta_3(a_2\tau_n), \ldots, \theta_3(a_m\tau_n)$$

are $\mathbb{Q}(\beta)$-linearly independent with 1.

First we note that in view of the result of Elsner, Luca and Tachiya, each of the Jacobi-theta values in Corollary 1.1 is transcendental. Also, note that if we take any two Jacobi-theta values in Corollary 1.1, then they are algebraically independent over $\mathbb{Q}$ but not when we take 3 distinct values. Therefore, the result stated in Corollary 1.1 is effective, when we consider more than three Jacobi-theta values.

By taking $f_i(n) = (-1)^n$ for all $i = 1, \ldots, m$ in Theorem 1.2, we have the following corollary.

Corollary 1.2. *Let $\beta > 1$, a_i's and τ_n be as in Corollary 1.1. Then for every integer $n \geq 1$, the m-theta values*

$$\theta_4(a_1\tau_n), \quad \theta_4(a_2\tau), \ldots, \theta_4(a_m\tau_n)$$

are $\mathbb{Q}(\beta)$-linearly independent with 1.

In [1], D. Bertand proved the following result; *If $\tau \in \mathbb{H}$ such that $e^{i\pi\tau}$ is algebraic, then the values $\theta_3(\tau), \theta_4(\tau)$ and $\theta_3'(\tau)$ are algebraically independent over $\mathbb{Q}$.* By this result, we see that each values in Corollary 1.2 is transcendental. In this case also assertion stated in Corollary 1.2 is effective when $m \geq 2$.

If we take $f_i(n) = (-1)^{i+1}$ for $i = 1, \ldots, m$ for an odd integer m in Theorem 1.2, we have the following corollary involving both $\theta_3(\tau)$ and $\theta_4(\tau)$.

Corollary 1.3. *Let $\beta > 1$, a_i's and τ_n be as in Corollary 1.1. Then for every integer $n \geq 1$, the m-theta values*

$$\theta_3(a_1\tau_n), \theta_4(a_2\tau), \ldots, \theta_4(a_{m_1}\tau), \theta_3(a_m\tau_n)$$

are $\mathbb{Q}(\beta)$-linearly independent with 1.

In the next corollary, we consider b-ary expansions with different bases and their $\mathbb{Q}$-linearly independence.

Corollary 1.4. *Let $b \geq 2$ be integer and let $(c_{i,n})_n$ be a sequences of positive integers defined over the alphabet $\{1, \ldots, b^i - 1\}$ for $i = 1, 2, \ldots, m$. Then the following base expansions*

$$\sum_{n=1}^{\infty} \frac{c_{1,n}}{b^{n^2}}, \quad \sum_{n=1}^{\infty} \frac{c_{2,n}}{b^{2n^2}}, \ldots, \sum_{n=1}^{\infty} \frac{c_{m,n}}{b^{mn^2}}$$

are $\mathbb{Q}$-linearly independent with 1.

The strategy of the proof of our main result is the following: First we prove the existence of infinitely many positive integers N such that the fractional part of $\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N$ lies in some fixed sub-interval of $[0, 1)$ for $i = 1, \ldots, m$

(which comes from Section 2). Set $N_1 = N$ and $N_i = \left[\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N\right]$ for all $i = 2, \ldots, m$. Suppose the assertion in Theorem 1.2 is not true. Then by these choices of N_i's, we get

$$-\beta^{a_1 N_1^2}\left[c_0 + \sum_{n=1}^{N_1} \frac{c_1}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m}{\beta^{a_m n^2}}\right] = -\beta^{a_1 N_1^2}\left[\sum_{n=N_1+1}^{\infty} \frac{c_1}{\beta^{a_1 n^2}} + \cdots + \sum_{n=N_m+1}^{\infty} \frac{c_m}{\beta^{a_m n^2}}\right],$$

where not all c_i's are zero. Finally using the fact that the left hand side of this equality is an algebraic integer and the right hand side tends to zero as $N \to \infty$, we see that left hand side quantity of this equality is zero and we will repeat this truncation process, till we reach $c_i = 0$ for all $i = 1, \ldots, m$, which hence proves the theorem.

The remaining part of our article is divided into two sections. In Sections 2, we collect all the tools to prove the main result in Theorem 1.2, and in Section 3, we complete the proof of the Theorem 1.2.

2. Preliminaries

Definition. Let $m \geq 1$ be an integer. We say that the sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^m$ is *uniformly distributed mod* 1, if for any subset $E = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m]$ of $[0, 1)^m$, we have

$$\lim_{N\to\infty} \frac{\operatorname{card}\{n | 1 \leq n \leq N, \{x_n\} \in E\}}{N} = \prod_{i=1}^{m} (b_i - a_i), \tag{2.1}$$

where $\{x_n\}$ denotes the fractional parts of each co-ordinates of x_n.

We need the following Theorem which can be found in [[6], pp 49].

Theorem 2.1. *Let $\mathbf{p}(x) = (p_1(x), ..., p_m(x))$, where all $p_i(x)$ are real polynomials, and suppose $\mathbf{p}(x)$ has the property that for each nonzero lattice point $\mathbf{h} = (h_1, \ldots, h_m) \in \mathbb{Z}^m$, the polynomial $(\mathbf{h}, \mathbf{p}(x)) = h_1 p_1(x) + \cdots + h_m p_m(x)$ has at least one non constant term with irrational coefficient. Then the sequence $(\mathbf{p}(n))_n$, $n = 1, 2, \ldots,$ is uniformly distributed mod 1 in $\mathbb{R}^m$.*

As a consequence of Theorem 2.1, we have the following important propositions, which are very crucial for the proof of Theorem 1.2.

Proposition 2.1. *Let $1 \leq \ell \leq m$ be integers and let $\alpha_1, \alpha_2, \ldots, \alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_m$ be real numbers satisfying the following*

(1) *$1, \alpha_1, \ldots, \alpha_\ell$ are $\mathbb{Q}$-linearly independent, and*
(2) *$\alpha_{\ell+j} \in \mathbb{Q}$, for $j = 1, 2, \ldots, m - \ell$.*

Then for any positive integer s, there exist infinitely many positive integers N_1 such that

$$\frac{1}{\sqrt{s+1}} < \{\alpha_i N_1\} < \frac{1}{\sqrt{s}} \quad \textit{for} \quad i = 1, \ldots, \ell$$

and

$$\{\alpha_{\ell+j} N_1\} = 0 \quad \textit{for} \quad j = 1, 2, \ldots, m - \ell.$$

Proof. We denote the common denominator of $\alpha_{\ell+1}, \alpha_{\ell+2}, \ldots, \alpha_m$ by d. Consider the following polynomials

$$q_i(x) = d\alpha_i x, \quad \text{for} \quad i = 1, 2, \ldots, \ell.$$

For any nonzero lattice point $\mathbf{h} = (h_1, h_2, \ldots, h_\ell)$, the quantity $(\mathbf{h}, \mathbf{q}(x))$ given by the polynomial

$$Q(x) = d(h_1\alpha_1 + \cdots + h_\ell \alpha_\ell)x.$$

Since $1, \alpha_1, \ldots, \alpha_\ell$ are $\mathbb{Q}$-linearly independent, the polynomial $Q(x) = (\mathbf{h}, \mathbf{q}(x)) = d(h_1\alpha_1 + \cdots + h_\ell\alpha_\ell)x$ has non constant irrational coefficient. Therefore the polynomial $\mathbf{q}(x)$ satisfies the hypothesis of Theorem 2.1 and hence we conclude that the sequence

$$(\mathbf{q}(n))_n = (q_1(n), \ldots, q_\ell(n))_n$$

is uniformly distributed mod 1. For a given positive integer s, let the subset

$$E = \left[\frac{1}{\sqrt{s+1}}, \frac{1}{\sqrt{s}}\right]^\ell \quad \text{of} \quad [0, 1]^\ell.$$

Since the quantity $\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s+1}}\right) > 0$ for all positive integer s, by (2.1), there exist infinitely many positive integers N such that

$$\frac{1}{\sqrt{s+1}} < \{Nd\alpha_i\} < \frac{1}{\sqrt{s}}, \quad \text{for} \quad k = 1, 2, \ldots, \ell.$$

Hence, by taking $N_1 = dN$, we get infinitely many positive integers N_1 such that $\frac{1}{\sqrt{s+1}} < \{\alpha_i N_1\} < \frac{1}{\sqrt{s}}$ for $i = 1, \ldots, \ell$, and $\{\alpha_{\ell+j} N_1\} = 0$ for $j = 1, 2 \ldots, m - \ell$. This proves the assertion. ☐

Proposition 2.2. *Let $1 \le \ell \le m$ be integers and let $\alpha_1, \ldots, \alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_m$ be positive real numbers satisfying the following*

(1) *$1, \alpha_1, \ldots, \alpha_\ell$ are $\mathbb{Q}$-linearly independent, and*

(2) *$\alpha_{\ell+j} = \sum_{k=1}^{\ell} b_{j,k}\alpha_k$ for all $j = 1, 2, \ldots, m - \ell$, where $b_{j,k} \in \mathbb{Q}$.*

Let $d > 1$ be a fixed positive integer which is divisible by the common denominator of $b_{j,k}$ for all j and k. Then there exist a sufficiently small real number $\epsilon > 0$, $\eta_k = \eta_k(b_{j,k})$ with $0 < \eta_k < 1$ for all $j = 1, 2, \ldots, m - \ell$ and $k = 1, \ldots, \ell$, positive integers $L_{\ell+1}, \ldots, L_m$ and infinitely many positive integers N such that

$$\eta_k < \{N\alpha_k\} < \eta_k + \epsilon^2 \quad \text{for } k = 1, 2, \ldots, \ell$$

and

$$A_{\ell+j}(\epsilon) := \frac{\epsilon L_{\ell+j} + r}{d} < \{N\alpha_{\ell+j}\} < \frac{\epsilon L_{\ell+j} + \epsilon^2 + r}{d} := B_{\ell+j}(\epsilon)$$

for some non-negative integer r with $0 \le r \le d$ and for all $j = 1, 2, \ldots, m - \ell$. Further, for a given positive integer h, we have

$$\frac{B_{\ell+j}(\epsilon)}{A_{\ell+j+i}(\epsilon)} < 1 + \frac{1}{2^h} \text{ for } \ell + 1 \le i < j \le m - \ell.$$

Proof. Consider the following polynomials

$$q_i(x) = \alpha_i x, \quad \text{for} \quad i = 1, 2, \ldots, \ell.$$

For any non-zero lattice point $\mathbf{h} = (h_1, h_2, \ldots, h_\ell)$, the quantity $(\mathbf{h}, \mathbf{q}(x))$ given by the polynomial

$$Q(x) = (h_1\alpha_1 + \cdots + h_\ell\alpha_\ell)x.$$

Since $1, \alpha_1, \ldots, \alpha_\ell$ are $\mathbb{Q}$-linearly independent, the polynomial $Q(x) = (\mathbf{h}, \mathbf{q}(x)) = (h_1\alpha_1 + \cdots + h_\ell\alpha_\ell)x$ has non constant irrational coefficient. Therefore, the polynomial $\mathbf{q}(x)$ satisfies the hypothesis of Theorem 2.1 and hence we conclude that the sequence

$$(\mathbf{q}(n))_n = (q_1(n), \ldots, q_\ell(n))_n$$

is *uniformly distributed mod* 1.

By hypothesis, note that $\alpha_j > 0$ for all j. Therefore, for any given $j = 1, 2, \ldots, m - \ell$, there exists k_0 such that $b_{j,k_0} > 0$. Therefore, for every $1 \le j \le m - \ell$, there exists positive integers $e_1, \ldots, e_\ell$ such that

$$\sum_{k=1}^{\ell} d b_{j,k} e_k > 0 \text{ for all } j = 1, 2, \ldots, m - \ell.$$

We choose least positive integers $e_1, e_2, \ldots, e_\ell$ such that the above is true. Choose ϵ such that

$$0 < \epsilon < \frac{1}{2^{h/2} d^2 e_1 e_2 \cdots e_\ell \prod_{j=1}^{m-\ell} \left(\sum_{k=1}^{\ell} d|b_{j,k}| e_k\right)}$$

and set $\eta_k = \epsilon e_k$ for all $k = 1, 2, \ldots, \ell$. Clearly by the choice of ϵ, we see that $0 < \eta_k < 1$. Also for all $j = 1, 2, \ldots, m - \ell$, we define the positive integer $L_{\ell+j}$ as follows;

$$\sum_{k=1}^{\ell} d b_{j,k} \eta_k = \epsilon \left(\sum_{k=1}^{\ell} d b_{j,k} e_k\right) := \epsilon L_{\ell+j} > 0.$$

Consider the subset E of $[0, 1]^\ell$ as

$$E = [\eta_1, \eta_1 + \epsilon^3] \times [\eta_2, \eta_2 + \epsilon^3] \times \cdots \times [\eta_\ell, \eta_\ell + \epsilon^3].$$

Since the quantity $\eta_k + \epsilon^3 - \eta_k = \epsilon^3 > 0$ for every k, by (2.1), there exists an infinite subset T of $\mathbb{N}$ such that

$$\eta_k < \{N\alpha_k\} < \eta_k + \epsilon^3, \quad \text{for all} \quad k = 1, 2, \ldots, \ell \quad \text{and for all } N \in T. \tag{2.2}$$

Now we work with this set T. For any $N \in T$, by hypothesis, consider

$$N\alpha_{\ell+j} = \sum_{k=1}^{\ell} b_{j,k} N\alpha_k = \sum_{k=1}^{\ell} b_{j,k} \left([N\alpha_k] + \{N\alpha_k\}\right).$$

Since d is common denominator of $b_{j,1}, \dots, b_{j,\ell}$, by multiplying d both sides to the above equality, we have

$$dN\alpha_{\ell+j} - \sum_{k=1}^{\ell} db_{j,k}[N\alpha_k] = \sum_{k=1}^{\ell} db_{j,k}\{N\alpha_k\}. \tag{2.3}$$

Notice that the second term on the left hand side of (2.3) is an integer. In order to prove the second assertion, we first claim the following.

Claim 1. *For every j, $1 \le j \le m - \ell$, we have*

$$\epsilon L_{\ell+j} < \left(\sum_{k=1}^{\ell} db_{j,k}\{N\alpha_k\}\right) < \epsilon L_{\ell+j} + \epsilon^2$$

holds for infinitely many positive integers $N \in T$.

By (2.2), we have

$$\sum_{k=1}^{\ell} db_{j,k}\eta_k < \sum_{k=1}^{\ell} db_{j,k}\{N\alpha_k\} < \sum_{k=1}^{\ell} db_{j,k}\eta_k + \sum_{k=1}^{\ell} db_{j,k}\epsilon^3. \tag{2.4}$$

Since

$$\sum_{k=1}^{\ell} db_{j,k}\eta_k = \epsilon\left(\sum_{k=1}^{\ell} db_{j,k}e_k\right) = \epsilon L_{\ell+j},$$

by substituting the estimate in (2.4), we get

$$\epsilon L_{\ell+j} < \sum_{k=1}^{\ell} db_{j,k}\{N\alpha_k\} < \epsilon L_{\ell+j} + \epsilon^2$$

by the choice of ϵ. This proves Claim 1.

By the choice of ϵ, we see that $0 < \eta_k < 1$ and $\epsilon L_{\ell+j} + \epsilon^2 < 1$, for all $j = 1, 2, \dots, m - \ell$. Thus from (2.3), we conclude that

$$\{Nd\alpha_{\ell+j}\} = \sum_{k=1}^{\ell} db_{j,k}\{N\alpha_k\},$$

and hence by Claim 1, we get

$$\epsilon L_{\ell+j} < \{Nd\alpha_{\ell+j}\} < \epsilon L_{\ell+j} + \epsilon^2, \quad \text{for} \quad j = 1, 2, \dots, m - \ell. \tag{2.5}$$

Therefore there exist infinitely many $N \in T$ satisfying (2.5); that is, for all natural number $N \in T$, we have

$$\frac{\epsilon L_{\ell+j}}{d} < \frac{\{Nd\alpha_{\ell+j}\}}{d} < \frac{\epsilon L_{\ell+j} + \epsilon^2}{d} \quad \text{for} \quad j = 1, 2, \dots, m - \ell.$$

Since

$$\{Nd\alpha_{\ell+j}\} = Nd\alpha_{\ell+j} - [Nd\alpha_{\ell+j}] = d[N\alpha_{\ell+j}] + d\{N\alpha_{\ell+j}\} - [Nd\alpha_{\ell+j}],$$

by re-writing the above equality, we have

$$\frac{\{Nd\alpha_{\ell+j}\}}{d} = \{N\alpha_{\ell+j}\} + \frac{d[N\alpha_{\ell+j}] - [Nd\alpha_{\ell+j}]}{d}.$$

Thus, from (2.5), we obtain that

$$\frac{\epsilon L_{\ell+j}}{d} < \{N\alpha_{\ell+j}\} + \frac{d[N\alpha_{\ell+j}] - [Nd\alpha_{\ell+j}]}{d} < \frac{\epsilon L_{\ell+j} + \epsilon^2}{d} \tag{2.6}$$

holds for $N \in T$. By Hermite's identity, for any natural number n, we know that

$$[n\alpha] = [\alpha] + \left[\alpha + \frac{1}{n}\right] + \dots + \left[\alpha + \frac{n-1}{n}\right] \le [\alpha] + [\alpha + 1] + \dots + [\alpha + 1] = n[\alpha] + n - 1$$

for all real number $\alpha \ge 1$ and hence $n[\alpha] \le [n\alpha] \le n[\alpha] + n - 1$.

First we note that in the inequality $n[\alpha] \le [n\alpha]$, the case $n[\alpha] = [n\alpha]$ occurs when $\left[\alpha + \frac{j}{n}\right] = [\alpha]$ for all $1 \le j \le n-1$, otherwise this inequality is strict. This can be seen as follows: if $n[\alpha] = [n\alpha]$, then by Hermite's identity, we get

$$n[\alpha] = [\alpha] + \cdots + [\alpha] = n[\alpha] = [\alpha] + \left[\alpha + \frac{1}{n}\right] + \cdots + \left[\alpha + \frac{n-1}{n}\right].$$

Then from this inequality combining with the fact $[\alpha] \le \left[\alpha + \frac{r}{n}\right]$ for $1 \le r \le n-1$, we conclude that equality $n[\alpha] = [n\alpha]$ holds if and only if $\left[\alpha + \frac{r}{n}\right] = [\alpha]$ for all $1 \le r \le n-1$.

In our case, we have $n = d$ which is a fixed integer and $\alpha = N\alpha_{\ell+j}$ where N varies over the elements of T. Since T is infinite, there exist an infinite subset $T_0 \subset T$ and an integer r with $0 \le r \le d-1$ such that

$$[dN\alpha_{\ell+j}] - d[N\alpha_{\ell+j}] = r \text{ for all } N \in T_0.$$

Thus from (2.5), for every $1 \le j \le m - \ell$, we have

$$\frac{\epsilon L_{\ell+j} + r}{d} < \{N\alpha_{\ell+j}\} < \frac{\epsilon L_{\ell+j} + \epsilon^2 + r}{d}$$

holds for all $N \in T_0$. Note that by the choice of ϵ, the quantity $(\epsilon L_{\ell+j} + \epsilon^2 + r)/d$ is much less than 1.

For the moreover part of the proposition, by the choice of ϵ, and by noting that $B_{\ell+j}(\epsilon)$ and $A_{\ell+j+i}(\epsilon)$ are lying in a tiny interval inside $(r/d, (r+1)/d)$, we can conclude the assertion. This proves the proposition. ☐

Lemma 2.1. *Let $b_1 < b_2 < \cdots < b_\ell$ be given positive integers and for each $i = 1, 2, \ldots, \ell$, let $f_i : \mathbb{N} \to \mathbb{Z}\backslash\{0\}$ such that $f_i(n) = O(n^k)$ for some non-negative integer k. Let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1. Let T be an infinite subset of natural numbers $\mathbb{N}$ and for all large enough integer $N \in T$, we let $N_1 = N_1(N)$ be a linear function of N and $N_i = [\sqrt{(b_1/b_i)}N_1]$ for all $i = 2, 3, \ldots, \ell$. Let*

$$X_{N_1} := \beta^{a_1 N_1^2}\left(c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{b_1 n^2}} + \cdots + \sum_{n=1}^{N_\ell} \frac{c_\ell f_\ell(n)}{\beta^{b_\ell n^2}}\right)$$

for some algebraic integers $c_i \in K = \mathbb{Q}(\beta)$ such that $|X_{N_1}| = O\left(\frac{1}{\beta^{N^c}}\right)$ for some positive constant c and for all large enough $N \in T$ with the implied constant in O does not depend on N. Then $X_{N_1} = 0$.

Proof. Note that by the definition of N_i's it is clear that X_{N_1} is an algebraic integer in K with the property that $|X_{N_1}| = O\left(\frac{1}{\beta^{N^c}}\right)$. If we prove that the norm of X_{N_1}, denoted by $\mathrm{norm}(X_{N_1})$, in K is 0, then it follows that $X_{N_1} = 0$. In order to prove $\mathrm{norm}(X_{N_1}) = 0$, we estimate all the conjugates of X_{N_1} and prove that their product is < 1. Being an integer, $X_{N_1} = 0$ follow at once.

Let $\sigma : K \to \mathbb{C}$ be a non trivial embedding of K. Now, we estimate the conjugate of $\sigma(X_{N_1})$ as

$$\begin{aligned} |\sigma(X_{N_1})| &= \left|\sigma(\beta)^{b_1 N_1^2}\left(\sigma(c_0) + \sum_{n=1}^{N_1} \frac{\sigma(c_1) f_1(n)}{\sigma(\beta)^{b_1 n^2}} + \cdots + \sum_{n=1}^{N_\ell} \frac{\sigma(c_\ell) f_\ell(n)}{\sigma(\beta)^{b_\ell n^2}}\right)\right| \\ &\le |\sigma(\beta)|^{\ell_1 N_1^2}|\sigma(c_0)| + \sum_{n=1}^{N_1} |f_1(n)\sigma(c_1)||\sigma(\beta)|^{b_1 N_1^2 - b_1 n^2} + \cdots + \sum_{n=1}^{N_\ell} |f_\ell(n)\sigma(c_\ell)||\sigma(\beta)|^{b_1 N_1^2 - b_\ell n^2}. \end{aligned}$$

Since $b_1 N_1^2 - b_i n^2 \ge 0$ for all $n = 1, \ldots, N_i$ and for all $i = 1, 2, \ldots, \ell$ and by the hypothesis $|\sigma(\beta)| \le 1$ for every embedding $\sigma \ne \mathrm{id} : K \to \mathbb{C}$, we obtain

$$|\sigma(X_{N_1})| \le H\left(1 + \sum_{n=1}^{N_1} |f_1(n)| + \cdots + \sum_{n=1}^{N_\ell} |f_\ell(n)|\right),$$

where $H = \max\{|\sigma(c_i)| : \sigma \ne \mathrm{id} : K \to \mathbb{C} \quad \text{and} \quad i = 0, 1, \ldots, \ell\}$. Using $f_i(n) = O(n^k)$ for some $k \ge 0$ with some absolute constant involved in O, we get the estimate for $|\sigma(X_{N_1})|$

$$|\sigma(X_{N_1})| = O\left(1 + \sum_{n=1}^{N_1} n^k + \cdots + \sum_{n=1}^{N_\ell} n^k\right).$$

By the Faulhaber's formula, it is well-known that.

$$\sum_{n=1}^{N_i} n^k = \frac{N_i^{k+1}}{k+1} + \frac{1}{2}N_i^k + \sum_{n=2}^{k} \frac{B_n}{n!}\frac{k!}{k-n+1}N_i^{k-n+1},$$

where B_n is the nth Bernoulli's number. Therefore for fix non-negative integer k, we have

$$\sum_{n=1}^{N_i} n^k = \frac{N_i^{k+1}}{k+1} + \frac{1}{2}N_i^k + \sum_{n=2}^{k} \frac{B_n}{n!}\frac{k!}{k-n+1}N_i^{k-n+1} = O(N_i^{k+1}),$$

where the constant involved in O depends only on k. By substituting this estimate, we obtain

$$|\sigma(X_{N_1})| = O(1 + N_1^{k+1} + \cdots + N_m^{k+1}) = O(N_1^{k+1}) \tag{2.7}$$

where the constant involved in O does not depend on N. Since

$$|\text{norm}(X_{N_1})| = |X_{N_1}| \prod_{\sigma \neq \text{id}: K \to \mathbb{C}} |\sigma(X_{N_1})|,$$

by (2.7), it follows that

$$|\text{norm}(X_{N_1})| = O\left((N_1^{k+1})^{d-1} \times \left(\frac{1}{\beta^{N_1^c}}\right)\right) = O\left(\frac{1}{\beta^{N_1^{c'}}}\right)$$

for some positive constant c' where d is the degree of K over $\mathbb{Q}$ and the constant involved in O depends only on k, ℓ, β and c_i's. Thus by choosing N_1 sufficiently large, we clearly see that $|\text{norm}(X_{N_1})| < 1$ and hence we get $\text{norm}(X_{N_1}) = 0$. □

We prove a technical proposition which roughly says that for linearly independence of series, it is enough to consider the partial sums, with mild conditions.

Proposition 2.3. *Let $b_1 < b_2 < \cdots < b_\ell$ be given positive integers and for each $i = 1, 2, \ldots, \ell$, let $f_i : \mathbb{N} \to \mathbb{Z}\backslash\{0\}$ such that $f_i(n) = O(n^k)$ for some non-negative integer k. Let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1. Let T be an infinite subset of natural numbers $\mathbb{N}$ and for all large enough integer $N \in T$, we let $N_1 = N_1(N)$ be a linear function in N and let $N_i = [\sqrt{(b_1/b_i)}N_1]$ for all $i = 2, 3, \ldots, \ell$ such that for any integer i with $1 \leq i \leq \ell$ and for any integer $r \geq 0$, suppose that*

$$b_i(N_i + r)^2 - b_1 N_1^2 \geq r(2N_1 + r). \tag{2.8}$$

If

$$c_0 + c_1 \sum_{n=1}^{\infty} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{\infty} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0 \tag{2.9}$$

for some algebraic integers $c_i \in \mathbb{Q}(\beta)$ for all $i = 0, 1, \ldots, \ell$, then we get

$$c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0 \tag{2.10}$$

for all large enough $N \in T$.

Proof. Let $N \in T$ be a large enough integer and N_1 is a positive integer defined as a function of N and $N_i = [\sqrt{(b_1/b_i)}N_1]$ for all $i = 2, 3, \ldots, \ell$ such that for any integer i with $1 \leq i \leq \ell$ and for any natural number r, we have (2.8) holds true. Then by (2.9), we get

$$c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = -c_1 \sum_{n=N_1+1}^{\infty} \frac{f_1(n)}{\beta^{b_1 n^2}} - \cdots - c_\ell \sum_{n=N_\ell+1}^{\infty} \frac{f_\ell(n)}{\beta^{b_\ell n^2}}.$$

Multiplying by $\beta^{b_1 N_1^2}$ on both sides, we get

$$\beta^{b_1 N_1^2}\left[c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell} \frac{f_\ell(n)}{\beta^{b_\ell n^2}}\right] = -\beta^{b_1 N_1^2}\left[c_1 \sum_{n=N_1+1}^{\infty} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=N_\ell+1}^{\infty} \frac{f_\ell(n)}{\beta^{b_\ell n^2}}\right]. \tag{2.11}$$

Let

$$X_{N_1} := \beta^{b_1 N_1^2}\left(c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{b_1 n^2}} + \cdots + \sum_{n=1}^{N_\ell} \frac{c_\ell f_\ell(n)}{\beta^{b_\ell n^2}}\right).$$

Then by the definition of N_i's, it follows that X_{N_1} is an algebraic integer in $K = \mathbb{Q}(\beta)$. If we prove that $|X_{N_1}| = O\left(\frac{1}{\beta^{N_1}}\right)$ for all large enough $N \in T$ with the implied constant in O does not depend on N, then by Lemma 2.1, it follows that $X_{N_1} = 0$ and the assertion follows.

Since $f_i(n) = O(n^k)$ for some integer $k \geq 0$ with an absolute constant involved in O, and by (2.8) with the fact that $N_i \leq N_1$, we have

$$\begin{aligned}\left|-\beta^{b_1 N_1^2}\left[c_i \sum_{n=N_i+1}^{\infty} \frac{f_i(n)}{\beta^{b_i n^2}}\right]\right| &\leq C|c_i|\left[\frac{(N_i+1)^k}{\beta^{2N_1+1}} + \frac{(N_i+2)^k}{\beta^{2(2N_1+2)}} + \cdots + \frac{(N_i+r)^k}{\beta^{2(2N_1+r)}} + \cdots\right] \\ &\leq C|c_i|\left[\frac{(N_1+1)^k}{\beta^{2N_1+1}} + \frac{(N_1+2)^k}{\beta^{2(2N_1+2)}} + \cdots + \frac{(N_1+r)^k}{\beta^{2(N_1+r)}} + \cdots\right] \\ &\leq \frac{C|c_i|}{\beta^{N_1}}\left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots\right) = O\left(\frac{1}{\beta^{N_1}}\right), \end{aligned} \tag{2.12}$$

because k is a given constant and the estimate is valid for all large enough N_1, where the implied O constant depends only on c_i's and β, not on the parameter N. Therefore, by Lemma 2.1, the proposition follows. □

We prove another technical lemma and a proposition which are useful in the proof of the main theorem.

Lemma 2.2. *Let $b_1 < b_2 < \cdots < b_\ell$ be given positive integers and for each $i = 1, 2, \ldots, \ell$, let $f_i : \mathbb{N} \to \mathbb{Z}\backslash\{0\}$ such that $f_i(n) = O(n^k)$ for some non-negative integer k. Let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1. Let T be an infinite subset of natural numbers $\mathbb{N}$ and for all large enough integer $N \in T$, we let $N_1 = N_1(N)$ be a linear function of N, $N_i = [\sqrt{(b_1/b_i)}N_1]$ for all $i = 2, 3, \ldots, \ell$ and $r_1 = r_1(N) \leq 2\sqrt{b_1 b_2}N_1 - 2$. Suppose that*

$$b_i N_i^2 - b_1 N_1^2 + r_1 \geq \kappa_2 N_1^{c''} \tag{2.13}$$

for all $i = 2, \ldots, \ell$ and for some positive constants $c'' \leq 1$ and κ_2. If

$$c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0, \tag{2.14}$$

for some algebraic integers $c_i \in \mathbb{Q}(\beta)$, not all zero, then

$$c_0 + c_1 \sum_{n=1}^{N_1-1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell-1} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0. \tag{2.15}$$

Proof. For a large enough integer $N \in T$, it is given that

$$c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0.$$

This implies

$$c_0 + c_1 \sum_{n=1}^{N_1-1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell-1} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = \frac{c_1 f_1(N_1)}{\beta^{b_1 N_1^2}} + \cdots + \frac{c_\ell f_\ell(N_\ell)}{\beta^{b_\ell N_\ell^2}}.$$

Multiplying this equality by $\beta^{b_1 N_1^2 - r_1}$, to get

$$\beta^{b_1 N_1^2 - r_1}\left(c_0 + c_1 \sum_{n=1}^{N_1-1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell-1} \frac{f_\ell(n)}{\beta^{b_\ell n^2}}\right) = \frac{c_1 f_1(N_1)}{\beta^{b_1 N_1^2 + r_1 - b_1 N_1^2}} + \cdots + \frac{c_\ell f_\ell(N_\ell)}{\beta^{b_\ell N_\ell^2 - b_1 N_1^2 + r_1}}.$$

Note that

$$b_1 N_1^2 - r_1 - b_i(N_i - 1)^2 = b_1 N_1^2 - b_i N_i^2 + 2b_i N_i - b_i - r_1$$

and since $b_1 N_1^2 - b_i N_i^2 \geq 0$ and $r_1 \leq 2\sqrt{b_1 b_2}N_1 - 2$, we see that the left hand side of this equality is an algebraic integer in $\mathbb{Q}(\beta)$. By the hypothesis (2.13) and $f_i(N_i) = O(N_i^k)$ for a fixed non-negative integer k, we see that the right hand side is $O\left(\frac{1}{\beta^{N^{c''}}}\right)$ for some positive constant $c'' \leq c$ with implied constant in O does not depend on N.

Therefore, by Lemma 2.1, we conclude that $c_0 + c_1 \sum_{n=1}^{N_1-1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell-1} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0$. This proves the lemma. □

Proposition 2.4. *Let $b_1 < b_2 < \cdots < b_\ell$ be given positive integers and for each $i = 1, 2, \ldots, \ell$, let $f_i : \mathbb{N} \to \mathbb{Z}\backslash\{0\}$ such that $f_i(n) = O(n^k)$ for some non-negative integer k. Let $\beta > 1$ be a real algebraic integer such that all its Galois conjugates are distinct from β and have absolute value less than or equal to 1. Let T be an infinite subset of natural numbers $\mathbb{N}$ and for all large enough integer $N \in T$, we let $N_1 = N_1(N)$ be a linear function in N, $N_i = [\sqrt{(b_1/b_i)}N_1]$ for all $i = 2, 3, \ldots, \ell$ and $r_1 = r_1(N_1)$ are as in Lemma 2.2 such that (2.13) is satisfied. If (2.14) is true for some algebraic integers $c_i \in \mathbb{Q}(\beta)$, not all zero, then $c_1 = 0$.*

Proof. By hypothesis and by Lemma 2.2, we get

$$c_0 + c_1 \sum_{n=1}^{N_1-1} \frac{f_1(n)}{\beta^{b_1 n^2}} + \cdots + c_\ell \sum_{n=1}^{N_\ell-1} \frac{f_\ell(n)}{\beta^{b_\ell n^2}} = 0.$$

Now, multiplying both sides by $\beta^{b_1(N_1-1)^2-r_1}$, we get

$$\frac{c_1 f_1(N_1-1)}{\beta^{r_1}} = -\beta^{b_1(N_1-1)^2-r_1}\left(c_0 + \sum_{n=1}^{N_1-2} \frac{c_1 f_1(n)}{\beta^{b_1 n^2}} + \cdots + \sum_{n=1}^{N_\ell-1} \frac{c_\ell f_\ell(n)}{\beta^{b_\ell n^2}}\right). \tag{2.16}$$

Since N_i's satisfies (2.13), we see that

$$b_1(N_1-1)^2 - r_1 - b_i(N_i-1)^2 \geq cN_1$$

for all $i = 2, 3, \ldots, m$. If we let the quantity on the right hand side of (2.16) be X_{N_1-1}, then X_{N_1-1} is an algebraic integer in $K = \mathbb{Q}(\beta)$. Since $X_{N_1-1} = c_1 f_1(N_1-1)/\beta^{r_1}$ and $f_1(n) = n^k$ for a fixed natural number k, it is clear that $|X_{N_1-1}| = O\left(\frac{1}{\beta^{N^{c''}}}\right)$ for some positive constant $c'' < 1$. Therefore, by Lemma 2.1, we get $X_{N_1-1} = 0$. Putting this information in (2.15), we get $c_1 = 0$. This proves the assertion. □

3. Proof of Theorem 1.2

Since $a_1, \ldots, a_m$ are distinct positive integers, we can assume, if necessary by rewriting the indices, that $a_1 < a_2 < \cdots < a_m$. Suppose to the contrary that the numbers in (1.1) are linearly dependent over $\mathbb{Q}(\beta)$. Then there exist algebraic integers $c_0, c_1, \ldots, c_m \in \mathbb{Q}(\beta)$ not all zero such that

$$c_0 + c_1 \sum_{n=1}^{\infty} \frac{f_1(n)}{\beta^{a_1 n^2}} + c_2 \sum_{n=1}^{\infty} \frac{f_2(n)}{\beta^{a_2 n^2}} + \cdots + c_m \sum_{n=1}^{\infty} \frac{f_m(n)}{\beta^{a_m n^2}} = 0. \tag{3.1}$$

We divide the proof of this theorem into three cases.

Case 1. $\sqrt{a_1/a_j} \in \mathbb{Q}$ for all $1 \leq j \leq m$.

Since $\sqrt{a_1/a_j} \in \mathbb{Q}$ for every j, multiplying by common denominator d, we have $\sqrt{a_1/a_j}dN$ is an integer for every integer $N \in \mathbb{N}$ and for $j = 2, 3, \ldots, m$. Choose a large positive integer N which is our parameter and set $N_1 = dN$. Then set $N_i = \left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1$ which is integer for $i = 2, 3, \ldots m$ and hence

$$a_1 N_1^2 - a_i N_i^2 = a_1 N_1^2 - a_i \left(\frac{a_1}{a_i}\right) N_1^2 = 0. \tag{3.2}$$

Also note that for all $i = 2, 3, \ldots, m$ and for any natural number ℓ, we have

$$a_i(N_i+\ell)^2 - a_1 N_1^2 = a_i\left(\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 + \ell\right)^2 - a_1 N_1^2 = 2\ell(a_1 a_i)^{\frac{1}{2}} N_1 + a_i \ell^2 \geq \ell(2N_1 + \ell). \tag{3.3}$$

By these choices of N_i, we truncate the series in (3.1) and by multiplying by $\beta^{a_1 N_1^2}$ on both sides to get

$$\beta^{a_1 N_1^2}\left[c_0 + c_1 \sum_{n=1}^{N_1} \frac{f_1(n)}{\beta^{a_1 n^2}} + \cdots + c_m \sum_{n=1}^{N_m} \frac{f_m(n)}{\beta^{a_m n^2}}\right] = -\beta^{a_1 N_1^2}\left[c_1 \sum_{n=N_1+1}^{\infty} \frac{f_1(n)}{\beta^{a_1 n^2}} + \cdots + c_m \sum_{n=N_m+1}^{\infty} \frac{f_m(n)}{\beta^{a_m n^2}}\right]. \tag{3.4}$$

Then by (3.2), we see that the quantity in the left hand side in (3.4) is an algebraic integer and lies in $\mathbb{Q}(\beta)$. By (3.3), we see that these choices of N_i satisfy the hypothesis of Proposition 2.3. Thus by Proposition 2.3, we get

$$c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \sum_{n=1}^{N_2} \frac{c_2 f_2(n)}{\beta^{a_2 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0 \tag{3.5}$$

valid for all large enough integer N_1. By (3.2), (3.5) and with any choice of $r_1 = [cN_1^{c'}]$ for some positive constants c and $c' \le 1$, we see that the choices of N_i's are satisfying the hypothesis of Lemma 2.2 to conclude that

$$c_0 + \sum_{n=1}^{N_1-1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \sum_{n=1}^{N_2-1} \frac{c_2 f_2(n)}{\beta^{a_2 n^2}} + \cdots + \sum_{n=1}^{N_m-1} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0,$$

holds for all large enough integer N_1. Therefore, by Proposition 2.4, we conclude that $c_1 = 0$.

Now, we replace the role of N_1 by N_2 and a_1 by a_2. Then by applying the same procedure, we get $c_2 = 0$. Thus, by continuing the same way, we can prove that $c_1 = 0 = c_2 = \cdots = c_m$ and hence $c_0 = 0$, which is a contradiction and hence the assertion follows in this case.

Case 2. $1, \sqrt{a_1/a_2}, \ldots, \sqrt{a_1/a_m}$ are $\mathbb{Q}$-linearly independent.

In this case, by Proposition 2.1, there exist infinitely many positive integers N_1 such that

$$\frac{1}{\sqrt{10^{a_m}+1}} < \left\{ \left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 \right\} < \frac{1}{\sqrt{10^{a_m}}} \quad \text{for} \quad i = 2, 3, \ldots, m. \tag{3.6}$$

If we set $N_i = \left[\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 \right]$ for $i = 2, 3, \ldots, m$, then, by (3.6), we have

$$\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 - \frac{1}{\sqrt{10^{a_m}}} < N_i < \left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 - \frac{1}{\sqrt{10^{a_m}+1}}, \quad i = 2, \ldots, m.$$

For all $i = 2, 3, \ldots, m$, we note that

$$a_1 N_1^2 - a_i N_i^2 \ge a_1 N_1^2 - a_i \left(\sqrt{\frac{a_1}{a_i}} N_1 - \frac{1}{\sqrt{10^{a_m}+1}} \right)^2 = \frac{2\sqrt{a_1 a_i}}{\sqrt{10^{a_m}+1}} N_1 - \frac{a_i}{\sqrt{10^{a_m}+1}} > 0. \tag{3.7}$$

Also, note that for $i = 2, 3, \ldots, m$ and for any integer $k \ge 1$, we have

$$\begin{aligned} a_i (N_i + k)^2 - a_1 N_1^2 &> a_i \left(\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 - \frac{1}{\sqrt{10^{a_m}}} + k \right)^2 - a_1 N_1^2 \\ &= a_i \left(k - \frac{1}{\sqrt{10^{a_m}}} \right)^2 + 2\sqrt{a_i a_1} N_1 \left(k - \frac{1}{10^{a_m}} \right) \ge k(2N_1 + k) \end{aligned} \tag{3.8}$$

holds for all sufficiently large values of N_1. With these choices of N_i, by (3.7) and (3.1), we see that the quantity in the left hand side, say, Y_{N_1} of the following equality

$$\beta^{a_1 N_1^2} \left[c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n)}{\beta^{a_m n^2}} \right] = -\beta^{a_1 N_1^2} \left[\sum_{n=N_1+1}^{\infty} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=N_m+1}^{\infty} \frac{c_m f_m(n)}{\beta^{a_m n^2}} \right]$$

is an algebraic integer in $K = \mathbb{Q}(\beta)$ and by (3.8), we see that $|Y_{N_1}| = O\left(\frac{1}{\beta^{N^c}}\right)$ for some positive constant $c \le 1$. Therefore, by Lemma 2.1, we get

$$c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0.$$

Now, by setting $r_1 = \left[\frac{2a_m}{10^{a_m/2}} N_1 \right]$, we see that

$$\begin{aligned} a_i N_i^2 - a_1 N_1^2 + r_1 \quad &\ge \quad a_i \left(\sqrt{\frac{a_1}{a_i}} N_1 - \frac{1}{\sqrt{10^{a_m}}} \right)^2 - a_1 N_1^2 + \frac{2a_m}{10^{a_m/2}} N_1 - 1 \\ &\ge \quad -2\sqrt{\frac{a_1 a_i}{10^{a_m}}} N_1 + \frac{2a_m}{10^{a_m/2}} N_1 - 1 \ge cN_1 \end{aligned}$$

for some positive constant c as $m \ge 2$. Therefore, by Lemma 2.2, we get

$$c_0 + \sum_{n=1}^{N_1-1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \sum_{n=1}^{N_2-1} \frac{c_2 f_2(n)}{\beta^{a_2 n^2}} + \cdots + \sum_{n=1}^{N_m-1} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0$$

and this holds true for infinitely many $N \in T$. Hence, by Proposition 2.4, we conclude that $c_1 = 0$.

Now, we replace the role of N_1 by N_2 and a_1 by a_2. Then by applying the same procedure, we get $c_2 = 0$. Thus, by continuing the same way, we can prove that $c_1 = 0 = c_2 = \cdots = c_m$ and hence $c_0 = 0$, which is a contradiction and hence the assertion follows in this case.

Case 3. $1, \sqrt{a_1/a_2}, \ldots, \sqrt{a_1/a_m}$ are $\mathbb{Q}$-linearly dependent.

By Case 1, we can always assume that $\sqrt{a_1/a_i}$ is irrational for some integer i with $2 \le i \le m$. In this case, for some integer $2 \le \ell \le m$, the numbers

$$1, \sqrt{a_1/a_2}, \ldots, \sqrt{a_1/a_\ell}$$

are $\mathbb{Q}$-linearly independent. We can assume that the set $\{\sqrt{a_1/a_2}, \ldots, \sqrt{a_1/a_\ell}\}$ is a maximal $\mathbb{Q}$-linearly independent subset of $\{\sqrt{a_1/a_j} : 2 \le j \le m\}$, by renaming the indices, if necessary. Thus we have

$$\sqrt{a_1/a_{\ell+i}} = \sum_{k=2}^{\ell} b_{i,k}\sqrt{a_1/a_k}, \quad \text{for all} \quad i = 1, 2, \ldots, m-\ell, \quad \text{where} \quad b_{i,k} \in \mathbb{Q}.$$

By Proposition 2.2, there exists $\epsilon > 0$ and there exist infinitely many positive integers N_1 such that

$$0 < \eta_i < \left\{ \left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 \right\} < \eta_i + \epsilon^2 < 1, \quad \text{for} \quad i = 2, \ldots, \ell, \tag{3.9}$$

for each $j = 1, \ldots, m-\ell$,

$$0 < A_{\ell+j}(\epsilon) = \frac{\epsilon L_{\ell+j}}{d} + \frac{r}{d} < \left\{ \left(\frac{a_1}{a_{\ell+j}}\right)^{\frac{1}{2}} N_1 \right\} < \frac{\epsilon L_{\ell+j} + \epsilon^2}{d} + \frac{r}{d} = B_{\ell+j}(\epsilon) < 1, \tag{3.10}$$

for some integer $0 \le r \le d-1$ where d is a positive integer such that $db_{\ell+j,k} \in \mathbb{Z}$. Here we are freely using the notations of Proposition 2.2 by replacing b_i by a_i. Also, from the construction of η_i, $(\epsilon L_{\ell+j} + \epsilon^2)/d$ in the proof of Proposition 2.2, we can make sure that $\eta_i, \dfrac{\epsilon L_{\ell+j} + \epsilon^2}{d} \le \dfrac{1}{2^h}$ for a given natural number h. For this case, we take $h = 10da_m$. We set

$$N_i = \begin{cases} \left[\left(\frac{a_1}{a_i}\right)^{\frac{1}{2}} N_1 \right], & \text{for } i = 2, \ldots, \ell; \\ \\ \left[\left(\frac{a_1}{a_{\ell+j}}\right)^{\frac{1}{2}} N_1 \right], & \text{for all } j = 1, \ldots, m-\ell \text{ with } 0 \le r \le d-1; \end{cases}$$

Then by the choices of N_i, we rewrite (3.1) as

$$\beta^{a_1 N_1^2} \left[c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n)}{\beta^{a_m n^2}} \right] = -\beta^{a_1 N_1^2} \left[\sum_{n=N_1+1}^{\infty} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=N_m+1}^{\infty} \frac{c_m f_m(n)}{\beta^{a_m n^2}} \right] \tag{3.11}$$

We shall prove that $a_1 N_1^2 - a_i N_i^2 \ge 0$ and for any integer $k \ge 1$, $a_i(N_i + k)^2 - a_1 N_1^2 \ge k(2N_1 + k)$ for all $i = 1, 2, \ldots, m$.

In this case, by definition, we get $a_1 N_1^2 - a_i N_i^2 \ge a_1 N_1^2 - a_i \frac{a_1 N_1^2}{a_i} = 0$ for all $i = 1, 2, \ldots, m$. Hence, for all $i = 1, 2, \ldots, \ell$ and for all natural number k, by (3.9), we have

$$\begin{aligned} a_i(N_i + k)^2 - a_1 N_1^2 &\ge a_i \left(\left(\frac{a_1}{a_i}\right)^{1/2} N_1 - \eta_i - \epsilon^2 + k \right)^2 - a_1 N_1^2 \\ &= 2\sqrt{a_1 a_i}(k - (\eta_i + \epsilon^2))N_1 + a_i(k - (\eta_i + \epsilon^2))^2 \ge k(2N_1 + k), \end{aligned}$$

as $\eta_i + \epsilon^2 < 1$ and $a_i \ge 2$. Now, for all $j = 1, 2, \ldots, m-\ell$ and for all natural number k, by (3.10), we have

$$\begin{aligned} a_{\ell+j}(N_{\ell+j} + k)^2 - a_1 N_1^2 &\ge a_{\ell+j} \left(\left(\frac{a_1}{a_{\ell+j}}\right)^{1/2} N_1 - B_{\ell+j}(\epsilon) + k \right)^2 - a_1 N_1^2 \\ &= 2\sqrt{a_1 a_{\ell+j}}(k - B_{\ell+j}(\epsilon))N_1 + a_{\ell+j}(k - B_{\ell+j}(\epsilon))^2 \ge k(2N_1 + k), \end{aligned}$$

where $B_{\ell+j}(\epsilon) = \dfrac{\epsilon L_{\ell+j} + \epsilon^2}{d} + \dfrac{r}{d} < \dfrac{1}{2^h} + \dfrac{r}{d} < 1$ and $a_{\ell+j} \geq 2$. Therefore, the quantity in the left hand side of (3.1), say, Z_{N_1} is an algebraic integer and $|Z_{N_1}| = O(\beta^{-N_1^c})$ for some positive constant $c \leq 1$. Therefore, by Lemma 2.1, we get

$$c_0 + \sum_{n=1}^{N_1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0. \tag{3.12}$$

Subcase 1. $r = 0$.

Set $r_1 = [2\sqrt{2}N_1]$ and for $i = 1, 2, \ldots, \ell$, we have

$$\begin{aligned} a_i N_i^2 + r_i - a_1 N_1^2 &\geq a_i \left(\sqrt{\frac{a_1}{a_i}} N_1 - (\eta_i + \epsilon^2) \right)^2 + r_1 - a_1 N_1^2 \geq -2\sqrt{a_1 a_i}(\eta_i + \epsilon^2) N_1 + a_i(\eta_i + \epsilon^2)^2 + N_1 \\ &= (1 - 2\sqrt{a_1 a_i}(\eta_i + \epsilon^2)) N_1 + a_i(\eta_i + \epsilon^2)^2. \end{aligned}$$

Since $\eta_i + \epsilon^2 < \dfrac{1}{2^{5a_m}}$, we see that $(1 - 2\sqrt{a_1 a_i}(\eta_i + \epsilon^2) > 0$. Hence for all large enough N_1's we have $a_i N_i^2 + r_1 - a_1 N_1^2 \geq cN_1$ for some positive constant c. Now, for all $j = 1, 2, \ldots, m - \ell$, we get

$$\begin{aligned} a_{\ell+j} N_{\ell+j}^2 + r_1 - a_1 N_1^2 &\geq a_{\ell+j} \left(\sqrt{\frac{a_1}{a_{\ell+j}}} N_1 - B_{\ell+j}(\epsilon) \right)^2 + r_1 - a_1 N_1^2 \\ &\geq -2\sqrt{a_1 a_{\ell+j}} B_{\ell+j}(\epsilon) N_1 + a_{\ell+j} B_{\ell+j}(\epsilon)^2 + 2\sqrt{2} N_1 - 1 \\ &= 2N_1(\sqrt{2} - \sqrt{a_1 a_{\ell+j}} B_{\ell+j}(\epsilon)) + a_{\ell+j} B_{\ell+j}(\epsilon)^2 - 1. \end{aligned}$$

Since in this subcase by (3.10), $B_{\ell+j}(\epsilon) = \frac{\epsilon L_{\ell+j} + \epsilon^2}{d}$, we see that by the choice of ϵ, the quantity $\sqrt{2} - \sqrt{a_1 a_{\ell+j}} B_{\ell+j} > 0$. Hence for all large values of N_1, we obtain $a_{\ell+j} N_{\ell+j}^2 + r_1 - a_1 N_1^2 \geq c_1 N_1$ for some positive constant c_1. Therefore, by Lemma 2.2, we get

$$c_0 + \sum_{n=1}^{N_1 - 1} \frac{c_1 f_1(n)}{\beta^{a_1 n^2}} + \sum_{n=1}^{N_2 - 1} \frac{c_2 f_2(n)}{\beta^{a_2 n^2}} + \cdots + \sum_{n=1}^{N_m - 1} \frac{c_m f_m(n)}{\beta^{a_m n^2}} = 0$$

and this holds true for infinitely many $N \in T$. Hence, by Proposition 2.4, we conclude that $c_1 = 0$.

Now, we replace the role of N_1 by N_2 and a_1 by a_2. Then by applying the same procedure, we get $c_2 = 0$. Thus, by continuing the same way, we can prove that $c_1 = 0 = c_2 = \cdots = c_m$ and hence $c_0 = 0$, which is a contradiction and hence the assertion follows in this subcase.

Subcase 2. r is non-zero.

In this subcase, we prove $c_i = 0$ in a different way as the above method doesn't work. Set $r_1 = [\sqrt{N_1}]$. Multiply $\beta^{a_1 N_1^2 - r_1}$ on both sides of (3.12) to get

$$c_0 \beta^{a_1 N_1^2 - r_1} + \sum_{n=1}^{N_1} \frac{c_1 f_1(n) \beta^{a_1 N_1^2 - r_1}}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n) \beta^{a_1 N_1^2 - r_1}}{\beta^{a_m n^2}} = 0.$$

Therefore, we get

$$\frac{c_1 f_1(N_1)}{\beta^{r_1}} = c_0 \beta^{a_1 N_1^2 - r_1} + \sum_{n=1}^{N_1 - 1} \frac{c_1 f_1(n) \beta^{a_1 N_1^2 - r_1}}{\beta^{a_1 n^2}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m f_m(n) \beta^{a_1 N_1^2 - r_1}}{\beta^{a_m n^2}}. \tag{3.13}$$

Note that

$$a_1 N_1^2 - r_1 - a_1(N_1 - 1)^2 = 2a_1 N_1 - r_1 - a - 1 \geq 2N_1,$$

and for all $i = 2, \ldots, \ell$ and by the choice of r_1, we have

$$a_1 N_1^2 - r_1 - a_i N_i^2 = a_1 N_1^2 - a_i \left(\left(\frac{a_1}{a_i} \right)^{\frac{1}{2}} N_1 - \eta_i \right)^2 - r_1 > 2\eta_i \sqrt{a_1 a_i} N_1 - \sqrt{N_1} - a_i \eta_i^2 > c_3(\epsilon) N_1$$

and for all $j = 1, \dots, m-\ell$, we have

$$a_1N_1^2 - r_1 - a_{\ell+j}N_{\ell+j}^2 = a_1N_1^2 - a_{\ell+j}\left(\left(\frac{a_1}{a_{\ell+j}}\right)^{\frac{1}{2}} N_1 - A_{\ell+j}(\epsilon)\right)^2 - r_1$$
$$> 2\sqrt{a_1a_{\ell+j}}A_{\ell+j}(\epsilon)N_1 - \sqrt{N_1} - a_{\ell+j}A_{\ell+j}(\epsilon)^2 > c_4(\epsilon)N_1,$$

holds for all large values of N_1 and for some positive constants $c_3(\epsilon)$ and $c_4(\epsilon)$ which do not depend on N_1. By these inequalities we see that the right hand side of the equality (3.13) is an algebraic integer and lies in $\mathbb{Q}(\beta)$ and it is $O\left(\beta^{-N_1^c}\right)$ for some positive constant $c \leq 1$. Therefore, by Lemma 2.1, we conclude that

$$c_0 + \sum_{n=1}^{N_1-1} \frac{c_1f_1(n)}{\beta^{a_1n^2}} + \dots + \sum_{n=1}^{N_\ell} \frac{c_\ell f_\ell(n)}{\beta^{a_mn^2}} + \dots + \sum_{n=1}^{N_m} \frac{c_mf_m(n)}{\beta^{a_mn^2}} = 0.$$

Consequently, from (3.12) we obtain $c_1 = 0$ and (3.12) becomes

$$c_0 + \sum_{n=1}^{N_2} \frac{c_2f_2(n)}{\beta^{a_2n^2}} + \dots + \sum_{n=1}^{N_m} \frac{c_mf_m(n)}{\beta^{a_mn^2}} = 0. \tag{3.14}$$

By continuing this way, we assume that $c_i = 0$ for all $i = 1, 2, \dots, \ell$. Thus in order to finish the proof, we need to prove that $c_{\ell+j} = 0$ for all $j = 1, \dots, m-\ell$ by assuming

$$c_0 + \sum_{n=1}^{N_{\ell+1}} \frac{c_{\ell+1}f_{\ell+1}(n)}{\beta^{a_{\ell+1}n^2}} + \dots + \sum_{n=1}^{N_m} \frac{c_mf_m(n)}{\beta^{a_mn^2}} = 0. \tag{3.15}$$

Multiplying by $\beta^{a_{\ell+1}N_{\ell+1}^2}$ on both sides of (3.15) to get

$$\frac{c_{\ell+1}f_{\ell+1}(N_{\ell+1})}{\beta_1^r} = -\beta^{a_{\ell+1}N_{\ell+1}^2-r_1}\left(c_0 + \sum_{n=1}^{N_{\ell+1}-1} \frac{c_{\ell+1}f_{\ell+1}(n)}{\beta^{a_{\ell+1}n^2}} + \dots + \sum_{n=1}^{N_m} \frac{c_mf_m(n)}{\beta^{a_mn^2}}\right). \tag{3.16}$$

Since

$$a_{\ell+1}N_{\ell+1}^2 - r_1 - a_{\ell+1}(N_{\ell+1}-1)^2 > 2a_{\ell+1}N_1 - \sqrt{N_1} - a_{\ell+1}^2 > 0,$$

and for all $j = 2, \dots, m-\ell$,

$$a_{\ell+1}N_{\ell+1}^2 - r_1 - a_{\ell+j}N_{\ell+j}^2 > a_{\ell+1}\left(\left(\frac{a_1}{a_{\ell+1}}\right)^{\frac{1}{2}} N_1 - B_{\ell+1}(\epsilon)\right)^2 - a_{\ell+j}\left(\left(\frac{a_1}{a_{\ell+j}}\right)^{\frac{1}{2}} N_1 - A_{\ell+j}(\epsilon)\right)^2 - r_1$$
$$= -2\sqrt{a_1a_{\ell+1}}B_{\ell+1}(\epsilon)N_1 + 2A_{\ell+j}(\epsilon)\sqrt{a_1a_{\ell+j}}N_1 + a_{\ell+1}B_{\ell+1}^2(\epsilon) - A_{\ell+j}(\epsilon)^2a_{\ell+j} - r_1$$
$$> 2\sqrt{a_1}\left(A_{\ell+j}(\epsilon)\sqrt{a_{\ell+j}} - B_{\ell+1}(\epsilon)\sqrt{a_{\ell+1}}\right)N_1 - \sqrt{N_1}.$$

In order to prove $a_{\ell+1}N_{\ell+1}^2 - r_1 - a_{\ell+j}N_{\ell+j}^2 > 0$, we need to show that $a_{\ell+j} > a_{\ell+1}\left(\frac{B_{\ell+1}(\epsilon)}{A_{\ell+j}(\epsilon)}\right)^2$. In the moreover part of Proposition 2.1, by taking $h = 10d\max\{a_1, \dots, a_m\}$, we can get $1 < \left(\frac{B_{\ell+1}(\epsilon)}{A_{\ell+j}(\epsilon)}\right)^2 < 1 + \frac{1}{2^{a_m}}$. This proves that $a_{\ell+j} > a_{\ell+1}\left(\frac{B_{\ell+1}(\epsilon)}{A_{\ell+j}(\epsilon)}\right)^2$. By these inequalities, we see that the right hand side of (3.16) is an algebraic integer. Since the left hand side $O(\beta^{-N_1^c})$ for some positive constant $c \leq 1$, by , we conclude that $c_{\ell+1} = 0$ and we get

$$c_0 + \sum_{n=1}^{N_{\ell+2}} \frac{c_{\ell+2}f_{\ell+2}(n)}{\beta^{a_{\ell+2}n^2}} + \dots + \sum_{n=1}^{N_m} \frac{c_mf_m(n)}{\beta^{a_mn^2}} = 0.$$

Hence, by continuing this process, we get $c_i = 0$ for all $i = \ell+1, \dots, m$ and hence $c_0 = 0$, a contradiction. Thus this proves the subcase 2 and Case 3. Hence the theorem follows.

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